The Standard Model

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Maxwell's Equations

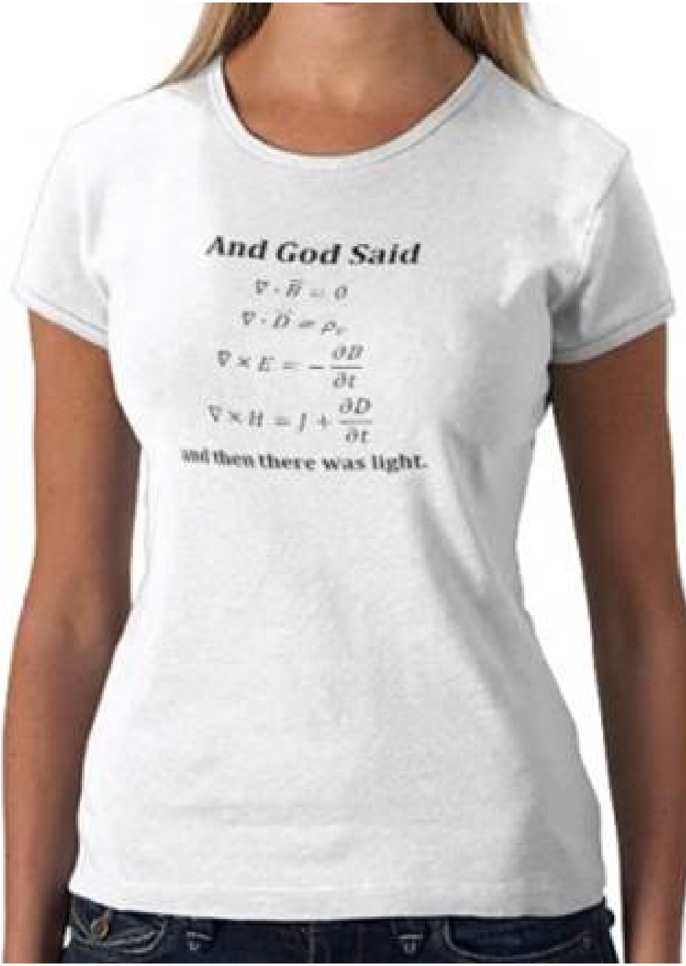
Maxwell's equations in the vacuum, with source terms, are given by

$$\mathrm{div}\vec{B} = 0 \qquad \mathrm{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\mathrm{div}\vec{E} = \frac{\rho}{\epsilon_0} \quad \mathrm{rot}\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{2}$$

The electric field $\vec{E}$ and magnetic induction $\vec{B}$ may be expressed in terms of the *electromagnetic potentials* V (electric) and $\vec{A}$ (magnetic) as

$$\begin{aligned} \vec{E} &= -\,\mathrm{grad}\; V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= -\,\mathrm{rot}\; \vec{A}\,. \end{aligned} \tag{3}$$


And God Said
$\nabla \cdot B = 0$
$\nabla \cdot D = \rho_f$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times H = J + \frac{\partial D}{\partial t}$
and then there was light.

Gauge transformations

The fields are physical, the potentials are not. The fields (and Maxwell's equations) are left unchanged if we transform the potentials as

$$\begin{aligned} V &\rightarrow V' = V + \frac{\partial \Lambda}{\partial t} \\ \vec{A} &\rightarrow \vec{A}' = \vec{A} - \text{grad}\,\Lambda\,, \end{aligned} \tag{4}$$

for a generic differentiable function Λ of the space-time coordinates t, $\vec{r}$.

These are called **GAUGE TRANSFORMATIONS**.

Gauge transformations

The theory is *invariant* under these transformations — it is said it possesses a Gauge Symmetry.

The whole of Electromagnetism can be obtained by starting from requiring that the Universe be invariant under this gauge symmetry.

For instance, invariance under this gauge symmetry directly implies electric charge conservation, expressed in terms of the *continuity equation*

$$\text{div}\,\vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{5}$$

Minkowski Notation

We can define the 4-vector A_μ (*4-potential*),

$$A_\mu = \left(V\,,\,-\vec{A}\right) \implies A^\mu = \left(V\,,\,\vec{A}\right)\,. \tag{6}$$

and the 4-vector J_μ (*4-current*)

$$J_\mu = \left(\rho\,,\,-\vec{J}\right) \implies J^\mu = \left(\rho\,,\,\vec{J}\right)\,. \tag{7}$$

HARD WORK $\Longleftrightarrow$ YOU DO IT!!!

Of course, it is necessary to show that A_μ and J_μ transform like 4-vectors for Lorentz transformations!

We then build the anti-symmetric tensor $F_{\mu\nu}$,

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu \,-\, \partial_\nu A_\mu \;=\; \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{8}$$

so that

$$F^{\mu\nu} \;=\; \partial^\mu A^\nu \,-\, \partial^\nu A^\mu \;=\; \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{9}$$

Due to the definition of $F_{\mu\nu}$, we automatically have (Exercise 1b)

$$\partial_\alpha F_{\beta\gamma} \,+\, \partial_\beta F_{\gamma\alpha} \,+\, \partial_\gamma F_{\alpha\beta} \;=\; 0\,. \tag{10}$$

Minkowski Notation

Maxwell's equations then may be written as (system of units with $\epsilon_0 = 1$)

PURE BEAUTY...

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad \textit{Homogenous equations}, \; (1)$$

$$\partial_\mu F^{\mu\nu} = J^\nu \quad \textit{Inhomogenous equations}, \; (2)$$

The second equation implies (because $F^{\mu\nu}$ is antisymmetric)

$$\partial\mu\partial_\nu F^{\mu\nu} = 0 \Longleftrightarrow \partial_\nu J^\nu = 0 \,, \tag{11}$$

which is the Minkowski formalism version of the charge-conservation equation, eq. (5).

Maxwell lagrangian

- The gauge transformations of eqs. (4) become

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda \tag{12}$$

- The tensor $F_{\mu\nu}$ is clearly invariant under these transformations,

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu + \partial_\mu \partial_\nu \Lambda - \partial_\nu A'_\mu - \partial_\nu \partial_\mu \Lambda = F_{\mu\nu} \tag{13}$$

- Maxwell's equations can be obtained from the *equations of motion* associated with the electromagnetic (Maxwell) lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \,. \tag{14}$$

The terms in F, which involve only derivatives of the A_μ fields, are called the *kinetic terms* of the lagrangian.

Proca lagrangian

The lagrangian for a spin-1 particle with mass M (with no electric charges present), described by the fields A_μ, is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A^\mu A_\mu \qquad (15)$$

For a generic gauge transformation, $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$, we see that:

- The first term in the lagrangian is invariant.
- The second term in the lagrangian, the **mass term**, is *NOT* invariant. (Exercise 1c)

THIS IS CRUCIAL!!

Therefore, we conclude that, for a theory of spin-1 particles to be **gauge invariant, no mass term can be present — those particles (called *Gauge Bosons*)have to be massless.**

Gauge symmetry implications

So, for electromagnetism, we see that:

GAUGE SYMMETRY

$\Longleftrightarrow$

CHARGE CONSERVATION

$\Longleftrightarrow$

MASSLESS PHOTON

Gauge symmetries

This is indeed a generic result, valid for *any* gauge symmetries:

GAUGE SYMMETRY

$$\Longleftrightarrow$$

CONSERVED “CHARGE”

$$\Longleftrightarrow$$

MASSLESS GAUGE BOSON(S)

Gauge symmetries Are Our Friends

- Each interaction/force between particles is *mediated* by particles, which are the *gauge bosons*.
- Each interaction/force is described by a different gauge symmetry.
- Each gauge symmetry is described by a different *symmetry (or gauge) group*.
- Each gauge group is characterized by a basis of *generators*, which may be described by matrices.
- *The number of generators is equal to the number of gauge bosons of the interaction*!!!
- $U(N)$ is the group of $N \times N$ unitary matrices. These groups are *abelian* (their generator matrices commute), and have N^2 generators.
- $SU(N)$ is the group of special (determinant $= 1$) $N \times N$ unitary matrices. These groups are *non-abelian* (their generator matrices don't commute), and have $N^2 - 1$ generators.

QED and QCD

- Electromagnetic interactions are described by the $U(1)_{em}$ gauge symmetry.
- The group has $1^2 = 1$ generators $\Longrightarrow$ QED is mediated by *one* gauge boson, the photon!
- The photon is massless $\Longleftrightarrow$ There is invariance under the gauge symmetry $U(1)_{em}$.

- Strong interactions are described by the $SU(3)_C$ gauge symmetry.
- The group has $3^2 - 1 = 8$ generators $\Longrightarrow$ QCD is mediated by *eight* gauge bosons, the eight gluons!
- The gluons are massless (???) $\Longleftrightarrow$ There is invariance under the gauge symmetry $SU(3)_C$.

Weak interactions

- Weak interactions are described by the $SU(2)_W$ gauge symmetry.
- The group has $2^2 - 1 = 3$ generators $\Longrightarrow$ The weak interactions are mediated by *three* electroweak gauge bosons!
- The electroweak gauge bosons have LARGE masses $\Longleftrightarrow$ There can be no invariance under the gauge symmetry $SU(2)_W$.

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- The electroweak gauge bosons have LARGE masses $\Longleftrightarrow$ There can be no invariance under the gauge symmetry $SU(2)_W$.

WTF!?!?!

Massive electroweak gauge bosons

How do we know electroweak gauge bosons have masses? Several experimental evidences point in that direction:

- The weak interaction has very *short range*, which suggests massive mediators, unlike QED – the massless photon indicates the electromagnetic interaction has infinite range.

  (exception: QCD, due to that mystery called "confinement")

- The weak interaction is, well, WEAK – which can be explained by an exchange of massive mediators, instead of a coupling constant that's extremely small.

- The theory developed by Fermi to successfully explain β-decay implied that high-energy reactions would have cross sections which tended to infinity at high energies – *violation of unitarity*.

Massive electroweak gauge bosons

- For instance, though Fermi's *contact interactions* could explain very well the low-energy decay of the neutron,

$$n \to p + e^- + \bar{\nu}_e \qquad (16)$$

the same theory predicted a cross section for the process

$$p + e^- \to n + \nu_e \qquad (17)$$

which grew with the square of the center-of-mass energy of the colliding particles.

- This pathology was completely solved if there were massive mediators involved in the weak interactions controlling these processes (see diagrams).

- Experimental evidence pointed to their masses being above 50 GeV...

Fermions

Quick reminder: fermions are spin 1/2 particles, described by spinors, which are solutions of Dirac's equation,

$$(i\gamma^\mu \partial_\mu - m)\,\psi \;=\; 0 \tag{18}$$

where the 4×4 gamma matrices are given by, in a given representation,

$$\gamma^0 \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \;,\; \gamma^i \;=\; \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \;, \tag{19}$$

and with $\gamma^5 \;=\; i\gamma^0\gamma^1\gamma^2\gamma^3$.

The lagrangian whence Dirac's equation emerges is given by

$$\mathcal{L}_\psi \;=\; \bar{\psi}\, i\gamma^\mu \partial_\mu \psi \;-\; m\bar{\psi}\,\psi\,. \tag{20}$$

Notice how the mass term involves $\bar{\psi}\,\psi$.

Fermions

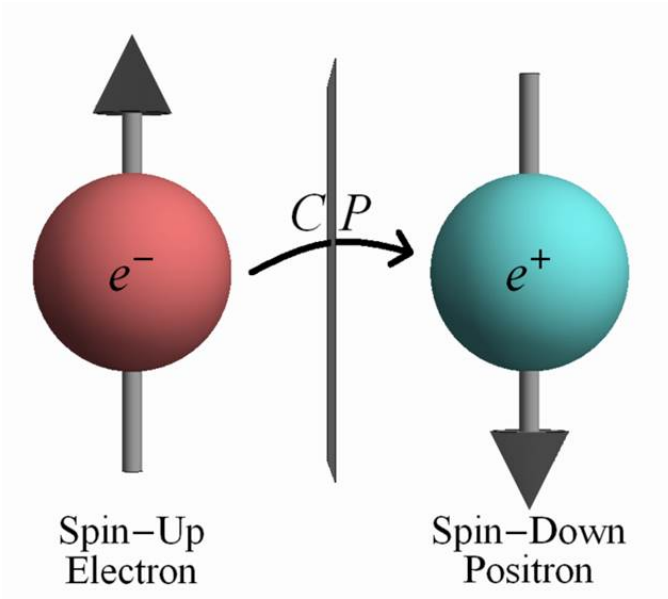
The **left projector** is defined as $P_L = (1-\gamma^5)/2$, and the **right projector** as $P_R = (1+\gamma^5)/2$. Any spinor can therefore be decomposed in its right and left components, defined as

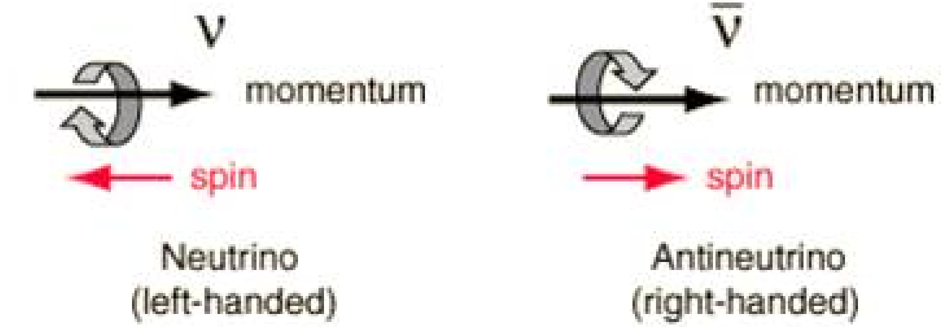
$$\psi = (P_L + P_R)\,\psi = P_L\,\psi + P_R\,\psi = \psi_L + \psi_R\,. \qquad (21)$$

Since P_L and P_R are projectors, they obviously obey $P_L^2 = P_L$, $P_R^2 = P_R$ and $P_L P_R = P_R P_L = 0$ (Exercise 3b).

- The weak interactions treat differently the Left and Right components of fermions.
- For instance, the W bosons only "see" the left components of fermions; the Z only interacts with the left components of neutrinos!

Left and Right





INTERLUDE - SCALAR QED

- Let us for now consider a theory with **scalars** (spin 0) and **photons** (spin 1) interacting among themselves.
- This theory is a *toy model*, it does not describe current particle physics — the only elementary scalar particle known, the Higgs boson, is *neutral* and does not interact (directly...) with photons.
- Nonetheless, we can learn the most important aspects of the Higgs mechanism with this simplified model.

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WE WANT TO BUILD A MODEL WHICH IS GAUGE INVARIANT BUT HAS A MASSIVE SPIN 1 PARTICLE.

Klein-Gordon equation and lagrangian

The first relativist quantum equation developed was Klein-Gordon's. It was developed as an attempt to produce a relativistic version of Schrödinger's equation – it failed. But it describes perfectly free scalar particles.

Consider a single real scalar field, φ, of mass m. It is described by the equation

$$\left(\partial_\mu \partial^\mu + m^2\right) \varphi(x) = \left(\Box + m^2\right) \varphi(x) = 0 \tag{22}$$

which is obtained as the equation of motion from the lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 \tag{23}$$

Again, the terms with the derivatives are called the kinetic terms.

Se um escalar chateia muita gente...

A *charged scalar* is described by two real scalars (one for each charge, "+" or "-", particle and antiparticle), or by a single complex scalar field, and its complex conjugate:

$$\phi(x) = \frac{1}{\sqrt{2}} (\varphi_1 + i\,\varphi_2) \tag{24}$$

The complex field still obeys the Klein-Gordon equation,

$$\left(\Box + m^2\right) \phi(x) = 0 \tag{25}$$

but it has a slightly different lagrangian,

$$\mathcal{L} = \partial_\mu \phi\, \partial^\mu \phi^* - m^2 \, |\phi|^2 \tag{26}$$

Why didn't anyone tell me this when I was a student?!?!

THIS IS NOT CRUCIAL BUT IT CAN SAVE YOU FROM GOING NUTS!!!

The factor of $1/\sqrt{2}$ in the definition of ϕ, in eq. (24), is there to make sure that each of the component scalar fields, φ_1 and φ_2, obey the respective Klein-Gordon equation (22), with lagrangians for each given by eq. (23).

$$\left(\Box + m^2\right)\,\phi(x) \,=\, 0 \quad\Longrightarrow\quad \left\{ \begin{array}{l} \left(\Box + m^2\right)\,\varphi_1(x) \\ \left(\Box + m^2\right)\,\varphi_2(x) \end{array} \right. \tag{27}$$

$$\mathcal{L} \,=\, \partial_\mu\phi\,\partial^\mu\phi^* - m^2\,|\phi|^2 \,=\, \frac{1}{2}\partial_\mu\varphi_1\,\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\,\partial^\mu\varphi_2 - \frac{1}{2}\,m^2\,(\varphi_1^2+\varphi_2^2)$$

Self-interacting scalar fields

The field ϕ can interact with itself, and the rules of quantum field theory dictate that the only interaction possible in this situation is proportional to $|\phi|^4$. The lagrangian for the scalar is therefore

$$\mathcal{L}_\phi \,=\, \partial_\alpha\phi\,\partial^\alpha\phi^* \,-\, V(\phi)\,, \tag{28}$$

where $V(\phi)$ is the interaction potential for the scalars,

$$V(\phi) \,=\, \mu^2\,|\phi|^2 \,+\, \lambda|\phi|^4 \tag{29}$$

I replaced the mass parameter "m" by a different parameter, "μ" – remember that.

The coupling constant λ has to be positive for the theory to have a state of minimum energy – a *vacuum state*...

Notice that $\mathcal{L}_\phi$ is invariant under the transformation $\phi \to e^{i\alpha}\phi$, if α is a constant.

Scalars and "photons" together

- A theory with only charged scalar fields has a lagrangian given by

$$\mathcal{L}_\phi = \partial_\alpha\phi\,\partial^\alpha\phi^* - \mu^2\,|\phi|^2 - \lambda|\phi|^4\,, \tag{30}$$

invariant under $\phi \to e^{i\alpha}\phi$ if α = constant.

- A gauge-invariant theory with only spin-1 fields has a lagrangian given by

$$\mathcal{L}_\mathcal{A} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{31}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the lagrangian is invariant under $A_\mu \to A_\mu + \partial_\mu\Lambda(x)$, for any well-behaved function $\Lambda(x)$.

The 0.000...001 $ question:

How do we build a theory with interacting gauge fields A_μ and scalar fields ϕ which is gauge invariant?

The recipe

It can be shown that the most general gauge-invariant lagrangian possible for interacting A_μ and ϕ can be obtained with the following procedure, called *minimal coupling*:

- Replace, in $\mathcal{L}_\phi$, the derivatives of ϕ by *covariant* derivatives, meaning

$$\partial_\mu \phi \rightarrow D_\mu \phi = (\partial_\mu - i\, e\, A_\mu)\, \phi\,, \tag{32}$$

  where e is a real number (the coupling constant of the $A - \phi$ interaction – it would be the elementary electric charge if this was electromagnetism).

- The resulting lagrangian is invariant under gauge transformations that affect simultaneously the A_μ and ϕ fields, given by

$$\phi \rightarrow \phi' = e^{i\, e\, \alpha(x)}\, \phi \quad , \quad A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha(x) \tag{33}$$

The Scalar QED lagrangian

- Notice that ϕ is now invariant under transformations with a generic function $\alpha(x)$ – these are called *local gauge transformations.*
- apply the recipe and the lagrangian one obtains is

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi\,(D^\mu\phi)^* - V(\phi) \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi\,\partial^\mu\phi^* - \mu^2\,|\phi|^2 - \lambda|\phi|^4 \\
&\quad +ie\left[\phi^*(\partial_\mu\phi) - \phi(\partial_\mu\phi^*)\right]A^\mu \,+\, e^2\phi^*\phi A^\mu A_\mu\,. \qquad (34)
\end{aligned}
$$

- The last line shows interactions terms between the A_μ and ϕ fields – triple and quartic vertices.

DO CALCULATIONS IN THE BOARD, PEDRO, IF THERE'S TIME...

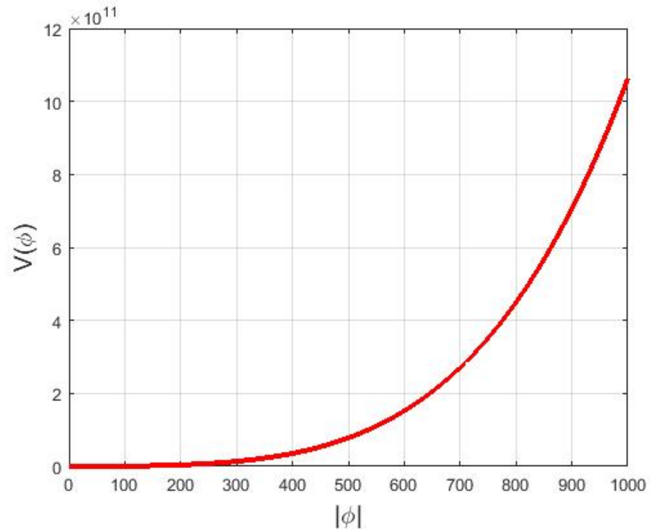
Spontaneous Symmetry Breaking

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi\,\partial^\mu\phi^* \;-\; V(\phi) \\ & +ie\left[\phi^*(\partial_\mu\phi) - \phi(\partial_\mu\phi^*)\right]A^\mu \;+\; e^2\phi^*\phi A^\mu A_\mu\end{aligned}$$

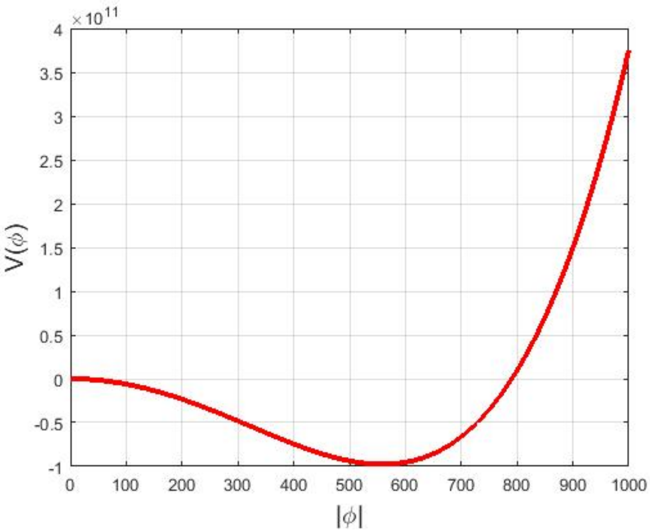
- Notice that there is no mass term for A_μ, due to gauge invariance. *But remember that last term in the lagrangean....*
- The vacuum of the theory – *i.e.* the state of lowest energy, around which small perturbations occur – is determined by the scalar potential $V(\phi)$.
- In fact, since the fields A_μ carries spin 1, in the vacuum state the value of these fields has to be zero, otherwise angular momentum conservation would be broken.
- We therefore say that *the vacuum expectation value* (vev) of the A_μ field is zero, $\langle A_\mu \rangle = 0$.

$$V(\phi) = \mu^2 |\phi|^2 + \lambda|\phi|^4$$

For what values of ϕ do we reach a minimum? This depends, crucially, **on the sign of the μ^2 parameter**.



$\mu^2 > 0$



$\mu^2 < 0$

Spontaneous Symmetry Breaking

- If $\mu^2 > 0$, the minimum of the potential is reached when $|\phi| = 0$, which means the vev of the field is zero.

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- If $\mu^2 < 0$, the minimum of the potential is reached when $|\phi| \neq 0$;

$$\frac{\partial V}{\partial |\phi|} = 0 \Leftrightarrow 2\,\mu^2\,|\phi| + 4\,\lambda\,|\phi|^3 = 0 \tag{35}$$

Spontaneous Symmetry Breaking

- If $\mu^2 > 0$, the minimum of the potential is reached when $|\phi| = 0$, which means the vev of the field is zero.
- If $\mu^2 < 0$, the minimum of the potential is reached when $|\phi| \neq 0$;

$$\frac{\partial V}{\partial |\phi|} = 0 \Leftrightarrow 2\,\mu^2\,|\phi| + 4\,\lambda\,|\phi|^3 = 0 \tag{35}$$

- The vacuum expectation value, vev, of the scalar field is its value at the minimum, thus

$$\langle |\phi|^2 \rangle = \frac{v^2}{2} = -\frac{\mu^2}{2\lambda} > 0 \tag{36}$$

- What's the consequence of this minimum away from the origin?

Spontaneous Symmetry Breaking

- If the minimum was at the origin ($|\phi| = 0$) then the vacuum of the theory would correspond to a fully symmetric state (invariant under all possible gauge transformations applied to the vacuum), but *the gauge boson A_μ would be massless.*
- With the minimum occurring for a value of $\langle\phi\rangle = v/\sqrt{2} \neq 0$, we first need to take care of a detail... The fields in a quantum field theory are supposed to described small (quantum) perturbations around the minimum – so it is to be expected that they assume small values, not $\phi \sim 246/\sqrt{2}$ GeV!
- So what we call the scalar field ϕ must be redefined:

$$\phi = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2) \rightarrow \phi = \frac{1}{\sqrt{2}}(v + h + i\,G) \tag{37}$$

- The new fields h and G have vacuum expectation values equal to zero, we just shifted the fields so as to express small variations around the vacuum (see drawing).

Let There Be MASS!

But this *shift* in the scalar ϕ – $\phi \to \phi + v/\sqrt{2}$ has dramatic consequences in the lagrangian...!!! For instance, in the potential $V(\phi)$, this makes a new term appear,

$$V(\phi) = \mu^2\,|\phi|^2 + \lambda|\phi|^4 \;\to \ldots + \frac{3}{4}\lambda v\,h^3 + \ldots \tag{38}$$

Meaning, spontaneous symmetry breaking predicts a new interaction, a triple-vertex involving the field h.

But the most dramatic consequence is in what happens in the interaction terms between ϕ and A_μ...

$$\begin{aligned} \mathcal{L} &= \ldots + ie\left[\phi^*(\partial_\mu\phi) - \phi(\partial_\mu\phi^*)\right]A^\mu + e^2\phi^*\phi A^\mu A_\mu \\ &\to \ldots + \frac{e^2}{2}h^2 A^\mu A_\mu + \frac{e^2}{2}v\,hA^\mu A_\mu + \frac{1}{2}\,e^2\,v^2\,A^\mu A_\mu \end{aligned} \tag{39}$$

Let There Be MASS!

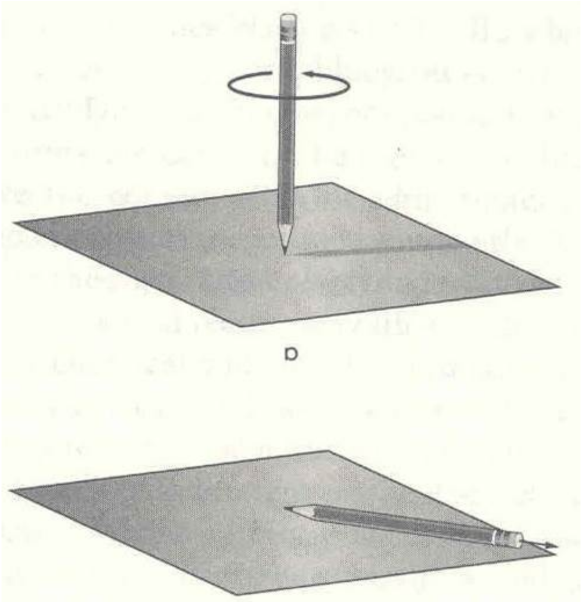
The term $\frac{1}{2}\, e^2\, v^2\, A^\mu A_\mu$ in the expanded lagrangean is a *mass term* for the gauge boson!

So we go from gauge symmetry forcing A_μ to be massless to a mass term being **spontaneously generated** when the lagrangian is written around the minimum.

What happened? Well, the potential exchanged *"being symmetric"* for *"being stable"* – the model chooses, **spontaneously**, a specific vacuum state. That vacuum state is no longer invariant under the gauge symmetry (no rephasing invariance), but the system is now at the state of lowest energy.

Spontaneous Symmetry Breaking

An **unstable** system which is symmetric (no preference for any specific state) will, *spontaneously*, choose a specific state, becoming stable, but **no longer being symmetric**.



Let There Be MASS!

The symmetry has been **spontaneously broken** and a mass term appears for the gauge boson, and the theory is still renormalizable. The mass of the gauge boson is thus (see eq. (15))

$$m_A = e\,v \tag{40}$$

There is a price to pay – the component G in ϕ "vanishes" from the theory, it is an unphysical field (a *Goldstone boson*) which will make up the necessary longitudinal polarization of the now-massive A_μ spin-1 field.

Let There Be MASS!

As for the complex scalar field, we are left with a single spin-0 particle, described by the real field h, with mass given by (see eq. (22))

$$m_h^2 = \left.\frac{\partial^2 V}{\partial h^2}\right|_{at\ the\ minimum} = \mu^2 + 3\lambda v^2 \tag{41}$$

which, using eqs. (35),(36), gives

$$m_h^2 = 2\,\lambda v^2 = -\,2\,\mu^2 \tag{42}$$

So "μ^2" is not the scalar's mass squared (could not be - it is negative!) but it's related to it.

And now for something completely different...

Let us now consider *non-abelian gauge symmetries*.

MATHEMATICS — the difference between *abelian* and *non-abelian* gauge symmetries relates to:

- a) The *generators* of abelian gauge groups commute (examples: $U(N)$), non-abelian groups have non-commuting generators. The simplest example of a non-abelian symmetry group is $SU(2)$, which is the group associated with *spin*, *nuclear spin (isospin)* and... The weak interactions. Its generators are $T_a = i\sigma_a/2$, expressed in terms of the well-known *Pauli matrices:*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{43}$$

- b) *General gauge transformations* of a field for a given group $\mathcal{G}$ (abelian or not) are given by

$$\phi \;\to\; \phi' = e^{i\alpha_a T_a}\phi \tag{44}$$

where the T_a are the N matrix generators of $\mathcal{G}$, α_a are the (infinitesimal) parameters of the transformation (constant for *global* transformations, functions of spacetime coordinates for *local* gauge theories), and sum on the a indices is assumed.

Recall that the exponential of a matrix is given by

$$e^{i\alpha_a T_a} = \sum_{i=1}^{\infty} \frac{(i\alpha_a T_a)^n}{n\,!} \simeq 1 + i\alpha_a T_a\,. \tag{45}$$

Non-abelian gauge symmetries

PHYSICS — the crucial difference between abelian and non-abelian gauge groups is:

- a) In abelian gauge theories, there are no vertices in which gauge bosons interact among themselves. Photons, for instance, do not interact with other photons (directly – at the one-loop level this already can occur).
- b) In non-abelian gauge theories, vertices in which the gauge bosons interact with each other become possible. Gluons, for instance, have tree-level triple and quartic interactions, with crucial relevance for the strong interactions (confinement and asymptotic freedom).

The Standard Model

A wealth of experimental evidence pointed out, in the 60's, that elementary particles had three interactions – nuclear strong, nuclear weak and electromagnetic – which seemed to emerge from a the gauge group $SU(3)_C \times SU(2)_W \times U(1)_Y$, where:

- a) $SU(3)_C$, with "C" standing for "colour", is the gauge group that describes the strong interactions between quarks, mediated by eight massless gauge bosons, the gluons. This symmetry is unbroken, which is why the gluons are massless (??) and the reason (??) why confinement occurs in nature.

- b) $SU(2)_W \times U(1)_Y$ (or $SU(2)_L$), with "W" standing for "weak" (or "L" for "left") and Y representing a quantum number called "hypercharge" is the direct product of the two groups that describe the **electroweak interactions**. ATTENTION: $U(1)_Y$ is NOT the electromagnetism $U(1)_{em}$ gauge group!

The Standard Model (Glashow, Weinberg, Salam) postulates these gauge groups, and predicts that the weak interactions and electromagnetism arise from a common gauge symmetry, $SU(2)_W \times U(1)_Y$. Thus the model is said to describe the **electroweak interactions**.

Forgetting about fermions for now, the model therefore should contain **twelve** spin-1 particles, the eight gluons and the electroweak gauge bosons:

- Three fields, $\{W^1_\mu, W^2_\mu, W^3_\mu\}$, associated with each of the three generators of $SU(2)_W$, $\{T_1, T_2, T_3\} = i\{\sigma_1, \sigma_2, \sigma_3\}/2$.
- One field, B_μ, associated with the hypercharge group $U(1)_Y$.

THE PROBLEM: *all four* electroweak gauge bosons would be massless if the gauge symmetry $SU(2)_W \times U(1)_Y$ is unbroken. Experimental evidence suggests only one – the photon – should have zero mass, the remaining three ought to be quite heavy.

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THE SOLUTION: in 1964, Brout, Englert and Higgs postulated the existence of a new scalar field, which would give mass to the electroweak gauge bosons leaving the photon and gluons massless, through *spontaneous symmetry breaking*.

Weak interactions in 20 seconds or less...

Fields which describe particles which sense the weak interactions are affected by the gauge group generators. Since those generators can be represented by 2×2 matrices, the "lowest irreducible representation" is a DOUBLET.

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(ANALOGY: $SU(2)$ also describes spin. Particles with spin 0 are scalars, invariant under rotation; particles with spin 1/2 are described by a vector with TWO possible states, up or down – a doublet.)

The Higgs Doublet

For very important (to be explained) reasons, the complex scalar Φ (spin 0) introduced by Brout, Englert and Higgs has no colour (it's a $SU(3)$ singlet!) but concerning the electroweak interactions it is *not* a gauge singlet, but rather a $SU(2)$ doublet, carrying hypercharge quantum number $Y_\Phi = 1$:

$$\Phi = \begin{pmatrix} \varphi^+ \\ \frac{1}{\sqrt{2}}\left(\varphi_1 + i\,\varphi_2\right) \end{pmatrix}, \tag{46}$$

where $\varphi^+ = (\varphi_3 + i\,\varphi_4)/\sqrt{2}$.

The *upper components* of a doublet have a "weak isospin projection" $I_3 = +1/2$; the *lower components* have $I_3 = -1/2$.

This is the analog, in quantum mechanics, of using the operator L – angular momentum – and its projection on the z-axis, L_z, to characterize different states.

The Higgs Potential

The most generic $SU(3)_C \times SU(2)_W \times U(1)_Y$ Higgs potential is very similar to eq. (29), being given by

$$V(\Phi) = \mu^2 \, |\Phi|^2 + \lambda |\Phi|^4 \,, \qquad (47)$$

where as before $\lambda > 0$ but where now we have (since Φ is a doublet) $|\Phi|^2 = \Phi^\dagger \Phi$.

As before, when $\mu^2 < 0$, this potential has a minimum away from the origin:

$$\frac{\partial V}{\partial \varphi_i} = 0 \Longleftrightarrow |\Phi|^2 = -\frac{\mu^2}{2\lambda} > 0 \,. \qquad (48)$$

Therefore, in the state of lowest energy the field Φ has a non-zero vacuum expectation value (vev):

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}}, \quad \text{with} \quad v = \sqrt{-\frac{\mu^2}{\lambda}} = 246\,\text{GeV}. \tag{49}$$

The value of v was first determined by measurements of muon decay. As before, we will shift the original field, redefining the component fields so that they represent small perturbations around the minimum:

$$\Phi \to \Phi = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + h + i\,G^0\right) \end{pmatrix}, \tag{50}$$

Gauge-Higgs interactions

As before, the "trick" to write the fully gauge-invariant model is to start with the scalar lagrangian and replace the partial derivatives ∂_μ by covariant derivatives which include the gauge fields, *i.e.*

$$\mathcal{L}_\Phi = (D_\mu \Phi)^\dagger D^\mu \Phi - V(\Phi)\,, \tag{51}$$

where for $SU(2)_W \times U(1)_Y$ the covariant derivative of the scalar doublet is

$$D_\mu \Phi = \left(\partial_\mu - i g T_a W^a_\mu - i \frac{g'}{2} Y_\Phi \mathbb{1} B_\mu\right) \Phi\,, \tag{52}$$

where g and g' are the coupling constants associated with the gauge groups $SU(2)_W$ and $U(1)_Y$, respectively. A sum on a (= 1, 2, 3) is implied

Recall that $\{T_1, T_2, T_3\} = i\{\sigma_1, \sigma_2, \sigma_3\}/2$.

Gauge-Higgs interactions

There are *lots* of different terms that come from the covariant derivatives of eq. (52). But we'll be interested in possible mass terms, which are quadratic in the gauge fields, and involve no derivatives:

$$\mathcal{L}_\Phi = \ldots + \frac{1}{4}\Phi^\dagger \left(g\,\sigma^a W^a_\mu + g'\,Y_\Phi B_\mu\right)\left(g\,\sigma^a W^{a\mu} + g'\,Y_\Phi B^\mu\right)\Phi \quad (53)$$

From this, it is possible to obtain (Exercise 2) that the terms quadratic in the fields may be rewritten as:

$$\mathcal{L}_\Phi = \ldots m_W^2 W^+_\mu W^{-\mu} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu + \frac{1}{2} m_A^2 A_\mu A^\mu\,, \quad (54)$$

where it is possible to obtain:

$$m_W^2 = \frac{1}{4} g^2\, v^2 \quad , \quad m_Z^2 = \frac{1}{4}(g^2 + g'^2)\, v^2 \quad , \quad m_A^2 = 0\,. \quad (55)$$

Physical fields

The physical fields are expressed in terms of the original W^a_μ, B_μ as:

$$\begin{aligned} W^\pm_\mu &= \frac{1}{\sqrt{2}}(W^1_\mu \mp i\,W^2_\mu) && \text{Charged } W \text{ electroweak bosons} \\ Z_\mu &= \frac{g\,W^3_\mu - g'\,B_\mu}{\sqrt{g^2 + g'^2}} && \text{Neutral } Z \text{ electroweak boson} \\ A_\mu &= \frac{g'\,W^3_\mu + g\,B_\mu}{\sqrt{g^2 + g'^2}} && \text{Massless photon} \end{aligned} \tag{56}$$

It can be shown that the A_μ field couples to all particles with a strength given by the elementary electric charge, e, times the quantity

$$Q = I_f + \frac{Y}{2} \tag{57}$$

where I_f is the weak isospin component of the field describing the particle, and Y its hypercharge.

Electric charge and Weinberg angle

Q is therefore the electric charge of the particle in cause!

For the scalar doublet, this means that the lower components have charge $Q_l = -1/2 + 1/2 = 0$ – they are NEUTRAL, the Higgs field has no electric charge! But the lower components would have charge $Q_u = 1/2 + 1/2 = +1$ – they carry one unit of electric charge, and correspond to the Goldstone bosons associated with the $W^\pm$ bosons! We also define the Weinberg angle θ_W, such that $\tan\theta_W = g'/g$. The Higgs mechanism therefore makes the following prediction:

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1 \tag{58}$$

Quantum corrections will allow small deviations from 1 for this quantity, but this remains one of the most powerful checks of the Higgs mechanism. Although other models predict the same thing...

Scalar mass matrix

In terms of the components φ_i of the doublet Φ, the scalar mass matrix is given by

$$[m_h^2]_{ij} = \left.\frac{\partial V}{\partial\varphi_i\partial\varphi_j}\right|_{minimum} = \begin{pmatrix} \mu^2 + 3\lambda v^2 & 0 & 0 & 0 \\ 0 & \mu^2 + \lambda v^2 & 0 & 0 \\ 0 & 0 & \mu^2 + \lambda v^2 & 0 \\ 0 & 0 & 0 & \mu^2 + \lambda v^2 \end{pmatrix} \quad (59)$$

from which we see (recall that $v^2 = -\mu^2/\lambda$) that three eigenvalues are zero – corresponding to the *Goldstone bosons* which were "absorbed" by the now-massive gauge bosons, to make up for the needed longitudinal polarizations.

Scalar mass matrix

The remaining eigenvalue, the non-zero one, corresponds to a physical particle – the Higgs boson!!! – with mass

$$m_h^2 = 2\,\lambda\,v^2 = -2\,\mu^2\,. \tag{60}$$

In 2012 the Higgs boson was finally discovered, and its mass measured to be about 125 GeV. This gives, for instance, a value of 0.2582 for λ.

Thus the Higgs mechanism gives mass to the three electroweak gauge bosons, leaving the photon massless, as well as the gluons. *Equally amazing is the effect of the Higgs particle on the fermions...*

Fermions

Quick reminder: fermions are spin 1/2 particles, described by spinors, which are solutions of Dirac's equation,

$$(i\gamma^\mu \partial_\mu - m)\,\psi \;=\; 0 \tag{61}$$

where the 4×4 gamma matrices are given by, in a given representation,

$$\gamma^0 \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \;,\; \gamma^i \;=\; \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \;, \tag{62}$$

and with $\gamma^5 \;=\; i\gamma^0\gamma^1\gamma^2\gamma^3$.

The lagrangian whence Dirac's equation emerges is given by

$$\mathcal{L}_\psi \;=\; \bar{\psi}\, i\gamma^\mu\, \partial_\mu\, \psi \;-\; m\bar{\psi}\,\psi\,. \tag{63}$$

Notice how the mass term involves $\bar{\psi}\,\psi$.

Fermions

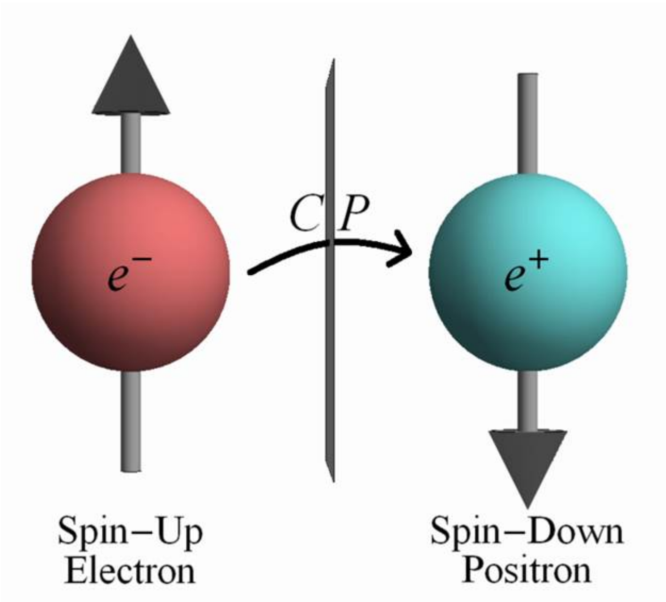
The **left projector** is defined as $P_L = (1-\gamma^5)/2$, and the **right projector** as $P_R = (1+\gamma^5)/2$. Any spinor can therefore be decomposed in its right and left components, defined as

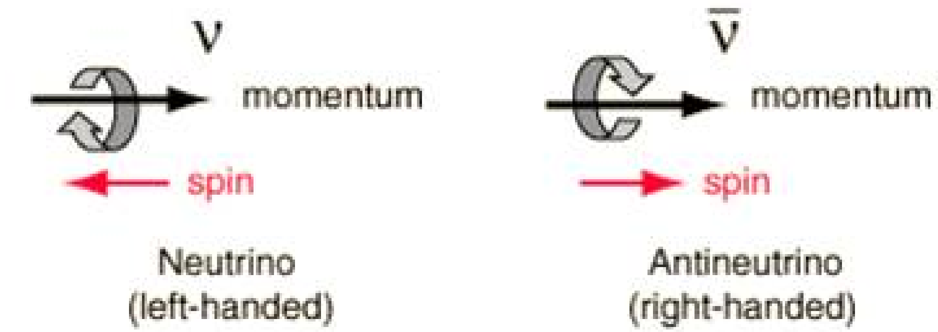
$$\psi = (P_L + P_R)\,\psi = P_L\,\psi + P_R\,\psi = \psi_L + \psi_R\,. \qquad (64)$$

Since P_L and P_R are projectors, they obviously obey $P_L^2 = P_L$, $P_R^2 = P_R$ and $P_L P_R = P_R P_L = 0$.

- The weak interactions treat differently the Left and Right components of fermions.
- For instance, the W bosons only "see" the left components of fermions; the Z only interacts with the left components of neutrinos!

Left and Right





Left and Right

Remember that the mass term of fermions in a lagrangian is given by $m\,\bar{\psi}\,\psi$. What is involved in terms of the left and right components? Remembering that $\bar{\psi} = \psi^\dagger \gamma^0$ and the projectors P_L and P_R are hermitian ($P_L^\dagger = P_L$, $P_R^\dagger = P_R$), we see that:

$$
\begin{aligned}
\overline{\psi_L} &= (P_L\psi)^\dagger\,\gamma^0 = \psi^\dagger\,P_L^\dagger\,\gamma^0 = \psi^\dagger\,\gamma^0\,P_R = \bar{\psi}\,P_R \\
\overline{\psi_R} &= (P_R\psi)^\dagger\,\gamma^0 = \psi^\dagger\,P_R^\dagger\,\gamma^0 = \psi^\dagger\,\gamma^0\,P_L = \bar{\psi}\,P_L
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\bar{\psi}\,\psi &= (\overline{\psi_L} + \overline{\psi_R})\,(\psi_L + \psi_R) = (\bar{\psi}\,P_R + \bar{\psi}\,P_L)\,(P_L\psi + P_R\psi) \\
&= \bar{\psi}\,P_R\,P_R\psi + \bar{\psi}\,P_L\,P_L\psi = \overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L\,. \qquad (65)
\end{aligned}
$$

Which means that the mass term for fermions needs is a product of both the LEFT and RIGHT components.

Left and Right

BUT:

- Since the W does not interact with the right-handed parts of the fermions, this means that the right-handed fermions are SINGLETS of $SU(2)_W$.
- On the other hand, the left-handed parts couple to the W and Z – they belong in a DOUBLET.

For instance, for leptons, they are organized in the following way:

Leptonic doublet (left fields): $L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$, hypercharge $Y_L = -1$.

Leptonic singlet (right fields): e_R, hypercharge $Y_e = -2$.

CHECK ELECTRIC CHARGES!

Yukawa interactions

- Since e_L is in a doublet and e_R in a singlet, it's *impossible* to make a scalar (number) with them! Gauge symmetry again forbids the mass term!
- There would be need for *another* doublet that could form a number with L via an internal product.
- Fortunately, there is, in the theory, another doublet: the Higgs field Φ!

The interaction terms between fermions and scalar are called the Yukawa interactions, and are given by

$$\mathcal{L}_Y = \lambda_e \, \overline{L} \, \Phi \, e_R + \text{h.c.} \tag{66}$$

After spontaneous symmetry breaking, the field Φ is shifted, and we obtain (for the lower components)

$$\mathcal{L}_Y = \ldots + \frac{\lambda_e}{\sqrt{2}} \bar{e}_L \, h \, e_R + \frac{\lambda_e}{\sqrt{2}} v \, \bar{e}_L \, e_R + \text{h.c.} \tag{67}$$

Yukawa interactions and fermion masses

The coefficient λ_e is an unknown dimensionless number, called *the electron Yukawa coupling*. The second term is a mass term for the fermions! Impossible to write in the original gauge-invariant theory due to gauge symmetries, but obtained naturally after spontaneously symmetry breaking. The mass of the electron is therefore given by

$$m_e = \frac{\lambda_e}{\sqrt{2}} v \tag{68}$$

Notice that, since λ_e is unknown, the Higgs mechanism can explain **HOW** the electrons gain mass, but cannot predict *the value* of that mass.

The first term is an interaction between the physical Higgs particle and the electrons! Notice that the strength of that interaction is proportional to λ_e — that is, proportional to the electron mass.

Yukawa interactions and fermion masses

- A similar procedure could be applied to the other leptons: there would be a Yukawa coupling for each of them, and the charged lepton masses are then given by

$$m_e = \frac{\lambda_e}{\sqrt{2}} v \ , \ m_\mu = \frac{\lambda_\mu}{\sqrt{2}} v \ , \ m_\tau = \frac{\lambda_\tau}{\sqrt{2}} v \ . \tag{69}$$

- For each fermion, there are vertices of the form $h \bar{f} f$, where the strength of the interaction is directly proportional to the particle's mass – *The Higgs Boson couples with fermions proportionally to the mass of the particles*.
- A similar mechanism could be used to give mass to neutrinos, but people don't like it and prefer more complicated ways...
- But the same mechanism is used to give mass to the quarks!

Yukawa interactions and quark masses

Quark doublet (left fields): $L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, hypercharge $Y_Q = \frac{1}{3}$.

Right-handed down quark: d_R, hypercharge $Y_d = -\frac{2}{3}$.

Right-handed up quark: u_R, hypercharge $Y_u = \frac{4}{3}$.

Yukawa lagrangian:

$$\mathcal{L}_Y = \lambda_d \overline{Q}\, \Phi\, d_R + \lambda_u \overline{Q}\, \tilde{\Phi}\, d_R + \text{h.c.} \qquad (70)$$

with $\tilde{\Phi} = i\sigma_2 \Phi^*$ the charge conjugate of Φ – the "antiparticle", with opposite hypercharge.

And again, the masses of the quarks are given by

$$m_u = \frac{\lambda_u}{\sqrt{2}} v \;,\; m_d = \frac{\lambda_d}{\sqrt{2}} v \;,\; m_s = \frac{\lambda_s}{\sqrt{2}} v \;,\; m_c = \frac{\lambda_c}{\sqrt{2}} v \;\; \ldots \qquad (71)$$

and the *h* field couples to each fermion with a strength proportional to its mass.