Hadron Electromagnetic Form Factors

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September 3, 2019

Foreword

Electromagnetic form factors are fundamental quantities that parametrize the internal structure of a composite particle and describe its dynamical properties. The most simple reactions that can be studied theoretically and experimentally are elastic electron-proton (neutron) scattering, and the crossed channels as nucleon-antinucleon to (or created by) an electron positron pair. Assuming that the colliding particles interact by exchange of one virtual photon, the transferred momentum squared probes the dynamical structure of the nucleon at the corresponding internal scale. The differential cross section and the polarization observables are expressed in terms of form factors.

The recent experimental achievements open the possibility of very precise measurements in an unexplored kinematical region. A wide program is ongoing and planned at all facilities in the GeV range: electron accelerators: Jefferson Lab (Newport News), electron-positron colliders: VEPPIII (Novosibirk), BEPCII (Beijing), and proton-antiproton colliders as the future FAIR facility, at Darmstadt. From the theoretical point of view, the precise knowledge of the form factors in a wide kinematical range allows to map the transition region, between the non perturbative domain where the nucleon is best described by constituent quarks and mesons, and the perturbative region where QCD can be applied and the nucleon appears as a confined system of quarks and gluons. Analytical and model independent properties of form factors are underlined as a guide for modeling the nucleon structure.

The lectures are devoted to elementary particle processes, the first to electron proton elastic scattering $e+p \to e+p$, and the second to annihilation reactions $e^+ + e^- \to \bar{p} + p$. Both lectures are structured in the same way: after a description of the interest of the specific process, the formulas for the matrix element, the unpolarized and the polarized cross section are derived in detail.

The Introduction traces selected steps in the comprehension of the nucleon structure as well as the meaning of hadronic electromagnetic currents and form factors. After an historical and pedagogical introduction to this field, a formal derivation of electromagnetic form factors for the scattering (Lecture I) and the annihilation channels (Lecture II) is given. The matrix element of these reactions, the unpolarized and polarized cross sections are formally derived. A section is devoted to the present and planned facilities, the experimental results are presented as well as the future plans. The present understanding of the electromagnetic nucleon structure in frame of a global view of form factors unifying the information obtained from the space and time-like regions will conclude the lectures. Exercises are proposed and the solution is given at the end.

The author is grateful for eventual corrections, suggestions and remarks, that should be sent to 'egle.tomasi@cea.fr'.

Contents

Chapter 1

Introduction

"Over a period of time lasting at least two thousand years, Man has puzzled over and sought an understanding of the composition of matter. It is no wonder that his interest has been aroused in this deep question because all objects he experiences, including, even his own body, are in a most basic sense special configurations of matter. The history of physics shows that whenever experimental techniques advance to an extent that matter, as then known, can be analyzed by reliable and proved methods into its "elemental" parts, newer and more powerful studies subsequently show that the "elementary particles" have a structure themselves. Indeed this structure may be quite complex, so that the elegant idea of elementarity must be abandoned."

These words are retranscripted from the Nobel Lecture of R. Hofstadter in 1961. He received the Nobel prize *"for his pioneering studies of electron scattering in atomic nuclei and for his thereby achieved discoveries concerning the structure of the nucleons"*. He lead a series of experiments at the Stanford SLAC accelerator, finding the Q^2 dependence of the form factors (FFs).

Since, the experimental and theoretical investigations of hadron FFs constitute an important field of hadron physics, that studies the properties of matter at the Fermi scale (1 fm= 10^{-15} m), with the aim to understand the dynamics of quarks and gluons confined in neutral color objects. Electromagnetic form factors characterize the internal structure of a composite particle. In a P- and T-invariant theory, the electromagnetic structure of a particle of spin S is characterized by $2S+1$ form factors. Proton and neutron have two FFs, electric G_E and magnetic G_M. They do not have to be equal. FFs are considered fundamental quantities because they are experimentally measurable through differential cross sections and polarization observables of elementary processes as $ep \to ep$ and $e^+e^- \leftrightarrow \bar{p}p$. On the theoretical side they enter in the expression of the electromagnetic current, provided that the reaction occurs through the exchange of a virtual photon. They constitute, therefore, a privileged playground for theory and experiment. Any nucleon model, after reproducing static properties as masses and magnetic moments, is tested on its predictions on FFs. Schematically, FFs at low q^2 probe the size of the nucleus, at high q^2 they test the quark content of the nucleon, where q^2 is the four momentum (the mass) of the virtual photon.

Let us recall selected milestones leading to the present knowledge of the nucleon. For a recent review and complete references, see Ref. [1]

Following the experiment of Geiger and Marsden, in 1909, the cross section for the scattering of electrons in the Coulomb field of a nucleus of charge Z, is given by the Rutherford formula (1911) [2][1]. The Rutherford formula applies to non relativistic, spin zero, pointlike particle scattering. It was used to measure the 'size' of the target and to introduce the concept of atomic nucleus[2]. The Rutherford formula is based on a totally new view of the atom. The atom is no more seen as the 'plum pudding' of W. Thomson 1904, where electrons are embedded in a positively charged matter, and counterbalance its charge, neither as the eternal, indivisible, indestructible particle of Democritus (470-360 B.C.).

In 1968 "deep inelastic scattering" (DIS) experiments at the Stanford Linear accelerator (SLAC,) in which very energetic electrons were scattered off protons showed that all the mass and charge of the proton is concentrated in smaller components, then called "partons". Partons were later identified with quarks (Friedman, Kendall and Taylor, Nobel Prize 1991). In 1929, N.F. Mott derives a formula for relativistic nuclei, that holds for scattering of spin 1/2 pointlike particles [3]. M.N. Rosenbluth extends the formalism to composite targets [4] (1950), on the basis of which R. Hofstadter receives the Nobel Prize [5]. An essential step in terms of precision was taken in the years 1958-1967, when A.I Akhiezer and M.P. Rekalo (Kharkov, Ukraine) pointed out the large sensitivity of the polarized cross section to a small G_E contribution. They explicitly derived the expressions of polarization observables for elastic ep scattering in terms of form factors [6, 7]. Their method not only brings a better precision on the measurements but allows also to determine the sign of FFs. It will be described in detail in Section 2. It has been applied experimentally only recently [8]. The precise data obtained in the scattering region and future plans are recalled in Section 4.1.2.

The second lecture is devoted to annihilation reactions. This domain is less explored as it requires electron-positron or proton-antiproton colliders, where the achieved luminosity was lower. In 1962, A. Zichichi, S. M. Berman, N. Cabibbo, R. Gatto derived cross section and single spin observables in terms of FFs in the annihilation process $p + \bar{p} \to e^+ + e^-$ [9]. The vector meson dominance model (VDM) of F. Iachello, A.D. Jackson, A. Landé was proposed in 1973, and it is still at the bases of a modelization of FFs in the whole kinematical region [10]. The first experiments with anti-protons were

[1] The observation of backscattered α particles, that can happen only if the target has smaller mass, was so surprising that, from the words of G. Marsden, : *Then I remember two or three days later Geiger coming to me in great excitement and saying " We have been able to get some of the alpha-particles coming backward...It was quite the most incredible event that ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you".*

[2] To get familiar with Rutherford scattering, different sites have been built to play with. See, for example, http://waowen.screaming.net/revision/nuclear/rssim.htm

done in the years 1983-1994, at LEAR(CERN) [11]. The PS170 collaboration measured firstly the annihilation cross section for $\bar{p}p \to e^+e-$ and the form factor ratio in this region. In 1998 the Frascati e^+e^- collider detected not only final state proton, but neutron too [12, 13]. Larger transferred momenta were reached in FermiLab, by the E835 collaboration during the years 1999-2003 [14]. After the years 2000 a new generation of e^+e^- colliders were built: BELLE in Japan, BES@BEPC recently upgraded to BESIII at BEPCII at Beijing. In particular BABAR@SLAC opened a new way by applying the initial state radiation (ISR) method [15]. Based on the quasi-real electron method [16], one can factorize out the radiator factor in the reaction $e^+ + e^- \to p + \bar{p} + \gamma$, that depends on the energy and angle of the emitted (hard) photon, tuning the momentum effectively transferred to the $e^+ + e^- \to p + \bar{p}$ system. Since, this is a method widely used. The state of the art of the experimental data is described in Section 4.2.

Outside the purpose of these lecture, one should mention a very important issue, related to radiative corrections that must be applied to the experimental cross section in order to recover the Born cross section, and extract FFs. Accelerated charged particles emit radiation, that modifies their energy and momentum and, hence, changing the value and kinematical dependence of the observables. In 1949, J. Schwinger calculated the photon emission from an electron in a Coulomb field, receiving Nobel prize in 1965 [17]. In 1969 L.W. Mo and Y.S.Tsai calculated radiation emission for electron scattering on hadrons at the first order in α ($\alpha = e^2/4\pi \simeq 1/137$ is the fine electromagnetic structure constant) [18]. This work was extended by L.C. Maximon and J.A. Tjon, including partly the structure of the proton (2000) [19]. High orders were calculated by E.A. Kuraev and V.V. Fadin, using the *electron structure function method*, in 1985 [20] and applied to elastic, deep inelastic scattering, and annihilation channels. Only recently, after the year 2000, radiative corrections were calculated for polarization phenomena [21].

1.1 Derivation of the Rutherford formula: analogy with optics

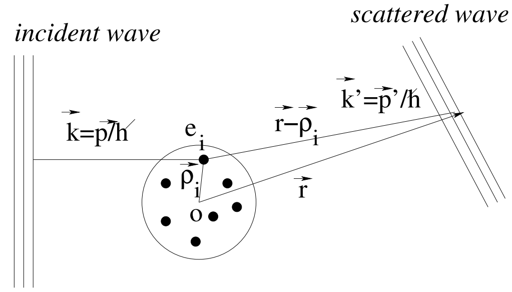


Figure 1.1: Schematic view of elastic scattering on a composite object.

In quantum mechanics, the particle-wave duality requires that a particle of three momentum $\vec{p}$ is associated to a plane wave vector $\vec{k} = \vec{p}/\hbar$. If a plane wave scatters off a charge e_i at a position ρ_i, it generates a spherical wave, that can be observed at large distances as a plane wave $\vec{k}' = \vec{p}\,'/\hbar$. The amplitude of the scattered wave in the point defined by $\vec{r}$ is:

$$A_i = fe_ie^{i\vec{k}\cdot\vec{\rho}_i}e^{i\vec{k}'\cdot(\vec{r}-\vec{\rho}_i)} = fe^{i\vec{k}'\cdot\vec{r}}e_ie^{iq\cdot\vec{\rho}_i} \tag{1.1}$$

where f is the amplitude on the unit charge, $f = Z_ae$, which is the same for all constituent particles, $\vec{r} - \vec{\rho}_i$ is the vector from the observation point to the charge i, and $\vec{q} = \vec{p}_i - \vec{p}_f$ is the momentum transfer. The factor $e^{i\vec{k}\cdot\vec{\rho}_i}$ defines the phase of the incident plane wave at the interaction point, and $e^{i\vec{k}'\cdot(\vec{r}-\vec{\rho}_i)}$ determines the phase of the scattered wave at the observation point. Similarly to optics, the total scattered amplitude on the nucleus can be taken as the sum of the amplitudes on the individual charges:

$$A = \sum_i A_i = fe^{i\vec{k}'\cdot\vec{r}}\sum_i e_ie^{iq\cdot\vec{\rho}_i}. \tag{1.2}$$

However, in quantum mechanics, $\vec{\rho}_i$ represent the position operators of the internal motion in the target. Therefore the last term should be replaced by the corresponding mean value in the ground state of the target. We define the form factor:

$$F(\vec{q}) = \frac{1}{Z_be} < i|\sum_i e_ie^{i\vec{q}\cdot\vec{\rho}_i}|i >, \tag{1.3}$$

and then the cross section on an extended nucleus becomes

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{pl} |F(\vec{q})|^2, \tag{1.4}$$

where we identified the cross section on a pointlike particle as:

$$\left(\frac{d\sigma}{d\Omega}\right)_{pl} = (Z_be)^2|f|^2 \propto (Z_aZ_be^2)^2. \tag{1.5}$$

The detailed and rigorous derivation of charge and magnetic FFs in a relativistic formalism is the content of these Lectures.

1.1.1 The charge form factor

If one wants to deduce the mean value of the charge density, in principle one can invert Eq. (1.3):

$$\rho(\vec{x}) =< \Psi_i|\hat{\rho}(\vec{x})|\Psi_i >= \frac{Z_be}{(2\pi)^3}\int d^3qF(\vec{q})e^{-i\vec{q}\cdot\vec{x}}. \tag{1.6}$$

However, in practice, $F(\vec{q})$ can not be determined for all values of $\vec{q}$, due to the limits of the kinematically accessible region. Moreover, at large q, cross sections are very small and difficult to measure. Furthermore, the cross section is sensitive to the FF modulus squared, and does not give access to the phase. Therefore, in general, one assumes a specific mathematical function for $\rho(\vec{x})$, and free parameters that are fitted to the experimental data.

For small values of q^2 one can develop $F(q^2)$ in a Taylor series expansion on $\vec{q}\cdot\vec{x}$:

$$\begin{aligned}
F(\vec{q}) &= \frac{1}{Z_b e}\int d^3\vec{x} e^{i\vec{q}\cdot\vec{x}}\rho(\vec{x}) \\
&= \frac{1}{Z_b e}\int d^3\vec{x}\left[1+i\vec{q}\cdot\vec{x}-\frac{1}{2}(\vec{q}\cdot\vec{x})^2+...\right]\rho(\vec{x}) \\
&\simeq \frac{1}{Z_b e}\int_0^\infty x^2 dx\int_0^{2\pi} d\varphi \\
&\quad \int_{-1}^{1} d\cos\theta\left[1+iqx\cos\theta-\frac{1}{2}q^2x^2\cos^2\theta\right]\rho(\vec{x}).
\end{aligned}$$

The normalization is $\int_\Omega d^3\vec{x}\rho(\vec{x}) = Z_b e$. The second term does not give any contribution, as $\vec{q}\cdot\vec{x} = qx\cos\theta$ and $\int_{-1}^{1}\cos\theta d\cos\theta = 0$. This is a general fact, as x is a odd quantity, whereas $\rho(\vec{x})$, which contains the square of the wave function, is an even quantity with respect to space parity.

In case of spherical symmetry,

$$F(q) \sim 1-\frac{1}{6}q^2 < r_c^2 > +O(q^2), \tag{1.7}$$

where we define the mean square root charge radius of the target, $< r_c^2 >$, as

$$< r_c^2 >= \frac{\int_0^\infty x^4\rho(x)dx}{\int_0^\infty x^2\rho(x)dx}.$$

1.1.2 Application to different charge distributions

Let us calculate $F(q)$ normalized to the full volume and charge:

$$F(q) = \frac{\int_\Omega d^3\vec{x} e^{i\vec{q}\cdot\vec{x}}\rho(\vec{x})}{\displaystyle\int_\Omega d^3\vec{x}\rho(\vec{x})}.$$

In case of spherical symmetry the denominator is:

$$D = 4\pi\int_0^\infty x^2\rho(x)dx$$

and the numerator:

$$N(q) = 2\pi\int_0^\infty x^2\rho(x)dx\int_{-1}^{1} d\cos\theta e^{iqx\cos\theta} = 2\pi\int_0^\infty x^2\rho(x)dx\frac{e^{iqx}-e^{-iqx}}{iqx}$$

Therefore:

$$F(q) = \frac{4\pi \int \frac{x}{q} \sin(qx)\rho(x)dx}{4\pi \int_0^\infty x^2 \rho(x)dx}. \tag{1.8}$$

The typical shapes of charge density, with spherical symmetry, and the corresponding form factors and radii are shown in Table 1.1.

density $\rho(r)$	Form factor $F(q^2)$	r.m.s. $< r_c^2 >$	comments
δ	1	0	pointlike
e^{-ar}	$\frac{a^4}{(q^2+a^2)^2}$	$\frac{12}{a^2}$	dipole
$\frac{e^{-ar}}{r}$	$\frac{a^2}{q^2+a^2}$	$\frac{6}{a^2}$	monopole
$\frac{e^{-ar^2}}{r^2}$	$e^{-q^2/(4a^2)}$	$\frac{1}{2a}$	gaussian
ρ_0 for $x \leq R$ 0 for $r \geq R$	$\frac{3(\sin X - X\cos X)}{X^3}$ $X = qR$	$\frac{3}{5}R^2$	square well

Table 1.1: Charge density, form factor and root mean squared radius.

1.1.3 Units and orders of magnitudes

The amplitude of the scattered wave is the sum of the amplitudes of the waves scattered from the individual constituents. An observer far from the target can see that the intensity of the scattered wave shows minima and maxima, as a function of the scattered angle, which correspond to interference among the different amplitudes A_i of the scattered waves. As in optics, one can introduce a resolving power δ:

$$\delta r[fm] = \frac{\hbar}{|\vec{q}|} \sim \frac{200}{c|\vec{q}|}, \tag{1.9}$$

The quantity δ defines the spatial region that can be accessed in an experiment where the transferred momentum is $|\vec{q}|$.

Let us compare $\hbar c$ to the dimension of an atom (Bohr radius): $\lambda \sim 10^5$[fm]:

$$\frac{\hbar c}{\lambda} \simeq \frac{200[\text{Mev}]\ [\text{fm}]}{10^5[\text{fm}]} \simeq 2 \cdot 10^{-3}\text{MeV}. \tag{1.10}$$

Therefore, to investigate the properties of the nucleon - that is smaller by 10^3 times an atom - and its constituents. a momentum in the GeV range at least, is necessary. For example $|\vec{q}|$= 1 GeV in ep scattering corresponds to $\delta r = 0.2$ fm.

1.2 Extensions of the Rutherford Formula

The Rutherford formula holds in frame of the Coulomb interaction between target and projectile, $U(r) = Z_1 Z_2 e^2/r$, at the lowest order of perturbation theory, (Born approximation), in a non relativistic approximation, for structureless and spinless particles.

The non relativistic approach is justified if the momenta of the particles are smaller than their masses ($p/m \ll 1$). The differential cross section for spinless and pointlike-particles, in the relativistic case and in the Born approximation, was derived by N. F. Mott, including recoil effects of the target nucleus of mass M [3]:

$$\left(\frac{d\sigma}{d\Omega}\right)^{Lab}_{Mott} = \frac{e^2}{4E^2}\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\frac{1}{1+\dfrac{2E}{M}\sin^2(\theta/2)}, \propto T_{fi}^2; T_{fi} \propto \frac{Z_1 Z_2 e^2}{|\vec{q}|^2}, \tag{1.11}$$

In the language of Feynman diagrams, it is easy to verify the main features of the Mott cross section. The transition amplitude is proportional to $Z_i e$, the vertices contribution, which does not depend on the particle momenta for pointlike particles, and to the photon propagator $1/q^2$:

$$T_{fi} \propto \frac{Z_1 Z_2 e^2}{|\vec{q}|^2}, \quad \left(\frac{d\sigma}{d\Omega}\right)^{Lab}_{Mott} \propto T_{fi}^2. \tag{1.12}$$

Further developments were given several years later. The extension of the Rutherford formula at the next order $\sim (Z\alpha)^3$ [22] showed that the scattering of electrons and positrons is no more equivalent, because the correction depends on the charge:

$$\frac{d\sigma^{\pm}}{d\Omega} = \frac{d\sigma_R}{d\Omega}\left[1 \pm \pi\alpha Z \sin(\theta/2)\right], \quad \frac{d\sigma_R}{dd\Omega} = \frac{(Z\alpha)^2}{4E^2\sin^4(\theta/2)}, \tag{1.13}$$

which leads to a charge asymmetry. Higher order corrections $\sim (Z\alpha)^n$ have been calculated more recently in the eikonal approximation for charge asymmetry and polarization phenomena. A non trivial universal angular dependence is predicted, whose sign depends on the charge, observable in electron and positron scattering. Further developments of the Rutherford formula include high energy scattering on heavy targets (also in the eikonal approximation) (see Ref. [23] and references therein).

Exercise 1
Verify the results of Table 1.1.

Chapter 2

Lecture I: the scattering region

In this section we derive step by step the unpolarized and polarized cross sections for elastic eN-scattering, $e + N \to e + N$, $N = p$ or n. Assuming that the reaction occurs through the exchange of one photon, specific polarization phenomena occur due to the fact that nucleon electromagnetic form factors (FF) are real functions of q^2, the momentum transfer squared. The result is that all one-spin polarization observables (being T-odd and P-even) vanish at any electron energy and scattering angle. The simplest non-zero polarization observables are two-spin correlations, which are P- and T-even. Generally, these observables are characterized by large absolute values and a weak dependence on the electron energy and on q^2, whereas the differential cross section (with unpolarized particles) shows a very steep decrease with q^2.

The calculation of polarization observables is simpler in the Breit system, which is defined as the system where the initial and final nucleon energies are the same. As a consequence, the energy of the virtual photon vanishes and its four-momentum squared, q^2, coincides with its three-momentum squared, $\mathbf{q_B}^2$, more exactly, $q^2 = -\mathbf{q_B}^2$. The derivation of the formalism in Breit system is therefore more simple and has some analogy with a non-relativistic description of the nucleon electromagnetic structure. The Breit system can be considered as the analogue of the center of mass system (CMS) for the annihilation reaction $e^+ + e^- \to p + \overline{p}$. Moreover, the definition of the electric G_E and magnetic G_M nucleon FFs for the nucleon electromagnetic current follows naturally in the Breit system. The space representation of the nucleon structure as Fourier transform of the Sachs FFs, G_E and G_M is valid only in the Breit system, at any value of momentum transfer.

2.1 Kinematics

2.1.1 Definition of four-momenta and reference systems

In the following we use units $\hbar = c = 1$. A four-momentum is the generalization of the classical three-momentum in space-time, allowing to build relativistic invariants.

A four-vectors has a scalar (time or energy) and a vector (position or momentum) component: $p = (p_0, \vec{p})$. The scalar product of two 4-momenta (invariant) is defined as: $p \cdot k = p_0 k_0 - \vec{p} \cdot \vec{k}$.

The mass of a particle (real or virtual) is defined as the square of its energy-momentum four vector: $p^2 = m^2 = p_0^2 - \vec{p}^2$. As the speed of a particle is limited to the speed of the light, the trajectory of a physical event is contained in a cone delimited by $|\vec{x}| \leq ct$ (Fig. 2.1).

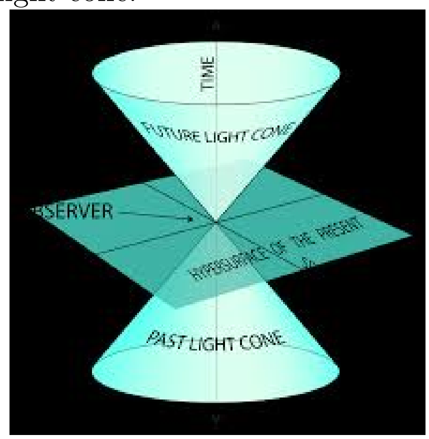


Figure 2.1: Illustration of the light cone.

A four momentum with negative (positive) four momentum squared: $p^2 < 0$ ($p^2 > 0$) is called space-like (time-like) vector. An example of space-like vector is the *four momentum transfer* for $ep \to ep$ elastic scattering ($m_1 = m_3 = m \simeq 0$, θ_e is the scattered electron angle):

$$\begin{aligned} t &= (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2p_1 \cdot p_3 \\ &= -2p_1^0 p_3^0 - 2\vec{p}_1 \cdot \vec{p}_3 \\ &= -4p_1^0 p_3^0 \sin^2 \theta_e/2 < 0. \end{aligned}$$

One can define combinations of four-momenta of the involved particles that are invariant by changing the frame, as the Mandelstam variables. For a two$\to$ two particle reaction, they are defined as in Table 2.1. The relation of momenta in a particle reaction is driven by energy and momentum conservation. The four momentum conservation implies the conservation of energy **and** momentum separately: $E_1 + E_2 = E_3 + E_4$, $\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4$.

Table 2.1: The Mandelstam variables

$a(p_1) + b(p_2) \to c(p_3) + d(p_4)$		
s	t	u
$(p_1 + p_2)^2$	$(p_1 - p_3)^2$	$(p_1 - p_4)^2$
$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$		

The center of mass (CMS) is defined as the system where the sum of particle momenta (in initial and final channel) is zero.

The Laboratory (Lab) system, where the target is at rest, is the most convenient from the experimental point of view.

The notations and the correspondence of the four-momenta in the different systems is given in Table 2.2.

	Lab	**CMS**	**Breit**
q	$(\omega, \mathbf{q})$	$(\widetilde{\omega}, \tilde{q})$	$(\omega_B = 0, \mathbf{q_B})$
k_1	$(\epsilon_1, \mathbf{k_1})$	$(\widetilde{\epsilon_1}, \widetilde{\mathbf{k_1}})$	$(\epsilon_{1B}, \mathbf{k_{1B}})$
p_1	$(M, 0)$	$(\widetilde{E_1}, -\widetilde{\mathbf{k_1}})$	$(E_{1B}, \mathbf{p_{1B}})$
k_2	$(\epsilon_2, \mathbf{k_2})$	$(\widetilde{\epsilon_2}, \widetilde{\mathbf{k_2}})$	$(\epsilon_{2B}, \mathbf{k_{2B}})$
p_2	$(E_2, \mathbf{p_2})$	$(\widetilde{E_2}, -\widetilde{\mathbf{k_2}})$	$(E_{2B}, -\mathbf{p_{1B}})$

Table 2.2: Notation of four-momenta in different reference frames.

2.1.2 Proton kinematics in the Breit system

From the energy conservation, and from the definition of the Breit system, one can find:

$$\omega_B = E_{1B} - E_{2B} = 0,$$

where all kinematical quantities in the Breit system are denoted with subscript B. The proton three-momentum can be found from the relation

$$E_{1B}^2 = E_{2B}^2 = \mathbf{p_{1B}}^2 + M^2 = \mathbf{p_{2B}}^2 + M^2, \ i.e.\ \mathbf{p_{1B}}^2 = \mathbf{p_{2B}}^2.$$

The physical solution of this quadratic relation is $\mathbf{p_{1B}} = -\mathbf{p_{2B}}$, as the Breit system moves in the direction of the outgoing proton. From the three-momentum conservation, in the Breit system $\mathbf{q_B} + \mathbf{p_{1B}} = \mathbf{p_{2B}}$:

$$\mathbf{p_{1B}} = -\frac{\mathbf{q_B}}{2}, \quad \mathbf{p_{2B}} = \frac{\mathbf{q_B}}{2}.$$

The proton energy can be expressed as a function of $\mathbf{q_B}^2$, and therefore of q^2:

$$E_{1B}^2 = E_{2B}^2 = M^2 + \frac{\mathbf{q_B}^2}{4} = M^2 - \frac{q^2}{4} = M^2(1+\tau),$$

where we replaced the three-momentum by the four-momentum and we introduced the dimensionless quantity $\tau = -\dfrac{q^2}{4M^2} \geq 0$.

2.1.3 Electron kinematics in the Breit system

The conservation of the four momentum, at the electron vertex, can be written, in any reference system, as: $k_1 = q + k_2$ (the virtual photon is radiated by the electron). In the Breit system, the energy and momentum conservation is:

$$\left\{ \begin{array}{ll} \epsilon_{1B} = & \omega_B + \epsilon_{2B} = \epsilon_{2B}, \\ \mathbf{k_{1B}} = & \mathbf{q_B} + \mathbf{k_{2B}}. \end{array} \right. \tag{2.1}$$

In order to proceed further, we must define a reference (coordinate) system: we choose the z-axis parallel to the photon three-momentum: $z \parallel \mathbf{q_B}$, and the xz-plane as the scattering plane. So we can write:

$$\begin{cases} \epsilon_{1B}^2 & = \epsilon_{2B}^2 \quad \rightarrow \quad m^2 + (k_{1B}^x)^2 + (k_{1B}^z)^2 = m^2 + (k_{2B}^x)^2 + (k_{2B}^z)^2 \\ k_{1B}^x = & k_{2B}^x \\ k_{1B}^y = & k_{2B}^y = 0 \\ k_{1B}^z = & q_B + k_{2B}^z \end{cases} \tag{2.2}$$

It follows $k_{1B}^z = -k_{2B}^z = \dfrac{q_B}{2}$ (the other possible solution $k_{1B}^z = k_{2B}^z$ would imply $q_B = 0$). A graphical representation for the conservation of three-momenta is given in Fig. 2.2.

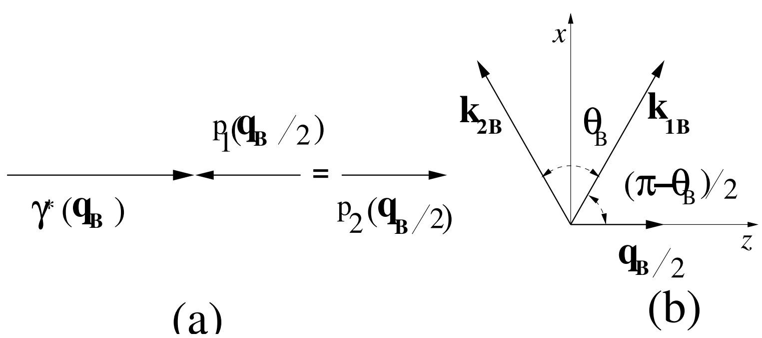


Figure 2.2: Proton (a) and electron (b) three-momenta representation for elastic eN-scattering in the Breit system.

We can then write, for the components of the initial and final electron three-momenta:

$$\mathbf{k_{1B}} = (k_{1B}^x, k_{1B}^y, k_{1B}^z) = \left(\frac{q_B}{2}\cot\frac{\theta_B}{2}, 0, \frac{q_B}{2}\right) = \frac{\sqrt{-q^2}}{2}\left(\cot\frac{\theta_B}{2}, 0, 1\right), \tag{2.3}$$

$$\mathbf{k_{2B}} = (k_{2B}^x, k_{2B}^y, k_{2B}^z) = \left(\frac{q_B}{2}\cot\frac{\theta_B}{2}, 0, -\frac{q_B}{2}\right) = \frac{\sqrt{-q^2}}{2}\left(\cot\frac{\theta_B}{2}, 0, -1\right), \tag{2.4}$$

The energy of the electron is (in the limit of zero electron mass) is given by:

$$\epsilon_{1B}^2 = \mathbf{k_{1B}}^2 = (k_{1B}^x)^2 + (k_{1B}^z)^2 = \frac{-q^2}{4\sin^2\dfrac{\theta_B}{2}} \text{ and } \epsilon_{2B} = \epsilon_{1B}.$$

One can prove the following relation between the electron scattering angles in the Lab system, θ_e and in the Breit system, θ_B (**Exercise 1**):

$$\cot^2\frac{\theta_B}{2} = \frac{\cot^2\theta_e/2}{1+\tau}. \tag{2.5}$$

One can derive the expression of $\sin\theta_B/2$ in terms of the energies in Lab system. Using the relation (2.5), one finds:

$$\frac{1}{\sin^2\dfrac{\theta_B}{2}} = 1 + \frac{\cot^2\dfrac{\theta_e}{2}}{1+\tau} = \frac{1}{1+\tau}\left[\tau + \frac{1}{\sin^2\dfrac{\theta_e}{2}}\right] = \frac{1}{1+\tau}\frac{1+\tau\sin^2\dfrac{\theta_e}{2}}{\sin^2\dfrac{\theta_e}{2}} \tag{2.6}$$

So

$$1+\tau\sin^2\frac{\theta_e}{2}=1+\frac{\frac{\epsilon_1^2}{M^2}\sin^4\frac{\theta_e}{2}}{1+2\frac{\epsilon_1}{M}\sin^2\frac{\theta_e}{2}}=\frac{(1+\frac{\epsilon_1}{M}\sin^2\frac{\theta_e}{2})^2}{1+2\frac{\epsilon_1}{M}\sin^2\frac{\theta_e}{2}}. \tag{2.7}$$

The final electron energy is:

$$\epsilon_2=\frac{\epsilon_1}{1+2\frac{\epsilon_1}{m}\sin^2\frac{\theta_e}{2}}, \tag{2.8}$$

therefore

$$1+\frac{\epsilon_1}{M}\sin^2\frac{\theta_e}{2}=\frac{1}{2}\frac{\epsilon_1+\epsilon_2}{\epsilon_2}. \tag{2.9}$$

Substituting (2.9) in (2.6), one finally finds:

$$\frac{1}{\sin^2\frac{\theta_B}{2}}=\frac{(\epsilon_1+\epsilon_2)^2}{(-q^2)(1+\tau)}. \tag{2.10}$$

2.2 Dynamics: relativistic formalism for ep elastic scattering

In this section we derive the elastic cross section and the polarization observables for electron proton scattering, in the Born approximation, in a fully relativistic formalism, taking into account that the proton has a spin and an internal structure. This derivation follows closely lecture notes earlier prepared with Prof. M. P. Rekalo [24].

2.2.1 Reminder about Dirac equation

The elastic eN scattering involves four particles, with spin 1/2. The relativistic description of the spin properties of each of these particles is based on the Dirac equation for particles (nucleon with momentum p_2) and antiparticles (antinucleon with momentum p_1):

$$\begin{aligned}\bar{u}(p_2)(\hat{p}_2-m)=0 \ &\Rightarrow\ \bar{u}(p_2)\hat{p}_2=\bar{u}(p_2)m,\\ (\hat{p}_1+m)u(-p_1)=0\ &\Rightarrow\ \hat{p}_1u(-p_1)=-u(-p_1)m.\end{aligned} \tag{2.11}$$

with $\hat{p}=p\gamma_\mu=pE\gamma_0-\mathbf{p}\cdot\vec{\gamma}$, where p is the particle four momentum, $p=(E,\mathbf{p})$ and $u(p)$ is a four-component Dirac spinor.

We shall use the following representation of the Dirac 4×4 matrixes:

$$\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \vec{\gamma}=\begin{pmatrix}0&\vec{\sigma}\\-\vec{\sigma}&0\end{pmatrix}, \tag{2.12}$$

where $\vec{\sigma}$ is the standard set of the Pauli 2×2 matrixes:

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \mathcal{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \tag{2.13}$$

with the following properties:

$$det\sigma_i = -1;\; Tr\sigma_i = 0;\; [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k;\; \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathcal{I}, \sigma_i\sigma_j = \delta_{ij}\mathcal{I} + i\epsilon_{ijk}\sigma_k.$$

On the basis of the Dirac equation one can write:

$$u(p) = \sqrt{E+m}\begin{pmatrix} \chi \\ \dfrac{\vec{\sigma}\cdot\mathbf{p}}{E+m}\chi \end{pmatrix}, \tag{2.14}$$

where χ is a two-component spinor. We used here the relativistic invariant normalization for the four-component spinors, $u^\dagger u = 2E$.

Useful properties of Dirac matrices

- The anticommutator is: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, where $g_{\mu\nu} = 1$ for $\mu, \nu = 0$ and $g_{\mu\nu} = 0$ for $\mu, \nu = x, y, z$ is the metric tensor of the Minkowski space-time: ;
- $\hat{a}\hat{b} + \hat{b}\hat{a} = 2ab$, $\hat{a}\gamma_\mu + \gamma_\mu\hat{a} = 2a_\mu$, where a and b are four vectors;
- $Tr\ \gamma_\alpha\gamma_\beta = 4g_{\alpha\beta}$;
- $Tr\ \gamma_\alpha\gamma_\beta\gamma_\gamma = 0$; (as well as the trace of the product of an odd number of matrices)
- $Tr\ \gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta = 4\left(g_{\alpha\beta}\gamma_{\gamma\delta} + \gamma_{\beta\gamma}\gamma_{\delta\alpha} - \gamma_{\gamma\alpha}\gamma_{\delta\beta}\right).$

Relativistic formulation for the spin The four vector of the electron spin, s_α, satisfies the following two conditions:

$$s\cdot p = 0, \quad s^2 = -1. \tag{2.15}$$

In terms of the three-vector $\vec{\chi}$ of the electron polarization at rest, i.e., with zero three-momentum, the four-vector s can be written as:

$$s = \left(\frac{\vec{\chi}\cdot\mathbf{p}}{m},\ \vec{\chi} + \frac{(\vec{\chi}\cdot\mathbf{p})\mathbf{p}}{m(\epsilon+m)}\right). \tag{2.16}$$

The condition $s^2 = -1$ corresponds to full electron polarization, so $s^2 = -|\vec{s}|^2 = -1$. Eq. (2.16) is simplified in case of relativistic electrons, $\epsilon \gg m$. In this case:

$$s_\alpha = \frac{\epsilon}{m}s_\ell(1, \mathbf{1}), \tag{2.17}$$

where $\mathbf{1}$ denotes the unit vector along $\mathbf{p}$ and $s_\ell = \vec{\chi}\cdot\mathbf{p}/|\mathbf{p}| \equiv \lambda$.

Taking into account that for relativistic electrons: $p_\alpha = \epsilon(1, \mathbf{1})$, Eq. (2.17) can be re-written in the form:

$$s_\alpha = \frac{p_{1\alpha}}{m}\lambda. \tag{2.18}$$

Applying the Dirac equation to the four-component spinor $u(p)$, of an electron with mass m, one can find the expressions for the density matrix of polarized electrons $\rho_{\alpha\beta}$:

$$\begin{aligned} \rho = \quad & u_\alpha(p)u^\dagger_\beta(p) = \frac{1}{2}(\hat{p}+m)\left(1-\gamma_5\frac{\hat{p}}{m}\lambda\right) = \frac{1}{2}(\hat{p}+m) + \frac{\lambda}{2}(\hat{p}+m)\frac{\hat{p}}{m}\gamma_5 \\ & = \frac{1}{2}(\hat{p}+m) + \frac{\lambda}{2}\left(p^2+m\hat{p}\right)\frac{1}{m}\gamma_5 = \frac{1}{2}(\hat{p}+m)(1+\lambda\gamma_5) \equiv \frac{1}{2}\hat{p}(1+\lambda\gamma_5), \end{aligned} \tag{2.19}$$

where we used the following property of the γ_5-matrix: $\hat{p}\gamma_5 + \gamma_5\hat{p} = 0$, for any four-vector p_α.

The density matrices $\rho = u(p)\bar{u}(p)$ for polarized and unpolarized particles and antiparticles are given in Table 2.3.

	particle	antiparticle
unpolarized	$\hat{p}+m$	$\hat{p}-m$
polarized	$(\hat{p}+m)\frac{1}{2}(1-\gamma_5\hat{s})$	$(\hat{p}-m)\frac{1}{2}(1-\gamma_5\hat{s})$

Table 2.3: The density matrices for unpolarized/polarized particles and antiparticles.

In some cases is possible to use a simpler representation in terms of σ matrices and two-component spinors χ. In this cases the density matrix $\rho = \chi(p)\bar{\chi}(p)$ becomes $\rho = \frac{1}{2}\,\mathcal{I}$ for unpolarized particles and $\rho = \dfrac{1}{2}\left(I + \vec{\sigma}\cdot\mathbf{P}\right)$ for a nucleon with polarization $\mathbf{P}$.

2.3 Cross section for ep elastic scattering

2.3.1 Cross section for a binary process

The cross section σ for a binary process

$$a(k_1) + b(p_1) \to c(k_2) + d(p_2), \tag{2.20}$$

(where the momenta of the particles are indicated in parenthesis), characterizes the probability that a given process occurs. The number of events issued from a definite reaction, N_F, is proportional to the number of incident particles N_B and to the number of the target particles N_T, where the constant of proportionality is the cross section σ:

$$N_F = \sigma N_a \times N_b. \tag{2.21}$$

The cross section can be viewed as an "effective area" over which the incident particle reacts. Therefore, its dimension is cm^2, but more often barn (1 barn=10^{-28} m^2), or fm^2 (1 fm=10^{-15} m) are used.

A useful quantity is the luminosity $\mathcal{L}$, defined as $\mathcal{L} = N_B$ [s^{-1}] N_T [cm^{-2}]. For simple counting estimations, $N_f = \sigma\mathcal{L}$. This is an operative definition, which is used in experimental physics.

On the other hand σ needs to be calculated theoretically for every type of process, as models have to be tested on observables as polarized and unpolarized cross section and on their dependence on the relevant kinematical variables. The present derivation is done in a relativistic approach. This means that :

1. The kinematics is relativistic,
2. The matrix element $\mathcal{M}$, which contains the dynamics of the reaction, is a relativistic invariant. In general it is better expressed as function of kinematical variables, also relativistic $\mathcal{M} = f(s, t \text{ or } u)$,
3. σ has to be written in a relativistic invariant form.

The starting point is the following expression for the cross section

$$d\sigma = \frac{\overline{|\mathcal{M}|^2}}{\mathcal{F}}(2\pi)^4\delta^{(4)}(k_1 + p_1 - k_2 - p_2)d\mathcal{P}, \tag{2.22}$$

which is composed of four terms:

1. The matrix element $\overline{|\mathcal{M}|^2}$ is calculated following a model (where the overline denotes the average of the polarization of the initial particles and the sum over the polarization of the final particles),
2. The flux of colliding particles $\mathcal{F}$,
3. The phase space for the final particles, $d\mathcal{P}$,
4. A term which insures the conservation of the four-momentum $\delta^{(4)}(k_1+p_1-k_2-p_2)$ which is the product of four δ functions, because each component has to be conserved separately.

Let us calculate in detail each term.

Definition of flux $\mathcal{F}$

The flux is defined through the relative velocity of incoming and target particles:

$$\mathcal{F} = n_B n_T v_{rel}, \tag{2.23a}$$

$$\mathcal{F} = 4\sqrt{(k_1 \cdot p_1)^2 - m^2M^2}, \tag{2.23b}$$

where $m(M)$ is the mass of the beam (target) particle, v_{rel} is the relative velocity between beam and target particles and the densities of the beam and target particles n_B, n_T are proportional to their energies as $n_i = 2E_i$.

Let us prove that the two expressions (2.23a) and (2.23b) are equivalent. It is more convenient to calculate $\mathcal{F}$ (Eq. 2.23) in the Lab frame where the target is at rest:

$$k_1 = (\epsilon_1, \mathbf{k_1}),\ p_1 = (M, 0),\ k_2 = (\epsilon_2, \mathbf{k_2}),\ p_2 = (M, \mathbf{p_2}), \tag{2.24}$$

$$|\mathbf{v_{rel}}| = |\mathbf{v_1} - \mathbf{v_2}| = \frac{|\mathbf{k_1}|}{\epsilon_1}\ n_B = 2|\epsilon_1|,\ n_T = 2M. \tag{2.25}$$

Replacing the equalities (2.25) in Eq. (2.23a):

$$\mathcal{F} = 2\epsilon_1 2M \frac{|\vec{k}_1|}{\epsilon_1} = 4M_2|\mathbf{k_1}|$$

and in Eq. (2.23b) :

$$(k_1 \cdot p_1)^2 - m^2M^2 = M^2\epsilon_1^2 - m^2M^2 = M^2(\epsilon_1^2 - m^2) = M^2|\mathbf{k_1}|^2,\ \text{thus}\ \mathcal{F} = 4M|\vec{k}_1|$$

and the equalities (2.23) are proved. Moreover, we prove also that the flux does not depend on the reference frame, because it can be written in a Lorentz invariant form.

Let us consider the center of mass system (CMS):

$$k_1 = (\tilde{\epsilon}_1, \widetilde{\mathbf{k}}),\ p_1 = (\tilde{E}_1, -\widetilde{\mathbf{k}}),\ k_1 \cdot p_1 = \epsilon_1 M + |\widetilde{\mathbf{k}}|^2,\ m_1^2 = \epsilon_1^2 - |\widetilde{\mathbf{k}}|^2,\ M = E_1^2 - |\widetilde{\mathbf{p}}|^2$$

and

$$\begin{aligned}(k_1 \cdot p_1)^2 - m^2M^2 &= \tilde{\epsilon}_1^2\tilde{E}_1^2 + 2\tilde{\epsilon}_1\tilde{E}_1|\widetilde{\mathbf{k}}|^2 + |\widetilde{\mathbf{k}}|^4 - \tilde{\epsilon}_1^2\tilde{E}_1^2 + |\widetilde{\mathbf{k}}|^2(\tilde{\epsilon}_1^2 + \tilde{E}_2^2) - |\widetilde{\mathbf{k}}|^4 \\ &= |\widetilde{\mathbf{k}}|^2(\tilde{\epsilon}_1 + \tilde{E}_1)^2 = |\widetilde{\mathbf{k}}|^2W^2. \end{aligned} \tag{2.26}$$

The flux, $\mathcal{F}$, can be written as

$$\mathcal{F} = 4|\widetilde{\mathbf{k}}|W, \tag{2.27}$$

where $W = \tilde{\epsilon}_1 + \tilde{E}_2$ is the total energy of the system in CMS.

2.3.2 Phase space

The phase space for a particle of energy E, mass M and four–momentum p (the number of states in the unit volume) can be written according to quantum mechanics in an invariant form:

$$d\mathcal{P} = \int \frac{d^4p\ \delta(p^2 - M^2)}{(2\pi)^3}\Theta(E),$$

where the δ function insures that the particle is on mass shell and the step function $\Theta(E)$ insures that only the solution with positive energy is taken into account. Note that the wave functions of all particles entering in the matrix element must be normalized to one particle per unit volume. In this case all these wave functions contain the factor $1/\sqrt{2\varepsilon}$, where ε is the particle energy. Usually these factors are explicitly taken into account in the expression for the cross section, we insert them into the phase space.

Extracting the term which depends on energy:

$$d^4p\ \delta(p^2 - M^2) = \delta^3\mathbf{p}dE\delta(E^2 - \mathbf{p}^2 - M^2),$$

and using the property of the δ function

$$\int \delta[f(x)]dx = \sum \frac{1}{|f'(x_i)|}, \tag{2.28}$$

(x_i are the roots of $f(x)$), with $f(E) = E^2 - \mathbf{p}^2 - M^2$, and $f'(E) = 2E$ one finds:

$$\int dE\delta(E^2 - \mathbf{p}^2 - M^2)\Theta(E) = \frac{1}{2E}.$$

Finally, the phase space of the two particle in the final state of the reaction (2.20), is:

$$d\mathcal{P} = \frac{d^3\mathbf{k_2}}{(2\pi)^3 2\epsilon_2}\frac{d^3\mathbf{p_2}}{(2\pi)^3 2E_2}.$$

The final formula for the total cross section in invariant form is:

$$\sigma = \frac{(2\pi)^4}{4\sqrt{(k_1\cdot p_1)^2 - m^2M^2}}\int |\mathcal{M}|^2\delta^{(4)}(k_1 + p_1 - k_2 - p_2)\frac{d^3\mathbf{k_2}}{(2\pi)^3 2\epsilon_2}\frac{d^3\mathbf{p_2}}{(2\pi)^3 2E_2}. \tag{2.29}$$

One can see that it corresponds to a six-fold differential, but four δ functions are equivalent to four integrations. So finally, for a $2 \to 2$ process one is left with two independent variables, initial energy & scattering angle (ϵ, θ) or total energy & transferred momentum:(s, t). For three particles, one has nine differentials, four integrations, *i.e.*, five independent variables.

The term $\delta^{(4)}(k_1 + p_1 - k_2 - p_2)$ can be split into an energy and a space part: $\delta^{(4)}(k_1 + p_1 - k_2 - p_2) = \delta(E_1 + E_2 - E_3 - E_4)\delta^{(3)}(\mathbf{k_1} + \mathbf{p_2} - \mathbf{k_2} - \mathbf{p_2})$.

Note that

$$\int \delta^{(3)}(\mathbf{k_1} + \mathbf{p_1} - \mathbf{k_2} - \mathbf{p_2})d^3\mathbf{p_2} = 1, \tag{2.30}$$

in any reference frame.

<u>**In CMS system**</u>

Let us use spherical coordinates in CMS ($\tilde{k}_2 = (\tilde{\epsilon}_2, \widetilde{\mathbf{k}})$, $\tilde{p}_2 = (\tilde{E}_2, -\widetilde{\mathbf{p}})$, $d^3\widetilde{\mathbf{p}} = |\widetilde{\mathbf{p}}|^2 d\Omega d\widetilde{\mathbf{p}}$)and consider the quantity A:

$$A = \delta(\tilde{\epsilon}_1 + M - \tilde{\epsilon}_2 - \tilde{E}_2)\frac{d^3\widetilde{\mathbf{p}}}{4\tilde{\epsilon}_2\tilde{E}_2} = \delta(W - \tilde{\epsilon}_2 - \tilde{E}_2)\frac{|\tilde{p}|^2 d\Omega d\widetilde{\mathbf{p}}}{4\tilde{\epsilon}_2\tilde{E}_2}, \tag{2.31}$$

where

$$\tilde{\epsilon}_2^2 = m^2 + |\widetilde{\mathbf{p}}|^2,\ \tilde{E}_2^2 = M^2 + |\widetilde{\mathbf{p}}|^2 \to \tilde{\epsilon}_2 d\tilde{\epsilon}_1 = \tilde{E}_2 d\tilde{E}_2 = |\widetilde{\mathbf{p}}|d\widetilde{\mathbf{p}}.$$

After integration, using the property (2.28):

$$A = \int \delta(W - \tilde{\epsilon}_2 - \tilde{E}_2)\frac{d\epsilon_2|\widetilde{\mathbf{p}}|d\Omega}{4\tilde{E}_2} = \frac{|\widetilde{\mathbf{p}}|d\Omega}{4\tilde{E}_2}\frac{1}{\left|\dfrac{d}{d\tilde{\epsilon}_2}\left(W - \tilde{\epsilon}_2 - \tilde{E}_2\right)\right|}, \tag{2.32}$$

where

$$\frac{d}{d\tilde{\epsilon}_2}(W - \tilde{\epsilon}_2 - \tilde{E}_2) = -1 - \frac{d\tilde{E}_2}{d\tilde{\epsilon}_2} = -1 - \frac{\tilde{\epsilon}_2}{\tilde{E}_2} = -\frac{W}{\tilde{E}_2} \tag{2.33}$$

and therefore

$$A = \frac{|\widetilde{\mathbf{p}}| d\Omega}{4W}. \tag{2.34}$$

Substituting Eqs. (2.27, 2.34) in Eq. (2.29) we find the general expression for the differential cross section of a binary process, in CMS:

$$\frac{d\sigma}{d\Omega} = \frac{\bar{|}\mathcal{M}|^2 |\widetilde{\mathbf{p}}|}{64\pi^2 W^2 |\widetilde{\mathbf{k}}|}, \tag{2.35}$$

and for the total cross section:

$$\sigma = \int \frac{|\mathcal{M}|^2 |\widetilde{\mathbf{p}}|}{64\pi^2 W^2 |\widetilde{\mathbf{k}}|} d\Omega. \tag{2.36}$$

In case of elastic scattering, $|\widetilde{\mathbf{k}}| = |\widetilde{\mathbf{p}}|$, therefore:

$$\frac{d\sigma}{d\Omega}^{el} = \frac{|\mathcal{M}|^2}{64\pi^2 W^2} = |\mathcal{A}^{el}|^2. \tag{2.37}$$

where $\mathcal{A}^{el}$ is the elastic amplitude.

In Lab system

To compare with experiments, it is more convenient to express the differential cross section in Lab system, $d\sigma/d\Omega_e$, where $d\Omega_e$ is the element of the electron solid angle in the Lab system. This can be done, integrating Eq. (2.29), using the properties of the δ^4 function.

First of all, let us integrate over the three-momentum $\mathbf{p_2}$, applying the three momentum conservation for the considered process:

$$\int d^3\mathbf{p_2} \delta^3(\mathbf{k_1} - \mathbf{k_2} - \mathbf{p_2}) = 1, \text{ with the condition } \mathbf{p_2} = \mathbf{k_1} - \mathbf{k_2}.$$

Using the definition $d^3\mathbf{k_2} \overset{m=0}{=} d\Omega_e \mathbf{k_2}^2 d|\mathbf{k_2}| \simeq d\Omega_e \epsilon_2^2 d\epsilon_2$, we can integrate over the electron energy, taking into account the conservation of energy:

$$\delta\left(\epsilon_1 + m - \epsilon_2 - E_2\right) d\epsilon_2 = \delta(\epsilon_1 + m - \epsilon_2 - \sqrt{m^2 + \mathbf{p_2}^2}\right) d\epsilon_2 =$$

$$\delta\left(\epsilon_1 + m - \epsilon_2 - \sqrt{m^2 + (\mathbf{k_1} - \mathbf{k_2})^2}\right) d\epsilon_2$$

Let us recall that:

$$\int \delta\left[f(\epsilon_2)\right] d\epsilon_2 = \frac{1}{|f'(\epsilon_2)|},$$

where $f(\epsilon_2) = \epsilon_1 + m - \epsilon_2 - \sqrt{m^2 + \epsilon_1^2 + \epsilon_2^2 - 2\epsilon_1\epsilon_2 \cos\theta_e}$. Therefore:

$$|f'(\epsilon_2)| = 1 + \frac{\epsilon_2 - \epsilon_1 \cos\theta_e}{E_2} = 1 + \frac{\epsilon_2^2 - \mathbf{k_1} \cdot \mathbf{k_2}}{\epsilon_2 E_2} = \frac{k_2 \cdot (k_1 + p_1)}{\epsilon_2 E_2},$$

where we multiplied by ϵ_2 the numerator and denominator, and we used the conservation of energy $\epsilon_2 + E_2 = \epsilon_1 + m$. But from the conservation of four-momentum, in the following form $k_1 + p_1 - k_2 = p_2$, we have:

$$(k_1 + p_1)^2 + k_2^2 - 2(k_1 + p_1) \cdot k_2 = m^2.$$

So $2(k_1 + p_1) \cdot k_2 = (k_1 + p_1)^2 - m^2 = 2k_1 \cdot p_1 = 2\epsilon_1 m$ (in Lab system). Finally

$$|f'(\epsilon_2)| = \frac{\epsilon_1}{\epsilon_2} \frac{m}{E_2}.$$

After substituting in Eq. (2.29), one finds the following relation between $\overline{|\mathcal{M}|^2}$ and the differential cross section in Lab system:

$$\frac{d\sigma}{d\Omega_e} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2} \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \frac{1}{m^2}. \tag{2.38}$$

Using the Dirac equation for the four-component spinors of the initial and final nucleon, Eq. (2.41) can be rewritten in a simpler form (see Exercise 2):

$$\mathcal{J}_\mu = \overline{u}(p_2) \left[(F_1 + F_2)\, \gamma_\mu - \frac{(p_1 + p_2)_\mu}{2m} F_2 \right] u(p_1), \tag{2.39}$$

which is also conserved.

2.3.3 The matrix element

The Feynman diagram for elastic ep-scattering is shown in Fig. 2.3, assuming one-photon exchange. The notations of the particle four-momenta are also shown in the Fig. 2 and in Table 2.2 (we will use in our calculation the system where $\hbar$=c=1).

Two vertexes are present in Fig. 2.3: (1) the electron vertex, which is described by QED-rules, (2) the proton vertex described by QCD and hadron electrodynamics.

The matrix element corresponding to this diagram, is written as:

$$\mathcal{M} = \frac{e^2}{q^2} \ell_\mu \mathcal{J}_\mu = \frac{e^2}{q^2} \ell \cdot \mathcal{J}, \tag{2.40}$$

where $\ell_\mu = \overline{u}(k_2)\gamma_\mu u(k_1)$ is the electromagnetic current of electron. The nucleon electromagnetic current, $\mathcal{J}_\mu$ describes the proton vertex and is generally written in terms of Pauli and Dirac FFs F_1 and F_2:

$$\mathcal{J}_\mu = \overline{u}(p_2) \left[F_1(q^2)\gamma_\mu - \frac{\sigma_{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p_1), \tag{2.41}$$

with

$$\sigma_{\mu\nu} = \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{2}.$$

Note that $\mathcal{J} \cdot q = 0$, for any values of F_1 and F_2, i.e. the current $\mathcal{J}_\mu$ is conserved[1].

The matrix element squared, $\overline{|\mathcal{M}|^2}$ that appears in the expressions of the cross section (2.38,eq:cms) can be written as:

$$\overline{|\mathcal{M}|^2} = \left(\frac{e^2}{q^2}\right)^2 \overline{|\ell \cdot \mathcal{J}|^2} = \left(\frac{e^2}{q^2}\right)^2 L_{\mu\nu} W_{\mu\nu}, \tag{2.42}$$

where:

$L_{\mu\nu} = \overline{\ell_\mu \ell_\nu^*}$ is the leptonic tensor;

$W_{\mu\nu} = \overline{\mathcal{J}_\mu \mathcal{J}_\nu^*}$ is the hadronic tensor.

The product of the tensors $L_{\mu\nu}$ and $W_{\mu\nu}$ is a relativistic invariant, therefore it can be calculated in any reference system.

The hadronic current

Eq. (2.39) (as well as (2.41)) is the expression of the nucleon electromagnetic current, which holds in any reference system. However, for the analysis of polarization phenomena, the Breit system is the most preferable. First of all, the explicit expression of the current $\mathcal{J}_\mu = (\mathcal{J}_0, \vec{\mathcal{J}})$ is simplified in the Breit system:

$$\begin{cases} \mathcal{J}_0 & = \overline{u}(p_2)\left[(F_1+F_2)\,\gamma_0 - \dfrac{(E_{1B}+E_{2B})}{2m}F_2\right]u(p_1),\ \ E_{1B}=E_{2B}=E, \\ \vec{\mathcal{J}} = & \overline{u}(p_2)\left[(F_1+F_2)\,\vec{\gamma} - \dfrac{(\mathbf{p_{1B}}+\mathbf{p_{2B}})}{2m}F_2\right]u(p_1) = (F_1+F_2)\,\overline{u}(p_2)\vec{\gamma}u(p_1). \end{cases}$$

Being $u(p_1)$ and $u(p_2)$ defined according to (2.14) we find, for the time component $\mathcal{J}_0$ of the current $\mathcal{J}_\mu$:

$$\begin{aligned} \mathcal{J}_0 &= (F_1+F_2)\,u^\dagger(p_2)u(p_1) - F_2\frac{E}{m}u^\dagger(p_2)\gamma_0 u(p_1) \\ &= (E+m)\Bigg\{(F_1+F_2)\,\chi_2^\dagger\left(1\ ,\ \frac{\vec{\sigma}\cdot\mathbf{q_B}}{2(E+m)}\right)\begin{pmatrix}\chi_1 \\ \dfrac{-\vec{\sigma}\cdot\mathbf{q_B}}{2(E+m)}\chi_1\end{pmatrix} \\ &-F_2\frac{E}{m}\chi_2^\dagger\left(1\ ,\ \frac{\vec{\sigma}\cdot\mathbf{q_B}}{2(E+m)}\right)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}\chi_1 \\ \dfrac{-\vec{\sigma}\cdot\mathbf{q_B}}{2(E+m)}\chi_1\end{pmatrix}\Bigg\} = \\ &= 2m\chi_2^\dagger\chi_1\,(F_1-\tau F_2)\,, \end{aligned} \tag{2.43}$$

where we used the definition:

$$\mathbf{p_{2B}}^2 = E^2 - m^2 = \frac{\mathbf{q_B}^2}{4},\ \text{so that}\ \frac{\mathbf{q_B}^2}{4(E+m)^2} = \frac{E-m}{E+m},$$

[1]This can be easily proved as follows. The term $\sigma_{\mu\nu}q_\mu q_\nu$ vanishes, because it is the product of a symmetrical and antisymmetrical tensors, and $\overline{u}(p_2)\hat{q}u(p_1) = \overline{u}(p_2)(\hat{p}_2-\hat{p}_1)u(p_1) = \overline{u}(p_2)(m-m)u(p_1) = 0$, as a result of the Dirac equation for both four-component spinors, $u(p_1)$ and $u(p_2)$. Note that the current (2.41) is conserved only when both nucleons (in initial and final states) are real, the form factor F_1 violates the current conservation, if one nucleon is virtual

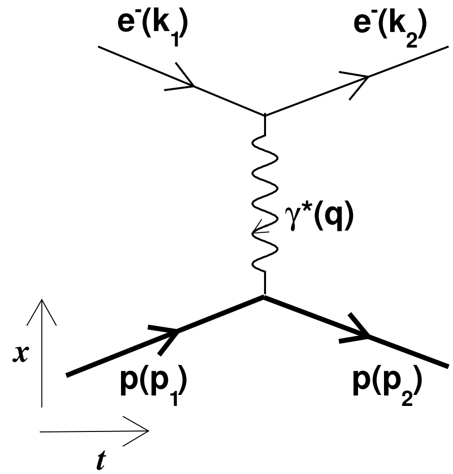


Figure 2.3: Feynman diagram representing elastic electron (e-)-proton (p) scattering. The reaction takes place in four dimensions : the time (horizontal axis) et the space (vertical axis). The proton is unchanged after receiving and sharing among its constituents a four-momentum q transferred through a virtual photon (γ^*) emitted by the incident electron.

and

$$\overline{u}(p_2) = u^\dagger(p_2)\gamma_0,\ \gamma_0^2 = 1 \text{ and } (\vec{\sigma}\cdot\mathbf{q})(\vec{\sigma}\cdot\mathbf{q}) = \mathbf{q}^2.$$

For the vector part $\vec{\mathcal{J}}$ of the nucleon electromagnetic current we can find similarly:

$$\begin{aligned}\vec{\mathcal{J}} &= (F_1+F_2)\,(E+m)\chi_2^\dagger\left(1\ ,\ \ -\frac{\vec{\sigma}\cdot\mathbf{q_B}}{2(E+m)}\right)\begin{bmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{bmatrix}\left(\begin{array}{c}\chi_1\\ \dfrac{-\vec{\sigma}\cdot\mathbf{q_B}}{2(E+m)}\chi_1\end{array}\right) = \\ &-\frac{1}{2}\,(F_1+F_2)\,\chi_2^\dagger\,(\vec{\sigma}\vec{\sigma}\cdot\mathbf{q_B}-\vec{\sigma}\cdot\mathbf{q_B}\vec{\sigma}) = (F_1+F_2)\,i\chi_2^\dagger\vec{\sigma}\times\mathbf{q_B}\chi_1.\end{aligned} \tag{2.44}$$

Finally:

$$\begin{aligned}\mathcal{J}_0 &= 2m\chi_2^\dagger\chi_1\,(F_1-\tau F_2)\\ \vec{\mathcal{J}} &= i\chi_2^\dagger\vec{\sigma}\times\mathbf{q_B}\chi_1\,(F_1+F_2)\,.\end{aligned} \tag{2.45}$$

These expressions for the different components of the current $\mathcal{J}_\mu$ are valid in the Breit frame only, and allow to introduce in a straightforward way the Sachs nucleon electric and magnetic FFs [25], which are written as:

$$G_E = F_1 - \tau F_2\ \ G_M = F_1 + F_2.$$

Such identification can be easily understood, if one takes into account that the time component of the current, $\mathcal{J}_0$, describes the interaction of the nucleon electric charge with the Coulomb potential. Correspondingly, the space component $\vec{\mathcal{J}}$ describes the interaction of the nucleon spin with the magnetic field.

Hadronic tensor $W_{\mu\nu}$

The hadronic tensor $W_{\mu\nu}$ is calculated in the Breit system, where the simple expression of the nucleon current, Eq. (2.45), can be rewritten in terms of the Sachs FFs as

$\mathcal{J}_\mu = \chi_2^\dagger F_\mu \chi_1$ with :

$$
\begin{aligned}
\mathcal{J}_0 \quad &= 2m\chi_2^\dagger\chi_1 \left(F_1 - \tau F_2\right) = 2m\chi_2^\dagger\chi_1 G_E, \\
\vec{\mathcal{J}} \quad &= i\chi_2^\dagger\vec{\sigma}\times\mathbf{q_B}\chi_1\left(F_1+F_2\right) = i\chi_2^\dagger\vec{\sigma}\times\mathbf{q_B}\chi_1 G_M
\end{aligned}
\tag{2.46}
$$

and the four components of F_μ, in terms of the FFs G_E and G_E, are:

$$
F_\mu = \begin{cases} 2mG_E & ,\mu=0 \\ i\sqrt{-q^2}G_M\sigma_y & ,\mu=x \\ -i\sqrt{-q^2}G_M\sigma_x & ,\mu=y \\ 0 & ,\mu=z \end{cases}
\tag{2.47}
$$

The hadronic tensor $W_{\mu\nu}$ can be written as follows:

$$
W_{\mu\nu} = \overline{(\chi_2^\dagger F_\mu\chi_1)(\chi_1^\dagger F_\nu^\dagger\chi_2)} = \frac{1}{2}TrF_\mu\rho_1F_\nu^\dagger\rho_2;
$$

where the averaging (summing) acts only on the two-component spinors, and we introduced density matrix for the nucleon: $\rho=\chi\chi^\dagger$, $\rho_{ab}=\chi_a\chi_b^*$, and $a,b=1,2$ are the spinor indexes. We included the statistical factor $1/(2s+1)=1/2$, for the initial nucleon.

In case of unpolarized particles $\rho=\frac{1}{2}\,\mathcal{I}$ (I is the unit matrix), $Tr\rho=1$, and

$$
W_{\mu\nu} = \frac{1}{2}TrF_\mu F_\nu^\dagger.
$$

Leptonic tensor $L_{\mu\nu}$

The leptonic tensor, which describes the electron vertex, is written as:

$$
L_{\mu\nu} = \overline{\ell_\mu\ell_\nu^*} = \overline{\overline{u}(k_2)\gamma_\mu u(k_1)\left[\overline{u}(k_2)\gamma_\nu u(k_1)\right]^*}.
$$

where the overline denotes the averaging over the polarizations of the initial electron and the summing over the polarizations of the final electrons. Recalling that

$$
\overline{u}=u^\dagger\gamma_0,\quad \overline{u}^\dagger=(u^\dagger\gamma_0)^\dagger=\gamma_0^\dagger u=\gamma_0 u,\quad \gamma_0\gamma_0=1,\ \gamma_0^\dagger=\gamma_0,
$$

we can write:

$$
\begin{aligned}
L_{\mu\nu} &= \overline{\overline{u}(k_2)\gamma_\mu u(k_1)u^\dagger(k_1)\gamma_\nu^\dagger\overline{u}^\dagger(k_2)} = \overline{\overline{u}(k_2)\gamma_\mu u(k_1)u^\dagger(k_1)\gamma_0\gamma_0\gamma_\nu^\dagger\gamma_0 u(k_2)} \\
&= \overline{\overline{u}(k_2)\gamma_\mu u(k_1)u^\dagger(k_1)\gamma_\nu^\dagger\gamma_0 u(k_2)} = \frac{1}{2}Tr\gamma_\mu\rho_e^1\gamma_\nu\rho_e^2.
\end{aligned}
\tag{2.48}
$$

From the Dirac theory we can write: $\overline{\overline{u}(k)u^\dagger(k)}=\hat{k}+m=\rho$. After performing the corresponding substitutions in Eq. (2.48), one finds (see Appendix 2):

$$
L_{\mu\nu} = \frac{1}{2}Tr\gamma_\mu(\hat{k_1}+m)\gamma_\nu(\hat{k_2}+m),
$$

from where we derive (neglecting the electron mass, and using $q^2 = (k_1 - k_2)^2 = -2k_1 \cdot k_2$:

$$\begin{aligned} L_{\mu\nu} &= 2k_{1\mu}k_{2\nu} + 2k_{1\nu}k_{2\mu} - 2g_{\mu\nu}k_1 \cdot k_2 \\ &= 2k_{1\mu}k_{2\nu} + 2k_{1\nu}k_{2\mu} - g_{\mu\nu}q^2 \end{aligned}$$

From this expression we see that the leptonic tensor which describes unpolarized electrons is symmetrical.

2.3.4 The Rosenbluth formula

Let us calculate explicitly the components for the hadronic tensor $W_{\mu\nu}$, in terms of the FFs G_E and G_M. Recalling the property that $Tr\vec{\sigma} \cdot \mathbf{A} = 0$, for any vector $\mathbf{A}$, we see that all terms for the components $W_{\mu\nu}$ which contain the product $G_E G_M$ vanish: this means that the unpolarized cross section of $eN-$scattering does not contain this interference term. The non-zero components of $W_{\mu\nu}$ are determined only by G_E^2 and G_M^2:

$$\begin{aligned} W_{00} &= 4m^2 G_E^2, \\ W_{xx} &= -q^2 G_M^2, \\ W_{yy} &= -q^2 G_M^2. \end{aligned}$$

Substituting these expressions in Eq. (2.42), one can find for the matrix element squared:

$$\left(\frac{q^2}{e^2}\right)^2 \overline{|\mathcal{M}|^2} = L_{00}W_{00} + (L_{xx} + L_{yy})W_{xx} = L_{00}4m^2G_E^2 + (L_{xx} + L_{yy})(-q^2)G_M^2. \quad (2.49)$$

The necessary components of the leptonic tensor $L_{\mu\nu}$, calculated in the Breit system, are:

$$\begin{aligned} L_{00} &= 4\epsilon_{1B}^2 + q^2 = -q^2 \cot^2 \frac{\theta_B}{2}, \\ L_{yy} &= -q^2, \\ L_{xx} &= 4k_{1x}^2 - q^2 = -q^2 \left(1 + \cot^2 \frac{\theta_B}{2}\right). \end{aligned}$$

Substituting the corresponding terms in Eq. (2.49) we have:

$$\overline{|\mathcal{M}|^2} = \left(\frac{e^2}{q^2}\right)^2 \left[-q^2 \cot^2 \frac{\theta_B}{2} 4m^2 G_E^2 + (-q^2 - q^2 \cot^2 \frac{\theta_B}{2})(-q^2 G_M^2)\right],$$

which becomes in the Lab system:

$$\overline{|\mathcal{M}|^2} = \left(\frac{e^2}{q^2}\right)^2 4m^2(-q^2) \left[2\tau G_M^2 + \frac{\cot^2 \frac{\theta_e}{2}}{1+\tau}(G_E^2 + \tau G_M^2)\right]. \quad (2.50)$$

We can then find the following formula for the cross section, $d\sigma/d\Omega_e$, in the Lab system, in terms of the electromagnetic FFs G_E and G_M (Rosenbluth formula [4], originallly written in terms of the Dirac (F_1) and Pauli (F_2) form factors):

$$\frac{d\sigma}{d\Omega_e} = \frac{\alpha^2}{-q^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \left[2\tau G_M^2 + \frac{\cot^2\frac{\theta_e}{2}}{1+\tau}\left(G_E^2+\tau G_M^2\right)\right], \tag{2.51}$$

where $\alpha = e^2/4\pi \simeq 1/137$ is the fine structure constant.

Note that the very specific $\cot^2\dfrac{\theta_e}{2}$-dependence of the cross section for eN-scattering results from the assumption of one-photon mechanism for the considered reaction [26]. The particular $\cot^2\dfrac{\theta_e}{2}$-dependence of the differential eN-cross section is at the basis of the method to determine both nucleon electromagnetic FFs, G_E and G_M, using the linearity of the *reduced* cross section:

$$\sigma_{red} = \frac{d\sigma}{d\Omega_e}\Big/\left[\frac{\alpha^2}{-q^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\right]$$

as a function of $\cot^2\frac{\theta_e}{2}$ (Rosenbluth fit or Rosenbluth separation). One can see that the backward eN-scattering ($\theta_e = \pi, \cot^2\frac{\theta_e}{2} = 0$) is determined by the magnetic FF only, and that the slope for σ_{red} is sensitive to G_E^2 (Fig. 2.4).

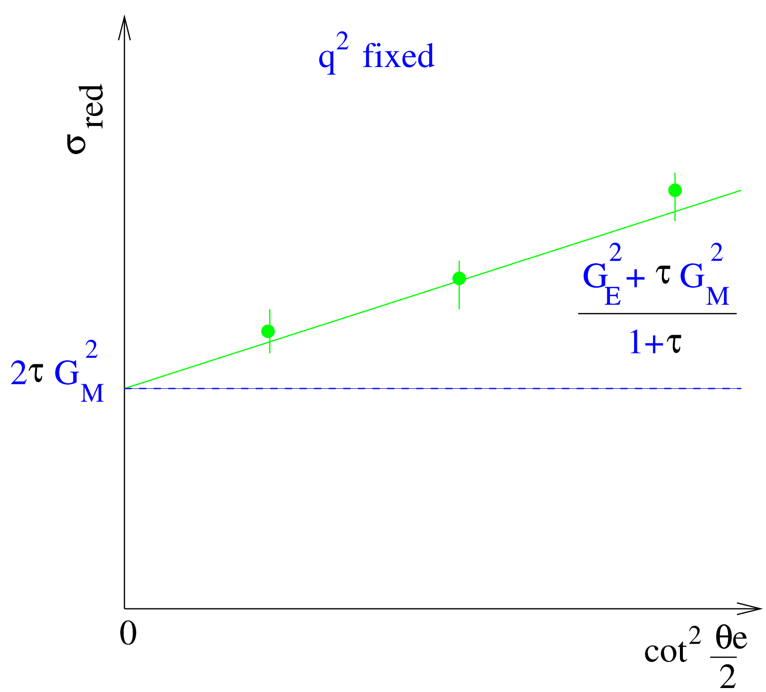


Figure 2.4: Illustration of the Rosenbluth separation for the elastic differential cross section for eN-scattering.

At large q^2, (such that $\tau \gg 1$), the differential cross-section $d\sigma/d\Omega_e$ (with unpolarized particles) is insensitive to G_E: the corresponding combination of the nucleon FFs, $G_E^2 + \tau G_M^2$ is dominated by the G_M contribution, due to the following reasons:

- $G_{Mp}/G_{Ep} \simeq \mu_p$, where μ_p is the proton magnetic moment, so $G_{Mp}^2/G_{Ep}^2 \simeq 2.79^2 \simeq 8$;
- The factor τ increases the G_M^2 contribution at large momentum transfer, where $\tau \gg 1$.

Therefore $ep-$scattering (with unpolarized particles) is dominated by the magnetic FF, at large values of momentum transfer. The same holds for $en-$scattering, even at relatively small values of q^2, due to the smaller values of the neutron electric FF.

As a result, for the exact determination of the proton electric FF, in the region of large momentum transfer, and for the neutron electric FF - at any value of q^2, polarization measurements are required and in particular those polarization observables which are determined by the product $G_E G_M$, and are, therefore, more sensitive to G_E.

In principle, there are some components of the depolarization tensor (characterizing the dependence of the final proton polarization on the target polarization (for the scattering of unpolarized electrons, $e + \vec{p} \to e + \vec{p}$) which are also proportional to $G_E G_M$, and therefore can be used for the determination of the nucleon electric FF [6].

There are at least two different classes of polarization experiments of such type: the scattering of longitudinally polarized electrons by polarized target (with polarization in the reaction plane, but perpendicular to the direction of the three-momentum transfer) $\vec{e} + \vec{p} \to e + p$, or the measurement of the ratio of transversal to longitudinal proton polarization (in the reaction plane) for the scattering of longitudinally polarized electrons by unpolarized target, $\vec{e} + p \to e + \vec{p}$. These two different polarization experiments bring the same physical information, concerning the electromagnetic FFs of proton. They have been realized, at some intermediate value of the transferred momentum. Note, however, that the experiments with polarized target do not allow, in principle, to reach higher momenta. The limitation is due to the fact that a depolarization of the target occurs, when the beam intensity if very large. High beam intensity is, however required to compensate the large decrease of the cross section with increasing momentum.

2.4 Polarization observables

In general the hadronic tensor $W_{\mu\nu}$, for ep elastic scattering can be decomposed in four terms, related to the four possibilities of polarizing the initial or the final proton or both:

$$W_{\mu\nu} = W_{\mu\nu}^{(0)} + W_{\mu\nu}(\mathbf{P_1}) + W_{\mu\nu}(\mathbf{P_2}) + W_{\mu\nu}(\mathbf{P_1}, \mathbf{P_2}),$$

where $\mathbf{P_1}$ ($\mathbf{P_2}$) is the polarization vector of the initial (final) proton. The first term corresponds to the unpolarized case, the second (third) term corresponds to the case when the initial (final) proton is polarized, and the last term describes the reaction when both protons (initial and final) are polarized.

Let us consider the case when only the final proton is polarized ($\mathbf{P} = \mathbf{P_2}$):

$$W_{\mu\nu}(\mathbf{P}) = \frac{1}{2} Tr F_\mu F_\nu^\dagger \vec{\sigma} \cdot \mathbf{P}$$

For the scattering of longitudinally polarized electrons (by unpolarized target), only the x and z components of the polarization vector $\mathbf{P}$ do not vanish. To find these components, let us calculate the tensors $W_{\mu\nu}(P_x)$ and $W_{\mu\nu}(P_z)$.

$$W_{\mu\nu}(P_x) = \frac{1}{2} Tr F_\mu F_\nu^\dagger \sigma_x.$$

Let us start [2] from the calculation of the components $F_\nu^\dagger$:

$$F_\nu^\dagger = \begin{cases} 2mG_E & ,\nu = 0 \\ -i\sqrt{-q^2}G_M\sigma_y & ,\nu = x \\ i\sqrt{-q^2}G_M\sigma_x & ,\nu = y \\ 0 & ,\nu = z \end{cases} \tag{2.52}$$

Therefore, one can find easily (using $\sigma_x\sigma_y = i\sigma_z$, $\sigma_y\sigma_z = i\sigma_x$,$\sigma_z\sigma_x = i\sigma_y$):

$$F_\nu^\dagger\sigma_x = \begin{cases} 2mG_E\sigma_x & ,\nu = 0 \\ -\sqrt{-q^2}G_M\sigma_z & ,\nu = x \\ i\sqrt{-q^2}G_M & ,\nu = y \\ 0 & ,\nu = z. \end{cases} \tag{2.53}$$

This allows to write:

$$F_\mu F_\nu^\dagger\sigma_x = \begin{cases} 2mG_E & ,\mu = 0 \\ i\sqrt{-q^2}G_M\sigma_y & ,\mu = x \\ -i\sqrt{-q^2}G_M\sigma_x & ,\mu = y \\ 0 & ,\mu = z \end{cases} \bigotimes \begin{cases} 2mG_E\sigma_x & ,\nu = 0 \\ -\sqrt{-q^2}G_M\sigma_z & ,\nu = x \\ i\sqrt{-q^2}G_M & ,\nu = y \\ 0 & ,\nu = z \end{cases} \tag{2.54}$$

As we have to calculate the trace, recalling that $Tr\sigma_{x,y,z} = 0$, we can see that the non-zero components of the hadronic tensor $W_{\mu\nu}(P_x)$ are:

$$\begin{aligned} W_{0y}(P_x) &= \ i\sqrt{-q^2}\ 2mG_EG_M, \\ W_{y0}(P_x) &= \ -i\sqrt{-q^2}\ 2mG_EG_M. \end{aligned} \tag{2.55}$$

So we proved here that only two components of $W_{\mu\nu}(P_x)$ are different from zero: they are equal in absolute value and opposite in sign: it follows that $W_{\mu\nu}(P_x)$ is an antisymmetrical tensor. Therefore, the product $L_{\mu\nu}W_{\mu\nu}(P_x)$ vanishes: $L_{\mu\nu}$ is a symmetrical tensor: the product of a symmetrical tensor and an asymmetrical tensor is

[2]We will take into account the fact that the FFs $G_E(q^2)$ and $G_M(q^2)$ are real functions of (q^2) in the space-like region, see later.

zero. This means that the polarization of the final proton vanishes, if the electron is unpolarized: **unpolarized electrons can not induce polarization of the scattered proton**. This is a property of the one-photon mechanism $for\ any\ elastic\ electron-hadron\ scattering$ and of the hermiticity of the Hamiltonian for the hadron electromagnetic interaction. Namely the hermiticity condition allows to prove that the hadron electromagnetic FFs are real functions of the momentum transfer squared in the space-like region. On the other hand, in the time-like region, which is scanned by the annihilation processes, $e^- + e^+ \leftrightarrow p + \overline{p}$, the nucleon electromagnetic FFs are complex functions of q^2, if $q^2 \geq 4m_\pi^2$, where m_π is the pion mass. This is due to the unitarity condition, which can be illustrated as in Fig. 2.5.

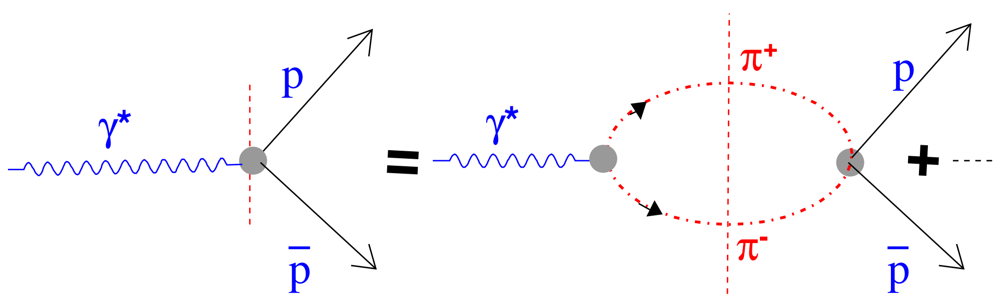


Figure 2.5: The unitarity condition for proton electromagnetic FFs in the time-like region of momentum transfer squared. Vertical line on the right side crosses the pion lines, describing real particles (on mass shell). The dotted line denotes other possible multi-pion states, in the chain of the following transitions: $\gamma^* \to n\pi \to p\overline{p}$, where n is the number of pions in the intermediate state.

The complexity of nucleon FF's (in the time-like region) results in specific polarization phenomena, for the the annihilation processes $e + e^- \leftrightarrow p + \overline{p}$, which are different from the case of elastic $ep-$scattering. For example, the polarization of the final proton (or antiproton) is different from zero, even in the case of collisions of unpolarized leptons: this polarization is determined by the product $\mathcal{I}mG_EG_M^*$ (and, therefore vanishes in the case of elastic ep-scattering, where the FFs are real).

Let us consider now the proton polarization in the z-direction:

$$W_{\mu\nu}(P_z) = \frac{1}{2}TrF_\mu F_\nu^\dagger \sigma_z.$$

Firstly we calculate the components of $F_\nu^\dagger \sigma_z$:

$$F_\nu^\dagger \sigma_z = \begin{cases} 2mG_E\sigma_z & ,\nu = 0 \\ \sqrt{-q^2}G_M\sigma_x & ,\nu = x \\ \sqrt{-q^2}G_M\sigma_y & ,\nu = y \\ 0 & ,\nu = z \end{cases} \tag{2.56}$$

Therefore we find:

$$F_\mu F_\nu^\dagger \sigma_z = \begin{cases} 2mG_E & ,\mu = 0 \\ i\sqrt{-q^2}G_M\sigma_y & ,\mu = x \\ -i\sqrt{-q^2}G_M\sigma_x & ,\mu = y \\ 0 & ,\mu = z \end{cases} \bigotimes \begin{cases} 2mG_E\sigma_z & ,\nu = 0 \\ \sqrt{-q^2}G_M\sigma_x & ,\nu = x \\ \sqrt{-q^2}G_M\sigma_y & ,\nu = y \\ 0 & ,\nu = z \end{cases} \tag{2.57}$$

We see that $W_{0\nu}(P_z) = W_{\nu 0}(P_z) = 0$, for any ν, and no interference term $G_E G_M$ is present. The nonzero components of $W_{\mu\nu}(P_z)$ are:

$$\begin{aligned} W_{xy}(P_z) =& \; -iq^2G_M^2, \\ W_{yx}(P_z) =& \; iq^2G_M^2, \end{aligned} \tag{2.58}$$

from where we see that $W_{\mu\nu}(P_z)$ is an antisymmetrical tensor, which depends on G_M^2.

Polarized electron

The leptonic tensor, $L_{\mu\nu}$, in case of unpolarized particles, contains only one term. For longitudinally polarized electrons, the polarization is characterized by the helicity λ, which takes values ± 1, corresponding to the direction of spin parallel or antiparallel to the electron three-momentum. The general expression for the leptonic tensor is:

$$L_{\mu\nu} = L_{\mu\nu}^{(0)} + L_{\mu\nu}(\lambda_1) + L_{\mu\nu}(\lambda_2) + L_{\mu\nu}(\lambda_1, \lambda_2). \tag{2.59}$$

where the first term, considered previously, describes the collision where the initial and final electrons are unpolarized, the second (third) term describes the case when the initial (final) electron is longitudinally polarized, and the last terms holds when both electrons are polarized.

Using the expression for the density matrix ρ Table 2.3, the leptonic tensor $L_{\mu\nu}(\lambda)$, corresponding to the scattering of longitudinally polarized electrons (neglecting the electron mass) is:

$$L_{\mu\nu}(\lambda) = \frac{1}{2}Tr\gamma_\mu \hat{k}_1(1+\lambda\gamma_5)\gamma_\nu \hat{k}_2 = \frac{1}{2}Tr\gamma_\nu \hat{k}_1\gamma_\nu \hat{k}_2 + \frac{\lambda}{2}Tr\gamma_\nu \hat{k}_1\gamma_5\gamma_\nu \hat{k}_2 = L_{\mu\nu}^{(0)} + \lambda L_{\mu\nu}^{(1)}. \tag{2.60}$$

The tensor $L_{\mu\nu}^{(0)}$ corresponds to the scattering of unpolarized electrons:

$$L_{\mu\nu}^{(0)} = 2k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu} - g_{\mu\nu}k_1 \cdot k_2. \tag{2.61}$$

The tensor $L^{(1)}_{\mu\nu}$, describing the dependence on the longitudinal electron polarization can be written in the following form:

$$L^{(1)}_{\mu\nu} = \frac{1}{2} Tr \gamma_\mu \hat{k}_1 \gamma_\nu \hat{k}_2 \gamma_5 = -\frac{1}{2} Tr \gamma_\mu \gamma_\nu \hat{k}_1 \hat{k}_2 \gamma_5 = 2i\epsilon_{\mu\nu\rho\sigma} k_{1\rho} k_{2\sigma}. \tag{2.62}$$

We applied another property of γ_5, that is:

$$Tr \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5 = -4i\epsilon_{\mu\nu\rho\sigma}.$$

Taking into account the conservation of four-momentum in the electron vertex: $k_1 = k_2 + q$, we can rewrite the tensor $L^{(1)}_{\mu\nu}$ in the following form, which is more convenient in this frame:

$$L^{(1)}_{\mu\nu} = 2i\epsilon_{\mu\nu\rho\sigma} q_\rho k_{1\sigma}. \tag{2.63}$$

The three-vector $\mathbf{q}$ has only nonzero $z-$component, in the Breit system. The tensor $\epsilon_{\mu\nu\rho\sigma}$ is defined in such way that $\epsilon_{xyz0} = +1$. If only the initial electron is polarized, $\lambda_1 = \lambda$, one can write for $L_{\mu\nu}$:

$$L_{\mu\nu}(\lambda) = 2i\lambda\epsilon_{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}. \tag{2.64}$$

The effect of the electron polarization is described by an antisymmetrical tensor $L_{\mu\nu}(\lambda)$. If the initial proton is unpolarized, again, being described by symmetrical tensor, the total result will be zero. This result holds because the FFs are real, so it does not apply in the time-like region.

The x-component : $L_{\mu\nu}(\lambda) W_{\mu\nu}(P_x)$

Let us consider the product of the leptonic $L_{\mu\nu}(\lambda)$ and hadronic $W_{\mu\nu}(P_x)$ tensors, for the x component of the final proton polarization:

$$\begin{aligned} L_{\mu\nu}(\lambda) W_{\mu\nu}(P_x) &= L_{0y}(\lambda) W_{0y}(P_x) + L_{y0}(\lambda) W_{y0}(P_x) \\ &= L_{0y}(\lambda) \left[W_{0y}(P_x) - W_{y0}(P_x) \right] = 2 L_{0y}(\lambda) W_{0y}(P_x). \end{aligned} \tag{2.65}$$

Taking into account that: $L_{0y} = 2i\lambda\epsilon_{0y\alpha\beta} k_{1\alpha} k_{2\beta}$ the only non-zero terms correspond to $\alpha = x$ and $\beta = z$ or $\alpha = z$ and $\beta = x$. Therefore:

$$L_{0y}(\lambda) = 2i\lambda \left(\epsilon_{0yxz} k_{1x} k_{2z} + \epsilon_{0yzx} k_{1z} k_{2x} \right) = 2i\lambda\epsilon_{0yxz} (k_{1x} k_{2z} - k_{1z} k_{2x}) = i\lambda q^2 \cot\frac{\theta_B}{2},$$

with $\epsilon_{0yxz} = 1$, and using Eqs. (2.3) and (2.4).

We finally find:

$$L_{\mu\nu}(\lambda) W_{\mu\nu}(P_x) = -4\lambda m q^2 \sqrt{-q^2} \cot\frac{\theta_B}{2} G_E G_M. \tag{2.66}$$

The z-component : $L_{\mu\nu}(\lambda) W_{\mu\nu}(P_z)$

Similarly, considering the antisymmetry of both tensors $L_{\mu\nu}(\lambda)$ and $W_{\mu\nu}(P_z)$, one can find:

$$\begin{aligned} L_{\mu\nu}(\lambda) W_{\mu\nu}(P_z) &= 2i\lambda\epsilon_{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} W_{\mu\nu}(P_z) = 4\epsilon_{xy0z} W_{xy}(P_z) \left(\epsilon_{1B} k^z_{2B} - \epsilon_{2B} k^z_{1B} \right) \\ &= 4\lambda q^2 \frac{G_M^2}{\sin\theta_B/2}. \end{aligned} \tag{2.67}$$

2.4.1 Final formulas

The polarization $\mathbf{P}$ of the scattered proton can be written as:

$$\mathbf{P}\frac{d\sigma}{d\Omega_e} = \frac{\alpha^2}{4\pi^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \frac{L_{\mu\nu}(\lambda)}{m^2}\vec{P}_{\mu\nu}.$$

with $\vec{P}_{\mu\nu} = \frac{1}{2}Tr(\mathcal{F}_\mu\mathcal{F}_\nu^\dagger\vec{\sigma})$, so that $P_{\mu\nu}^{(z)} = W_{\mu\nu}(P_z)$ and $P_{\mu\nu}^{(x)} = W_{\mu\nu}(P_x)$

Using Eq. (2.5) one can find the following expressions for the components P_x and P_z of the proton polarization vector (in the scattering plane) - in terms of the proton electromagnetic FFs:

$$\begin{aligned} DP_x &= -2\lambda\cot\frac{\theta_e}{2}\sqrt{\frac{\tau}{1+\tau}}G_E G_M, \\ DP_z &= \lambda\frac{\epsilon_1+\epsilon_2}{m}\sqrt{\frac{\tau}{1+\tau}}G_M^2, \end{aligned} \tag{2.68}$$

where D is proportional to the differential cross section with unpolarized particles:

$$D = 2\tau G_M^2 + \cot^2\frac{\theta_e}{2}\frac{G_E^2+\tau G_M^2}{1+\tau}. \tag{2.69}$$

So, for the ratio of these components one can find the following formula:

$$\frac{P_x}{P_z} = \frac{P_t}{P_\ell} = -2\cot\frac{\theta_e}{2}\frac{m}{\epsilon_1+\epsilon_2}\frac{G_E(q^2)}{G_M(q^2)} \tag{2.70}$$

which clearly shows that a measurement of the ratio of transverse and longitudinal polarization of the recoil proton gives is a direct measurement of the ratio of electric and magnetic FFs, $G_E(q^2)/G_M(q^2)$.

In the same way it is possible to calculate the dependence of the differential cross section for the elastic scattering of the longitudinally polarized electrons by a **polarized** proton target, with polarization $\mathcal{P}$, in the above defined coordinate system:

$$\frac{d\sigma}{d\Omega_e}(\mathcal{P}) = \left(\frac{d\sigma}{d\Omega_e}\right)_0 \left(1+\lambda\mathcal{P}_x A_x + \lambda\mathcal{P}_z A_z\right), \tag{2.71}$$

where the asymmetries A_x and A_z (or the corresponding analyzing powers) are related in a simple and direct way, to the components of the final proton polarization:

$$\begin{aligned} A_x &= P_x, \\ A_z &= -P_z. \end{aligned} \tag{2.72}$$

This holds in the framework of the one-photon mechanism for elastic $ep-$scattering. Note that the quantities A_x and P_x have the same sign and absolute value, but the components A_z and P_z, being equal in absolute value, have opposite sign.

Note that the P_y-component of the proton polarization vanishes in the scattering of polarized and unpolarized electrons, as well. This results from the one-photon mechanism and the reality of form factors G_E and G_M. For the same reasons, the corresponding analyzing power, A_y, also vanishes.

Summarizing this discussion, let us stress once more that these results for polarization observables in elastic ep-scattering hold in the framework of the one-photon mechanism.

Still in the framework of the one-photon mechanism, there are at least two different sources of corrections to these relations:

- the standard radiative corrections;
- the electroweak corrections.

These last corrections arise from the interference of amplitudes, corresponding to the exchange of γ and $Z-$boson. The relative value of these contributions is characterized by the following dimensionless parameter:

$$G_{eff} = \frac{G_F}{2\sqrt{2}\alpha\pi}|q^2| \simeq 10^{-4}\frac{|q^2|}{\mathrm{GeV}^2},$$

where G_F is the standard Fermi constant of the weak interaction, $G_F \simeq 10^{-5}/m^2$.

This formalism equally applies to en-elastic scattering, too, in the case of free neutron. As typically a target like d or ^3He is used, specific considerations apply, which are outside the present notes. This formalism is valid in case of elastic $e+^3He$ and $e+^3H$ scattering, and, in general, for elastic scattering of electrons on any spin 1/2 target.

Polarization phenomena for elastic positron scattering and for elastic scattering of positive and negative muons are the same as in case of electron scattering.

Exercise 1

Prove the relation $s+t+u = m_1^2+m_2^2+m_3^2+m_4^2$, in CMS or Lab system.

Exercise 2

Prove the following relation between the electron scattering angles in the Lab system, θ_e and in the Breit system, θ_B:

$$\cot^2\frac{\theta_B}{2} = \frac{\cot^2\theta_e/2}{1+\tau}. \tag{2.73}$$

Exercise 3

Derive the following relation:

$$\overline{u}(p_2)\frac{\sigma_{\mu\nu}q_\nu}{2m}u(p_1) = \overline{u}(p_2)\left[\gamma_\mu - \frac{(p_1+p_2)_\mu}{2m}\right]u(p_1). \tag{2.74}$$

with $q = p_2 - p_1$, $\hat{a} = a_\mu\gamma_\mu$ and $\sigma_{\mu\nu} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2$.

Chapter 3

Lecture II: The annihilation channel $\bar{p} + p \to e^+ + e^-$

In this section we derive the unpolarized and polarized cross section for the annihilation reactions $e^+ + e^- \leftrightarrow p + \bar{p}$. The formalism can be applied, of course, to the reaction $e^- + e^- \to n + \bar{n}$, accessible at electron colliders.

3.1 The annihilation channel $\bar{p} + p \to e^+ + e^-$

The measurement of the differential cross section for the process $\bar{p} + p \to \ell^+ + \ell^-$ at a fixed value of the total energy s, and for two different angles $\tilde{\theta}$, allows the separation of the two FFs, $|G_M|^2$ and $|G_E|^2$, and is equivalent to the Rosenbluth separation for the elastic ep-scattering. In TL region, this procedure is simpler, as it requires to change only one kinematical variable, $\cos\tilde{\theta}$, whereas, in SL region it is necessary to change simultaneously two kinematical variables: the energy of the initial electron and the electron scattering angle, fixing the momentum transfer squared, Q^2. Due to the limited statistics, the individual determination of the $|G_E|^2$ and $|G_M|^2$ contributions could not be realized earlier, only few data on the FF ratio were given by the BABAR [15] and the PS170 collaborations [11]. Only very recently, the BESIII collaboration could extract separately $|G_E|$ and $|G_M|$ with a meaningful error in the near threshold region [27].

In the TL region, the determination of a generalized FF requires to integrate the differential cross section over a wide angular range. One typically assumes that the G_E contribution plays a minor role in the cross section at large q^2 and the experimental results are usually given in terms of $|G_M|$, under the hypothesis that $G_E = 0$ or $|G_E| = |G_M|$. The first hypothesis is an arbitrary one. The second hypothesis is strictly valid at threshold only, i.e., for $\tau = q^2/(4M^2) = 1$, but there is no theoretical argument which justifies its validity at any other momentum transfer, where $q^2 \neq 4M^2$. The $|G_M|$ values depend, in principle, on the kinematics where the measurement was performed and the angular range of integration. However, it turns out that these two assumptions for G_E lead to comparable values for $|G_M|$ at large q^2.

In annihilation channel, The CMS system is more preferable to perform the calculations. The constrain on the vanishing total momentum in initial and final channels strongly simplifies the formalism.

3.1.1 Observables for $\bar{p} + p \to e^+ + e^-$

The derivation given below is simplified by the use of 2×2 Pauli matrix, and 2-rank spinors, instead of 4×4 Dirac matrices and 4-rank spinors. It is a rigorous and simple derivation in frame of one-photon exchange. The full derivation in the Dirac formalism can be found in Ref. [28], where additional (odd) amplitudes are added. Such amplitudes arise in frame of different reaction mechanisms, as Z-boson exchange or two photon exchange.

Let us consider the annihilation reaction

$$\bar{p}(p_1) + p(p_2) \to e^-(k_1) + e^+(k_2) \tag{3.1}$$

in the CMS system, where an antiproton with three-momentum $\mathbf{p_1} = \mathbf{p}$ annihilates with a proton with three-momentum $\mathbf{p}_2 = -\mathbf{p}$. The transferred momentum is $t = s = (k_1 + k_2)^2 = 4E^2$ and (assuming $m = 0$) one has $\mathbf{k} = \mathbf{k_1} = -\mathbf{k_2}$; $E = |\mathbf{k}|$. We choose a reference system with the z axis along the beam momentum, and xz is the scattering plane. In this system the unit vectors are: $\hat{\mathbf{p}} = (0, 0, 1)$ and $\hat{\mathbf{k}} = (\sin\tilde{\theta}, 0, \cos\tilde{\theta})$, with $\mathbf{p} \cdot \mathbf{k} = \cos\tilde{\theta}$.

The following relation holds (neglecting the electron mass):

$$\frac{\vec{\sigma} \cdot \mathbf{k}}{E + m} = \frac{\vec{\sigma} \cdot \mathbf{k}}{|\mathbf{k}|} = \vec{\sigma} \cdot \hat{\mathbf{k}} \tag{3.2}$$

The starting point of the analysis of the reaction $\bar{p} + p \to e^+ + e^-$ is the standard expression of the matrix element in framework of one-photon exchange mechanism:

$$\mathcal{M} = \frac{e^2}{q^2} \overline{v}(k_2) \gamma_\mu u(k_1) \overline{u}(p_2) J_\mu v(p_1), \tag{3.3}$$

with

$$J_\mu = \left[F_1(q^2)\gamma_\mu - \frac{\sigma_{\mu\nu} q_\nu}{2M} F_2(q^2) \right] = \left[F_1(q^2) + F_2(q^2) \right] \gamma_\mu - \frac{(-p_1 + p_2)_\mu}{2M} F_2(q^2),$$

where p_1, p_2, k_1 and k_2 are the four-momenta of initial antiproton and proton and the final electron and positron respectively, $q^2 > 4M^2$, $q = k_1 + k_2 = p_1 + p_2$. F_1 and F_2 are the Dirac and Pauli nucleon electromagnetic FFs, which are complex functions of the variable q^2 - in the TL region of momentum transfer. The spinors for particles (electron and proton) and antiparticles (positron and antiproton) are denoted with u and v, respectively.

The matrix element is written as the product of the leptonic and hadronic currents:

$$\mathcal{M} = \frac{e^2}{q^2} L_\mu J_\mu = \frac{e^2}{q^2}(L_0 J_0 - \vec{L}\cdot\vec{J}) = -\frac{e^2}{q^2}\vec{L}\cdot\vec{J}, \tag{3.4}$$

where $L_0 J_0 = 0$, due to the conservation of the leptonic and hadronic currents. The conservation of the current implies that $L \cdot q = 0$, i.e., $L_0 q_0 - \vec{L}\cdot\vec{q} = 0$, but $\mathbf{q} = \mathbf{k_1} + \mathbf{k_2} = 0$ in CMS. Therefore, $L_0 q_0 = 0$ for any energy q_0, i.e., $L_0 = 0$.

Let us reduce the expressions of the current in terms of σ (Pauli) matrices instead of Dirac γ matrices $J_\mu \to \varphi_2 \tilde{J}_\mu \varphi_1$ (we keep in mind a global factor $(E+M)$).

$$\begin{aligned}
J_\mu &= (F_1+F_2)\left(\varphi_2,\ -\frac{\vec{\sigma}\cdot(-\mathbf{p})}{E+M}\varphi_2\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\dfrac{\vec{\sigma}\cdot\mathbf{p}}{E+M}\varphi_1\\ \varphi_1\end{pmatrix}\\
&\quad+\left(\varphi_2,\ \frac{\vec{\sigma}\cdot(-\mathbf{p})}{E_1+M}\varphi_2\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{2\mathbf{p}}{2M}F_2\begin{pmatrix}\dfrac{\vec{\sigma}\cdot\mathbf{p}}{E_1+M}\varphi_1\\ \varphi_1\end{pmatrix}\\
&= (F_1+F_2)\left(\varphi_2,\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E+M}\varphi_2\right)\begin{pmatrix}\vec{\sigma}\varphi_1\\ -\vec{\sigma}\dfrac{\vec{\sigma}\cdot\mathbf{p}}{E+M}\varphi_1\end{pmatrix}\\
&\quad+\frac{\mathbf{p}}{M}F_2\varphi_2\left(\frac{\vec{\sigma}\cdot\mathbf{p}}{E+m}+\frac{\vec{\sigma}\cdot\mathbf{p}}{E+M}\right)\varphi_1\\
&= (F_1+F_2)\left[\vec{\sigma} - \frac{1}{(E+M)^2}\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\vec{\sigma}\cdot\mathbf{p}\right] + \frac{2\mathbf{p}}{M}F_2\varphi_2\frac{\vec{\sigma}\cdot\mathbf{p}}{E+M}\varphi_1.
\end{aligned}$$

Using the relation $p^2 = E^2 - M^2$, introducing the unit vectors $\hat{\mathbf{p}}$ and applying the following properties of σ matrices:

$$(2\hat{\mathbf{p}} - \vec{\sigma}\vec{\sigma}\cdot\hat{\mathbf{p}})\vec{\sigma}\cdot\hat{\mathbf{p}} = 2\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}} - \vec{\sigma},$$

one finds

$$\begin{aligned}
J_\mu &= (F_1+F_2)\left(\vec{\sigma} - 2\frac{E-M}{E+M}\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}} + \frac{E-M}{E+M}\vec{\sigma}\right) + \frac{2(E-M)}{M}F_2\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}\\
&= (F_1+F_2)\left(\vec{\sigma} + \frac{E-M}{E+M}\vec{\sigma}\right) - 2\left[(F_1+F_2)\frac{E-M}{E+M} - \frac{E-M}{M}F_2\right]\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}\\
&= \frac{2E}{E+M}(F_1+F_2)\vec{\sigma} - \frac{2(E-M)}{M(E+M)}[MF_1 + MF_2 - EF_2 - MF_2]\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}\\
&= \frac{2E}{E+M}(F_1+F_2)\vec{\sigma} - 2E(F_1+F_2)\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}} + 2M\left(F_1 + \frac{E^2}{M^2}F_2\right)\\
&= \frac{2E}{E+M}\left[G_M(\vec{\sigma} - \hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}})\right] + 2MG_E\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}.
\end{aligned}$$

Finally (reminding the global factor) we find for the hadronic current:

$$\vec{J} = \sqrt{q^2}\varphi_2^\dagger\left[G_M(q^2)(\vec{\sigma} - \hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}) + \frac{1}{\sqrt{\tau}}G_E(q^2)\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}\right]\varphi_1, \tag{3.5}$$

where φ_1 and φ_2 are the two-component spinors of the antiproton and the proton, $\hat{\mathbf{p}}$ is the unit vector along the three momentum of the antiproton in CMS. The expression for the leptonic current is:

$$\vec{L} = \sqrt{q^2}\varphi_2^\dagger(\vec{\sigma} - \hat{\mathbf{k}}\vec{\sigma}\cdot\hat{\mathbf{k}})\varphi_1, \tag{3.6}$$

where $\varphi_1(\varphi_2)$ is the two-component spinor of the electron (positron), $\hat{\mathbf{k}}$ is the unit vector along the final electron three-momentum.

Note that Eq. (3.6) holds for the production of unpolarized lepton (sum over the lepton polarization). From this expression one can see the physical meaning of the particular relation between the nucleon electromagnetic FFs at threshold:

$$G_E(q^2) = G_M(q^2),\ q^2 = 4M^2.$$

The structure $\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}$ describes the $\overline{p}+p$ annihilation from D-wave, i.e., with angular momentum ℓ=2. At threshold, where $\tau \to 1$, the finite radius of the strong interaction allows only the S-state, and $G_M(q^2) - \dfrac{1}{\sqrt{\tau}}G_E(q^2) = 0$.

From Eqs. (3.4), (3.5), and (3.6) one can find the formulas for the unpolarized cross section, the angular asymmetry and all the polarization observables.

3.1.2 The cross section

To calculate the cross section when all particles are unpolarized, one has to sum over the polarization of the final particles and to average over the polarization of initial particles:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 q^2}\frac{|\mathbf{k}|}{|\mathbf{p}|},\ |\mathbf{k}| = \frac{\sqrt{q^2}}{2},\ |\mathbf{p}| = \sqrt{\frac{q^2}{4} - M^2},$$

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{4}\frac{e^4}{q^4}L_{ab}J_{ab},\ L_{ab} = L_a L_b^*,\ J_{ab} = J_a J_b^*.\\ \overline{L_{ab}} &= \overline{L_a L_b^*} \sim Tr(\sigma_a - \hat{k}_a\vec{\sigma}\cdot\mathbf{k})(\sigma_b - \hat{k}_b\vec{\sigma}\cdot\mathbf{k}) = 2(\delta_{ab} - k_a k_b).\end{aligned} \tag{3.7}$$

Let us decompose the contribution to $\mathcal{M}$ in four terms classifying along FFs:
1) - $|G_M|^2$:

$$\begin{aligned}&\frac{1}{2}Tr(\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})(\sigma_b - p_b\sigma\cdot p) =\\ &\delta_{ab} - \sigma_a p_a p_b\vec{\sigma}\cdot\mathbf{p} - p_a\vec{\sigma}\cdot\mathbf{p}\sigma_b + p_a p_b\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\mathbf{p} = \delta_{ab} - p_a p_b.\end{aligned} \tag{3.8}$$

Therefore $|G_M|^2$ contributes to the cross section with:

$$(\delta_{ab} - p_a p_b)(\delta_{ab} - k_a k_b) = \delta_{ab}\delta_{ab} - p^2 - k^2 - (\mathbf{p}\cdot\mathbf{k}) = 3 - 1 - 1 + \cos^2\tilde{\theta}. \tag{3.9}$$

2) - The term $G_E G_M^*$ vanishes:

$$\frac{1}{2}Tr(p_a\vec{\sigma}\cdot\mathbf{p}\sigma_b - p_a p_b\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\mathbf{p}) = \frac{1}{2}(p_a p_b - p_a p_b) = 0. \tag{3.10}$$

3) - The term $G_M G_E^*$ similarly vanishes:

$$\frac{1}{\tau}p_a\vec{\sigma}\cdot\mathbf{p}(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p}). \tag{3.11}$$

This shows that no interference term will be present in the cross section.
4) - $|G_E|^2$:

$$(\sigma_a - p_a\sigma\cdot p)(\sigma_b - p_b\sigma\cdot p) = \frac{1}{\sqrt{\tau}}\vec{\sigma}\cdot\mathbf{p}\frac{1}{\sqrt{\tau}}\vec{\sigma}\cdot\mathbf{p} = \frac{1}{\sqrt{\tau}}p_a p_b \tag{3.12}$$

Therefore $|G_E|^2$ contributes to the cross section with:

$$\frac{1}{\sqrt{\tau}}p_a p_b(\delta_{ab} - k_a k_b) = \frac{1}{\sqrt{\tau}}[1-(\mathbf{p}\cdot\mathbf{k})^2] = \frac{1}{\tau}(1-\cos^2\tilde{\theta}) = \frac{1}{\tau}\sin^2\tilde{\theta}. \tag{3.13}$$

We took into account the properties of σ matrices: $\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\mathbf{p} = p^2 = 1$, and $Tr\vec{\sigma}\cdot\vec{a}\vec{\sigma}\cdot\vec{b}\vec{\sigma}\cdot\vec{c} = i\vec{a}\cdot\vec{b}\times\vec{c}$.

Using the expressions (3.5) and (3.6), the formula for the cross section in CMS is:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \mathcal{N}\left[(1+\cos^2\tilde{\theta})|G_M|^2 + \frac{1}{\tau}\sin^2\tilde{\theta}|G_E|^2\right], \tag{3.14}$$

where $\mathcal{N} = \dfrac{\alpha^2}{4\sqrt{q^2(q^2-4M^2)}}$, $\alpha = e^2/(4\pi) \simeq 1/137$, is a kinematical factor. This formula was firstly obtained in Ref. [9]. Note that the normalization factor is inessential for the calculation of the polarization phenomena.

The angular dependence of the cross section, Eq. (3.14), results directly from the assumption of one-photon exchange, where the photon has spin 1 and the electromagnetic hadron interaction satisfies the $P-$invariance. One can prove that this very specific $\cot^2\dfrac{\theta_e}{2}$-dependence of the scattering cross section for eN-scattering is directly related to the $\cos^2\tilde{\theta}$ annihilation cross section by (**Exercise1**):

$$\cos^2\tilde{\theta} = \frac{\cot^2\theta_e/2}{1+\tau} + 1 \tag{3.15}$$

Therefore, the measurement of the differential cross section at three angles (or more) would also allow to test the presence of 2γ exchange.

The electric and the magnetic FFs are weighted by different angular termss in the cross section, Eq. (3.14). One can define an angular asymmetry, $\mathcal{R}$, with respect to the differential cross section measured at ($\tilde{\theta} = \pi/2$:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \sigma(\tilde{\theta} = \pi/2)\left(1 + \mathcal{R}\cos^2\tilde{\theta}\right), \tag{3.16}$$

where $\mathcal{R}$ can be expressed as a function of FFs:

$$\mathcal{R}=\frac{\tau|G_M|^2-|G_E|^2}{\tau|G_M|^2+|G_E|^2},\ \sigma(\tilde{\theta}=\pi/2)=\mathcal{N}\left(|G_M|^2+\frac{1}{\tau}|G_E|^2\right). \tag{3.17}$$

This observable is very sensitive to the different underlying assumptions on FFs, therefore, a precise measurement of this quantity, which does not require polarized particles, would be very interesting. A deviation of the differential cross section from a linearity in $\cos^2\theta_e$ would be the signature of mechanisms beyond one photon exchange (similarly to a deviation form linearity in the Rosenbluth plot).

The q^2 dependence of the total cross section can be presented as follows:

$$\sigma(q^2)=\mathcal{N}\frac{8}{3}\pi\left(2|G_M|^2+\frac{1}{\tau}|G_E|^2\right). \tag{3.18}$$

3.1.3 Polarization observables

Polarization phenomena will be especially important in $\bar{p}+p\to\ell^++\ell^-$, as FFs are complex. One spin polarization allows to access the relative phase.

The dependence of the cross section on the polarizations $\vec{P}_1$ and $\vec{P}_2$ of the colliding antiproton and proton can be written as follows:

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)(\vec{P}_1,\vec{P}_2) &= \left(\frac{d\sigma}{d\Omega}\right)_0[1+A_y(P_{1y}+P_{2y})+\\ &A_{xx}P_{1x}P_{2x}+A_{yy}P_{1y}P_{2y}+\\ &A_{zz}P_{1z}P_{2z}+A_{xz}(P_{1x}P_{2z}+P_{1z}P_{2x})],\end{aligned} \tag{3.19}$$

where the coefficients A_i and A_{ij} $(i,j=x,y,z)$, analyzing powers and correlation coefficients, depend on the nucleon FFs. Their explicit form is given below. The dependence (3.19) results from the P-invariance of hadron electrodynamics. The polarized hadronic tensor reads:

$$W_{ab}(\vec{P}_1,\vec{P}_2)=\frac{1}{2}TrJ_a\vec{\sigma}\cdot\vec{P}_1J_b^*\vec{\sigma}\cdot\vec{P}_2,$$

and the cross section with unpolarized electrons is proportional to $L_{ab}\overline{W_{ab}}$.

3.1.4 Single spin polarization observables

In case of polarized antiproton beam with polarization $\vec{P}_1$, the contribution to the cross section can be calculated as:

$$\begin{aligned}&\left(\frac{d\sigma}{d\Omega}\right)_0\vec{A}_1\sim -L_{ab}\frac{1}{4}TrJ_a\vec{\sigma}J_b^*=\\ &[(\sigma_a-p_a\vec{\sigma}\cdot\mathbf{p})G_M+\frac{1}{\tau}G_Ep_a\vec{\sigma}\cdot\mathbf{p}](-\vec{\sigma}\cdot\vec{P}_1)\\ &[(\sigma_b-p_b\vec{\sigma}\cdot\mathbf{p})G_M^*+\frac{1}{\tau}G_E^*p_b\vec{\sigma}\cdot\mathbf{p}](\delta_{ab}-k_aK_b).\end{aligned} \tag{3.20}$$

1. The term in $|G_M|^2$:

$$[1]: \ (\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p})\delta_{ab} - \tag{3.21}$$
$$[2]: \ (\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p})\hat{k}_a\hat{k}_b. \tag{3.22}$$

The first contribution (3.21) reduces to:

$$\begin{aligned}[1]: &\quad \sigma_a\vec{\sigma}\cdot\vec{P}_1\sigma_a - \sigma_a\vec{\sigma}\cdot\vec{P}_1 p_a\vec{\sigma}\cdot\mathbf{p} - p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\sigma_a + p_a^2\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p} \\ = &\quad -p_a(a\cdot P_1\times\mathbf{p} + p\cdot P_1\times\vec{a}) + p_a^2\vec{\sigma}\cdot\vec{P}_1 = 0.\end{aligned}$$

The second contribution (3.22) becomes:

$$\begin{aligned}[2]: \quad & (\vec{\sigma}\cdot\mathbf{k} - \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1(\vec{\sigma}\cdot\mathbf{k} - \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}) \\ & \vec{\sigma}\cdot\mathbf{k}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{k} - \vec{\sigma}\cdot\mathbf{k}\vec{\sigma}\cdot\vec{P}_1\mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p} - \\ & \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{k} + (\mathbf{p}\cdot\mathbf{k})^2\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p} \\ & = -\cos\tilde{\theta}(\sigma\cdot\mathbf{k}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p} + \sigma\cdot\mathbf{p}\sigma\cdot\vec{P}_1\sigma\cdot\mathbf{k} \\ & = -\cos\tilde{\theta}[(\mathbf{k}\cdot\vec{P}_1\times\mathbf{p} + \mathbf{p}\cdot\vec{P}_1\times\mathbf{k}] = 0\end{aligned}$$

due to the antisymmetric terms in first parenthesis and the fact that the σ matrices have zero trace.

2. The term $|G_E|^2$:

$$\frac{1}{\tau}\left[p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1 p_a\vec{\sigma}\cdot\mathbf{p} - (\mathbf{p}\cdot\mathbf{k})^2\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p}\right] = 0.$$

3. The term $G_M G_E^*$

$$\begin{aligned} & \frac{1}{2}Tr\frac{1}{\tau}[(\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1 p_b\vec{\sigma}\cdot\mathbf{p}](\delta_{ab} - k_a k_b) \\ = & \frac{1}{\tau}[(\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1 p_a\vec{\sigma}\cdot\mathbf{p} - (\vec{\sigma}\cdot\mathbf{k} - \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}].\end{aligned} \tag{3.23}$$

Let us decompose explicitly the components:

$$\begin{aligned} & \frac{1}{\tau}[(\sigma_x\vec{\sigma}\cdot\vec{P}_1 p_x\sigma_z + \sigma_y\vec{\sigma}\cdot\vec{P}_1 p_y\sigma_z) \\ & -(\sigma_x\sin\tilde{\theta} + \sigma_z\cos\tilde{\theta} - \sigma_z\cos\tilde{\theta})\vec{\sigma}\cdot\vec{P}_1\cos\tilde{\theta}\sigma_z] \\ = & -\sigma_x\sin\tilde{\theta}\cos\tilde{\theta}\vec{\sigma}\cdot\vec{P}_1\sigma_z = -i\sin\tilde{\theta}\cos\tilde{\theta}P_{1y},\end{aligned}$$

$$\boxed{G_M G_E^* \to -i\sin\tilde{\theta}\cos\tilde{\theta}P_{1y}}$$

4. Similarly for the term in $G_E G_M^*$ one finds:

$$\begin{aligned} & [p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p})](\delta_{ab} - k_a k_b) \\ = & [p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}_a - p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1 p_a\vec{\sigma}\cdot\mathbf{p} - \\ & \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{k} - (\mathbf{p}\cdot\mathbf{k})^2\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p}] \\ = & i[p_a\vec{a}\cdot\mathbf{p}\times\vec{P}_1 - \cos\tilde{\theta}\mathbf{p}\cdot\vec{P}_1\times\mathbf{k}].\end{aligned}$$

Let us calculate the mixte product:

$$\vec{a} \cdot \mathbf{p} \times \vec{P}_1 \to p_x = p_y = 0; zp_z \times P_1 = 0.$$

More explicitly:

$$\left(\begin{array}{c|ccc} p & 0 & 0 & 1 \\ P & P_{1x} & P_{1y} & P_{1z} \\ k & \sin\tilde{\theta} & 0 & \cos\tilde{\theta} \end{array} \right)$$

$$\boxed{G_E G_M^* \to \frac{i}{\sqrt{\tau}} \cos\tilde{\theta} \sin\tilde{\theta} P_{1y}}$$

In the calculation of the single spin polarization the terms related to $|G_E|^2$ and $|G_M|^2$ vanish. We add a global sign as the term for polarization of an antiparticle contains a "-" sign: $-\vec{\sigma} \cdot \mathbf{p}$.

For the interference terms, the only non zero analyzing power is related to the normal polarization P_y:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 A_{1,y} = -\frac{i\mathcal{N}}{\sqrt{\tau}} \sin\tilde{\theta} \cos\tilde{\theta} [G_M G_E^* - G_E G_M^*] = \frac{\mathcal{N}}{\sqrt{\tau}} \sin 2\tilde{\theta} Im(G_M G_E^*). \qquad (3.24)$$

Other observables can be obtained with some algebra in similar way. When the target is polarized, one writes:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 \vec{A}_2 = L_{ab} \frac{1}{4} Tr J_a J_b^* \vec{\sigma}.$$

Again the terms related to $|G_E|^2$ and $|G_M|^2$ vanish. Moreover, one can find $\vec{A}_2 = \vec{A}_1 = \vec{A}$.

Eq. (3.24) has been proved also in Ref. [9]. One can see that this analyzing power, being T-odd, does not vanish in $\bar{p} + p \to \ell^+ + \ell^-$, even in one-photon approximation, due to the fact FFs are complex in time-like region. This is a principal difference with elastic *ep* scattering. Let us note also that the assumption $G_E = G_M$ implies $A_y = 0$, independently from any model taken for the calculation of FFs.

3.1.5 Double spin polarization observables

The contribution to the cross section, when both colliding particles are polarized is calculated through the following expression:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 A_{ab} = -\frac{1}{4} L_{mn} Tr J_m \sigma_a J_n^\dagger \sigma_b,$$

where a and $b = x, y, z$ refer to the $a(b)$ component of the projectile (target) polarization. Among the nine possible terms, $A_{xy} = A_{yx} = A_{zy} = A_{yz} = 0$, and the nonzero components are:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 A_{xx} = \sin^2\tilde{\theta}\left(|G_M|^2+\frac{1}{\tau}|G_E|^2\right)\mathcal{N},$$
$$\left(\frac{d\sigma}{d\Omega}\right)_0 A_{yy} = -\sin^2\tilde{\theta}\left(|G_M|^2-\frac{1}{\tau}|G_E|^2\right)\mathcal{N},$$
$$\left(\frac{d\sigma}{d\Omega}\right)_0 A_{zz} = \left[(1+\cos^2\tilde{\theta})|G_M|^2-\frac{1}{\tau}\sin^2\tilde{\theta}|G_E|^2\right]\mathcal{N},$$
$$\left(\frac{d\sigma}{d\Omega}\right)_0 A_{xz} = \left(\frac{d\sigma}{d\Omega}\right)_0 A_{zx}=\frac{1}{\sqrt{\tau}}\sin 2\tilde{\theta} Re G_E G_M^*\mathcal{N}. \tag{3.25}$$

One can see that the double spin observables depend on the moduli squared of FFs, except A_{xz} (A_{zx}). Therefore, in order to determine the relative phase of FFs, in TL region, the interesting observables are A_y, and A_{xz}, which contain respectively the imaginary and the real part of the product $G_E G_M^*$.

Exercise II.1: Crossing symmetry

Derive the specific kinematical relation that shows in a natural way the origin of the angular dependence of the differential cross section for the scattering and annihilation elementary reactions.

$$\cos^2\tilde{\theta}=\frac{\cot^2\tilde{\theta}/2}{1+\tau}+1 \tag{3.26}$$

where $\tilde{\theta}$ is the laboratory scattering angle of the electron in elastic ep scattering and $\tilde{theta}$ is the CMS angle of the antiproton produced in the annihilation: $e^-+e^+\to\bar{p}+p$ with respect to the beam direction. Hints:

1)Define the reactions and particle momenta as.

$$e^-(k_1)+p(p_1) \rightarrow e^-(k_2)+p(p_2)$$
$$e^-(k_1)+e^+(-k2) \rightarrow \bar{p}(-p_1)+p(p_2)$$

2) express the considered angles in terms of the relativistic invariants s and t.

3) prove that

$$(1)=\frac{(s-M^2)^2+ts}{t\left(\frac{t}{4}-M^2\right)}+1$$

M is the proton mass, $\tau=-t/(4M^2)$.

Chapter 4

Experimental status

Since the years 2000, the Akhiezer-Rekalo polarization method was preferentially applied for the extraction of the electromagnetic FFs, thanks to the availability of the high intensity electron beams, the very performant detection with large solid angle spectrometers equipped by hadron polarimeters, and/or polarized targets. In particular, the progress driven by dedicated polarimetry in the GeV region increased essentially the performances of the experiments at Jefferson Lab.

4.1 Experiments in the scattering region

4.1.1 Polarimetry

A particle of spin S has $2S+1$ quantified values of the spin projection on a quantization axis. A proton and a neutron have spin $S = 1/2$ and can be in two states: up ($\uparrow$) and down ($\downarrow$). A beam of protons and neutrons is not polarized when the two spin directions are equally probable. The vector polarization of a proton beam, say, along the vertical (y) axis (it is the most common orientation, as it is the direction of the magnetic field in an accelerator) is defined as:

$$P_y = \frac{N(\uparrow) - N(\downarrow)}{N(\uparrow) + N(\downarrow)}.$$

The reaction products are emitted with cylindrical symmetry around the only defined direction, the beam axis. If the beam is polarized, not only the beam direction, but also the direction related to the polarization affects the emitted particles and modifies the differential cross section, creating a left-right or up-down asymmetry.

The working principle of a polarization measurement is the precise determination of the azimuthal asymmetry of the emitted proton(or neutron) that interacts on a secondary (the polarimeter) target.

Therefore, polarization measurements are lengthy, requiring a secondary scattering. Therefore, for each experimental conditions, namely the kinematics, careful studies should be done to find the best configuration, minimizing the beam time.

The precision on the track reconstruction is obtained after a careful alignment of the tracking detectors. Typically a mechanical alignment at the level of 1 mm is obtained with help of a laser, but it is possible to achieve at least ten times better precision by a software calibration under beam, using the particles that do not interact.

The optimization of a polarimeter is based not only on the detection efficiency and acceptance, but also on the choice of the secondary reaction, which should be: 1) simple to detect (minimizing the complexity of the detection system and of the data analysis), 2) have a large cross section (minimizing statistical errors) and 3) large analyzing power (minimizing systematic errors).

A polarized proton beam of known polarization interacts with a thick target, ideally with rich hydrogen content, and one charged particle is detected in the final state. This is the calibration step, that allows to determine analyzing powers, and compare different geometries and choices of material. Once that the polarimeter is optimized, it can be used to measure the polarization of incident particles with unknown polarization.

The number of particles emitted in a solid angle around a direction defined by the angles (θ, φ) is:

$$N^{\pm}(\theta, \varphi) = N_0(\theta, \varphi)[1 \pm P_y A_y(\theta) \cos\varphi],$$

(where $\pm$ refers to the polarization state of the beam). Then, the asymmetry $a(\theta) = P_y A_y(\theta)$ is extracted from the ratio:

$$R(\theta, \varphi) = \frac{N^+(\theta, \varphi) - N^-(\theta, \varphi)}{N^+(\theta, \varphi) + N^-(\theta, \varphi)} = a(\theta) \cos\varphi.$$

The particle polarization and the analyzing power of the reaction play a symmetric role in the asymmetry. Knowing the beam polarization allows to extract the analyzing power (calibration); knowing the analyzing power allows to extract the beam polarization (experiment):

$$A_y(\theta) = \frac{a(\theta)}{P_y}, \ or \ P_y = \frac{a(\theta)}{A_y(\theta)}.$$

Quantitatively, the performance of a polarimeter is quantified by the Figure of Merit, F:

$$F^2 = \sum_\theta \epsilon(\theta) A_y^2(\theta)$$

where the differential efficiency ϵ is defined as the useful fraction of events in a bin θ, $N_{useful}(\theta)$: $\epsilon(\theta) = \frac{N_{useful}(\theta)}{N_{incident}}$. The error on the measured polarization ΔP_y (or on the analyzing power) is related to the figure of merit by:

$$\Delta P_y = \sqrt{\frac{2}{N_{incident} F^2}}.$$

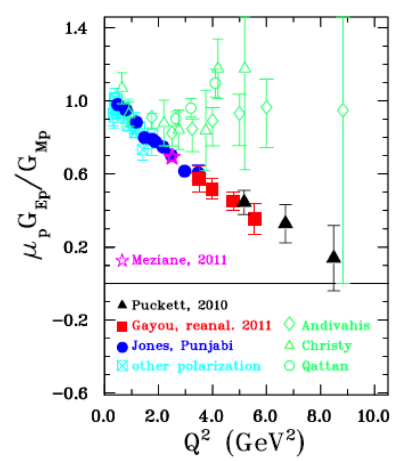
Systematic studies were performed with polarized hadron beams at Saturne National Laboratory in France [29] and at the Laboratory of High Energies, in Dubna (JINR-VBLHEP) [30].

Different configurations were used during the years, thin targets [31], leaving the place to thicker hydrogen rich CH_2 targets [32] . A larger liquid hydrogen target was also used to measure the deuteron (vector and tensor) polarization up to 2 GeV [33]. The polarimeter is usually set in the focal plane of a spectrometer. The scattered particle from the primary reaction, which polarization has to be measured, is magnetically analyzed by the spectrometer. Its track is reconstructed by the focal plane detection of the spectrometer. The particle then scatters on the polarimeter target and a larger detection gives the outgoing track. The angles (θ, φ) can then be calculated knowing the two directions of the incident and outgoing track from the polarimeter target.

4.1.2 Polarization experiments in ep scattering

Form factor measurements using the polarization method were carried on since the years 2000 at the Jefferson Laboratory (JLab), situated in Newport News,Virginia, mostly by the GEp collaboration (see [8] and references therein). A high intensity electron beam is available, at first at 3.5 GeV energy. Successive upgrades during the years brought the energy to 11 GeV. The feasibility of the experiments at the different energies had to be demonstrated beforehand by experiments at other accelerators with polarized hadron beams. The collected data are shown in Fig. 4.1.

Figure 4.1: Ratio of the electric to the magnetic proton form factor, as a function of the transferred momentum, Q^2. Different colors refer to different polarization experiments. The green points are (some of) the data obtained with the Rosenbluth method.



One can see that the data on the ratio of the electric to magnetic FF obtained with the polarization method (different colors) have a better precision in comparison to the data obtained with the Rosenbluth method (green symbols), as expected. The surprising issue is that the electric charge and magnetic distribution inside the proton have a very different dependence with Q^2, contrary to what was previously commonly assumed. These data raised several questions and number of theoretical and experimental papers. In particular a constant ratio, compatible with the dipole approximation for both FFs, was compatible with scaling laws in frame of PQCD [34, 35]. A dipole approximation is also obtained by a Fourier Transform of an exponential charge density (See Table 1.1). The faster decrease of G_E (it is assumed that G_M is well measured by the unpolarized data, as the magnetic contribution to the unpolarized cross section is $> 90\%$ for $Q^2 \leq 2$ GeV^2) is not uniquely explained.

Some nucleon models (vector meson dominance [10], soliton models [36]) predicted indeed the decrease of this ratio. But very few models can reproduce all FFs: electric, magnetic, for proton and neutron in both space and time-like regions. FFs derived in frame of VDM contain the necessary properties to be analytically continued in the time-like region [37]. This model is based the assumption that the three bare quarks are concentrated in a small volume inside the proton and are surrounded by a cloud of vector mesons. More recently, a model was proposed where the suppression of G_E is interpreted as the evidence of the existence of a color neutral region inside the proton, with quantum vacuum properties [38].

Two other issues, related to these data, give rise to various interpretations and different explanations. One is the difference between polarized and unpolarized data: the Rosenbluth and the Akhiezer-Rekalo methods are based on the same formalism and assume the exchange of one virtual photon. No shortcoming were found in the experiments and data analysis. It was suggested that this difference might arise from the importance of two photon exchange. This mechanism induces additional amplitudes, that are complex function of two variables and prevents the extraction of FFs[1], that are real functions of one variable, q^2. New experiments were performed searching for this effect. It seems more probable that other explanations, such as correlations of the Rosenbluth parameters - that prevent the extraction of an electric component of size comparable or smaller than the error of the cross section- or the applied radiative corrections - that are calculated at first order and not taken accurately into account (see Ref. [40] and references therein).

The decrease of the ratio could lead to a zero of the electric form factor and even to a change of sign. This has to be confirmed by future experiments that are already planned at JLab. The proposal E12-07-109 in JLab, Hall A, will extend the measurement to 15 GeV^2, requiring a large acceptance lead glass calorimeter to detect the electrons [41]. The polarization of the emitted proton, after momentum analysis in the SuperBigBite spectrometer, will be measured by a double polarimeter, with two large CH_2 targets. A hadron calorimeter will be added downstream to select high energy events.

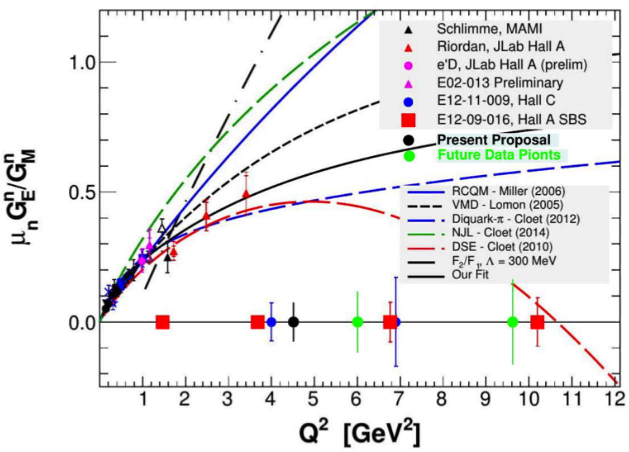
Let us briefly mention the situation with neutron FFs. The electric FF is small, and was assumed to be zero, while the magnetic form factor does follow the dipole behavior.

Polarization measurements are possible in Mainz and JLab, with longitudinally polarized electron beams, using a polarized target, or measuring the polarization of the outgoing neutron. It turns out that the neutron electric FF is measurable, and differs from zero. As a neutron target does not exist, (polarized) deuteron or ^3He targets are used, implying corrections due to the interaction of the neutron in these bound systems.

[1] More exactly, one can prove that it is always possible to extract proton FFs, also in presence of two photon exchange, but this would require the measurement of three time-odd or five time-even polarization observables, including triple polarizations, very difficult to measure as they are of the order of percent (see, for example, Ref. [39]).

The experimental data are shown in Fig. 4.2, indicating an increase of the neutron electric FF with Q^2. Also shown the expected data from the planned experiments as well as several model predictions. This is an example of the dispersion of the theories when they are not constrained by the data.

Figure 4.2: Electric neutron form factor, as a function of the transferred momentum, Q^2. Selected data, obtained with the Akhiezer-Rekalo polarization method are shown. The expected error of future measurements is set at the corresponding Q^2, along the axis. Some of model predictions are also shown.



Some references of future experiments for further reading are reported below:

- E12-07-108: measurement of G_{Mp} from the elastic $p(e,e')p$ cross section at 2% level, using the Super Big Byte high Resolution Spectrometer, up to $Q^2 = 16$ (GeV/c)^2, in frame of the SBS program of nucleon electromagnetic FFs measurements [41];

- E12-09-019: measurement of G_{Mn}/G_{Mp} by measuring the ratio $d(e,e'n)/d(e,e'p)$ up to $Q^2 = 18$ (GeV/c)^2;

- E12-09-016: measurement of G_{En}/G_{Mn} with polarized beam & ^3He target [42];

- E12-07-109: measurement of G_{Ep}/G_{Mp} with polarized beam & recoil proton polarimetry up to $Q^2 = 14.5$ (GeV/c)^2 [41];

- E12-17-004: measurement of G_{En}/G_{Mn} with polarized beam & recoil neutron polarimetry [43].

4.2 Experiments in the annihilation region

FFs are accessible in the time-like region through collider experiments. The luminosity of e^+e^- colliders as DAPHNE@Frascati, VEPP@Novosibirsk, LAL@Orsay as well as $\bar{p}p$ colliders at LEAR@CERN and FERMILab@Chicago was sufficient to extract the annihilitation cross section. From the total cross section, the effective FF was derived, but precise angular distributions, that give access to the individual FFs were not available. First attempts to extract the FF ratio can be found in Refs. [11, 44], but more recently the collaboration BABAR@SLAC, using initial state radiation (*i.e.*, the process $e^+ + e^- \to \bar{p} + p + \gamma$) opened the way to precise measurements in a large kinematical range [45, 46, 15], followed by BESIII@BEPCII, in Beijing [47, 27]. The world data on the effective FF are shown in Fig. 4.3, togeher with phenomenological fits from Ref. [48]. One can see that the best fitting function is based on the following form:

$$|F(q^2)| \;=\; \frac{\mathcal{A}}{(1+q^2/m_a^2)\,[1-q^2/0.71]^2}, \;\mathcal{A}=7.7\ \mathrm{GeV}^{-4},\;\; m_a^2=14.8\ \mathrm{GeV}^2, \qquad (4.1)$$

which is a modification of the dipole function, suggesting a faster decrease of FFs, similarly to electric FF in the space-like region. From the insert, one can also notice

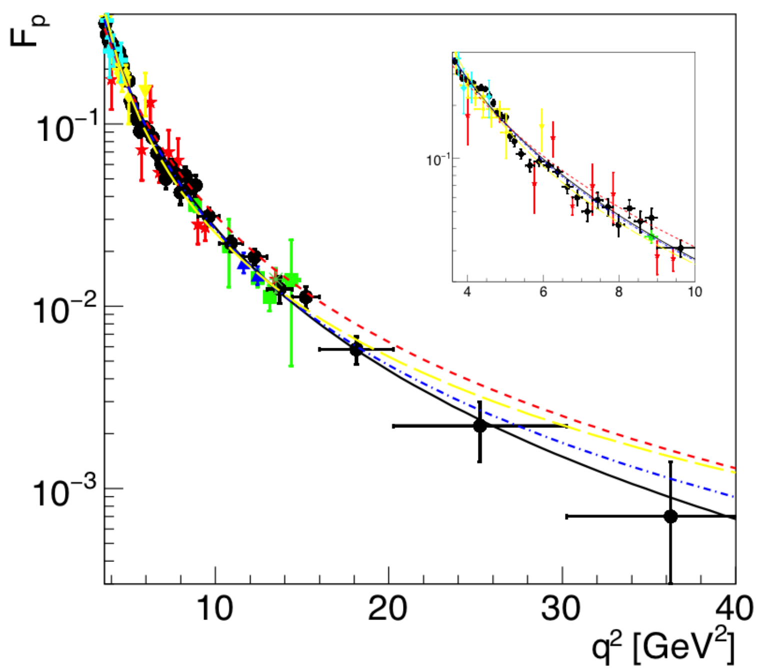


Figure 4.3: World data on the TL proton generalized FF as a function of q^2, together with the calculation pQCD (blue dash-dotted line), pQCD modified (red dashed line), and dipole modified (black solid line), see Ref. [48]. The insert magnifies the near threshold region.

some structures, that become regular and equally spaced if plotted as a function of the relative momentum of the outgoing hadrons. In Ref. [48] this was interpreted as an interference phenomenon in spacetime, with competition between processes involving well separated regions with different properties and could be related to the time of the $q\bar{q}$ formation from the quantum vacuum.

The individual determination of electric and magnetic FFs can be done only through the precise measurement of the angular distribution of one of the emitted particles and was achieved only recently [27]. The results are shown in Fig. 4.4. The results confirm the observed oscillatory behavior of the effective FF. The results show also that G_E is larger up to 1.5 times G_M, in the near threshold region, becoming of similar size for $Q^2 \geq 2.5$ GeV^2. For symmetry reasons $G_E = G_M$ at threshold.

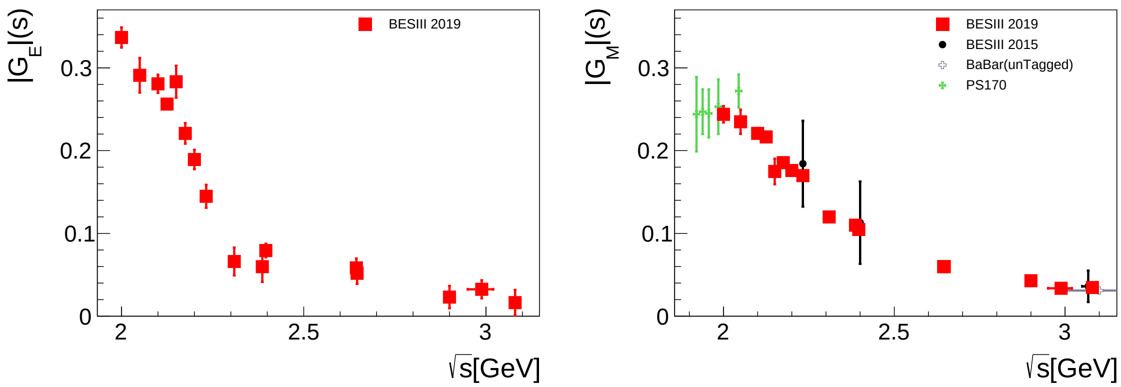


Figure 4.4: First measurement of electric and magnetic FF, in the time-like region from Ref. [27].

4.3 Towards a unified interpretation of form factors

Based on the analytical properties of FFs, attempts to look for a unified description of these functions in all kinematical domain were suggested comparing different models in Ref. [49], in frame of VDM [50, 51, 52] and in frame of PQCD at large momenta [53].

PQCD can not calculate the cross section for exclusive processes, but gives scaling laws where the Q^2 dependence of nucleon FFs is $\simeq (q^2)^2$, and in general, is driven by the number of constituents participating to the process [34, 35]. In order to explain the faster decrease of the electric FF, in Ref. [38] it was suggested a generalization of the definition of FFs:

$$F(q^2) = \int_{\mathcal{D}} d^4x e^{iq_\mu x^\mu} \rho(x),\ q_\mu x^\mu = q_0 t - \vec{q} \cdot \vec{x} \tag{4.2}$$

where $\rho(x) = \rho(\vec{x}, t)$ is the space-time distribution of the electric charge in a space-time volume $\mathcal{D}$. In the scattering channel, $e+p \to e+p$ and in the Breit frame, one recovers the usual definition of FFs:

$$F(q^2) = \delta(q_0) F(Q^2),\ Q^2 = -(q_0^2 - \vec{q}^2) > 0, \tag{4.3}$$

where zero energy transfer is implied. In the annihilation channel, one has:

$$F(q^2) = \int_{\mathcal{D}} dt e^{i\sqrt{q^2}t} \int d^3\vec{r} \rho(\vec{r}, t) = \int_{\mathcal{D}} dt e^{i\sqrt{q^2}t} \mathcal{Q}(t), \tag{4.4}$$

where $\mathcal{Q}(t)$ describes the time evolution of the charge distribution in the domain $\mathcal{D}$.

It is usually assumed that the nucleon is an antisymmetric state of colored quarks. The proton is considered as a system of three valence quark and of a neutral sea consisting of gluons and quark-antiquark pairs.

The main idea of Ref. [38] is that, contrary to VDM where the three valence quarks are concentrated in a small volume at short relative distances, the spatial center of the nucleon (proton and neutron) is electrically neutral. In the inner region of the nucleon a strong gluonic field may create a gluonic condensate of clusters with a randomly oriented chromo-magnetic field. At smaller distances the gluonic field as well as the number of gluons increases. Therefore, the strength of the chromo-electric field increases too: in the region of strong chromo-magnetic field the color quantum number of quarks does not play any role, due to stochastic averaging. Then, due to Pauli principle, uu (or dd) quarks are pushed outside the internal region of the proton (or the neutron). The third quark is attracted by one of the identical quarks and forms a compact diquark. In the regions of not so intense gluonic field the color state of quarks is restored. The creation of a quark-diquark dipole system occurs when the attraction force exceeds the stochastic force of the gluon field.

Quark counting rules apply to the vector part of the potential, and derive from the interaction of the virtual photon with three independent quarks. The present model leaves unchanged the prediction for the magnetic FF. An additional suppression mechanism of the electric FF is provided by the 'central' region of the hadron. A similar effect to the screening of a charge in plasma may take place.

The distribution in momentum space, applying the Fourier transform, shows that the central region provides an additional suppression for the electric FF:

$$G_E(Q^2) = \frac{G_M(Q^2)}{\mu_p}\left(1 + Q^2/q_1^2\right)^{-1} \tag{4.5}$$

with

$$G_M(Q^2) = G_D(Q^2) = \left[1 + Q^2/(0.71\ \mathrm{GeV}^2)\right]^{-2}. \tag{4.6}$$

q_1 should be considered as a fitting parameter, different in principle for proton and neutrons. This induces the observed monopole decreasing for the form factor ratio, $R_{p,n}(Q^2)$ in SL region. A similar picture can be applied in the time-like region, and the functions $\rho(x)$ and $\mathcal{Q}(t)$ should be understood as the projection of the generalized FF, Eq. (4.2) in the space and time coordinates.

In Fig. 4.5 we report the results of this model in the SL region as a green line for G_E, a blue line for G_M and as a black line in the TL region (where the data are extracted under the assumption that $G_E = G_M$), while the orange line represents a VDM based fit. These models, that contain a small number of parameters having a definite physical meaning, may reproduce the main trends of the FFs, in the whole region. In future, efforts will be focussed on such global descriptions of FFs data and

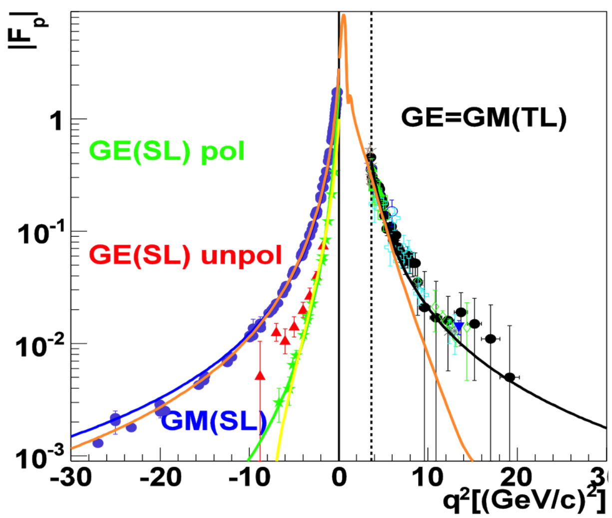


Figure 4.5: World data on proton FFs as function of q^2. **Space-like region:** $G_M(Q^2)$ data (blue circles), dipole function (blue line) from Eq. (4.6); electric FF, $G_E(Q^2)$, from unpolarized measurements (red triangles) and from polarization measurements (green stars). The green line is the model prediction from Eq. (4.5). **Time-like region:** world data for $|G_E| = |G_M|$ (various symbols for $q^2 > 4M_p^2$ and model prediction (black line) from Eq. (4.6). The orange line is a VDM based fit prediction.

attention will be payed to those models that contain the necessary properties to be applied in the whole kinematical region.

4.4 Conclusion

We have given here a formal derivation of unpolarized cross section and polarization observables for the case of ep elastic scattering in the Breit system and $\bar{p}p$ annihilation into a (massless) lepton pair in CM system, where the calculation is simplified.

The results are formulated as model independent expressions of (polarized and unpolarized) experimental observables as functions of FFs. they hold at any energy, under the assumption of one photon exchange mechanism and obeyi to the symmetries and conservation laws of the electromagnetic and strong interactions.

Polarization observables play an important role as they contain the interference of FFs, whereas only the (moduli) squared FFs contribute to the unpolarized cross section. The possibility to accelerate polarized antiproton beams, or extend the use of polarized targets in colliders motivate a series of experiments. Precise data will strongly constrain nucleon models. Several experiments are planned or ongoing at electron accelerators as JLab, Mainz and colliders as VEPP (Novosibirsk), BES (Beijing), and Panda at FAIR (Darmstadt). In SL region, the main purpose is to reach higher transferred momenta or better precisions. In TL region the individual determination of the electric and magnetic FFs at least in the region over threshold has been recently possible. Forthcoming results will be also ' a premiére' in case of neutron. The measurements at the highest possible momentum transfer will allow to study asymptotic properties, where predictions exist from QCD and analyticity.

The modelisation of the nucleon structure is formulated in different possible parametrizations of FFs. Several models have been developed in the recent years. In future, the interest will be focused on those models which can describe coherently all four nucleon FFs, proton and neutron, electric and magnetic, in SL and TL regions. The bridge between meson and nucleon FFs is not straightforward by now. We have highlighted an attempt of generalizing the definition of form factors, in order to understand more deeply the nucleon electromagnetic structure and the mechanism of the creation of matter from the dynamics of annihilation and scattering reactions. We did not focus the lectures on nucleon models, as several model of the nucleon exist (for a review, see [1]) based on different, some times contradicting assumptions on the nucleon structure.

Further constrains will be given by the forthcoming data, with increased precision and in a wider kinematical range, for protons and neutrons. Other hadron FFs as strange or charm hyperons (where one light quark is replaced by a strange or charmed quark) are also of great interest. In particular the Λ polarization can be measured through its weak decay, giving access to the relative phase of FFs in the time-like region [54].

4.5 Corrections of Exercises

Introduction Exercise 1

Verify the results of Table 1.1.

As an example, let us calculate the radius corresponding to an exponential charge density, $\rho(x) = e^{-ax}$. First, we recall the following integrals:

$$\Gamma(\frac{1}{2}) = \int_0^\infty dz e^{-z} z^{-1/2} = \sqrt{\pi}, \tag{4.7}$$

$$\Gamma(x) = \int_0^\infty dz e^{-z} z^{x-1},\ \Gamma(x+1) = x\Gamma(x), \tag{4.8}$$

$$n! = \int_0^\infty dx x^n e^{-x} dx. \tag{4.9}$$

The radius is given by:

$$< r_c^2 >= \frac{\int_0^\infty x^4 e^{-ax} dx}{\int_0^\infty x^2 e^{-ax} dx} = \frac{a^{-5}\int_0^\infty (ax)^4 e^{-ax} d(ax)}{a^{-3}\int_0^\infty (ax) e^{-ax} d(ax)}$$

and the form factor:

$$F(q) = \frac{\displaystyle\frac{1}{q}\int_0^\infty x\sin(qx)e^{-ax}dx}{\int_0^\infty x^2 e^{-ax}dx}. \tag{4.10}$$

Applying (4.9), the denominator in Eq. (4.10) is:

$$D = \int_0^\infty x^2 e^{-ax}dx = \frac{2}{a^3}.$$

The numerator:

$$\begin{aligned} N &= \frac{1}{2iq}\int_0^\infty x(e^{iqx} - e^{-iqx})e^{-ax}dx \\ &= \frac{1}{2iq}\int_0^\infty x\left[e^{-(-iq+a)x} - e^{-(iq+a)x}\right]dx \\ &= \frac{1}{2iq}\left[\frac{1}{(a-iq)^2}\int_0^\infty ye^{-y}dy - \frac{1}{(a+iq)^2}\int_0^\infty ye^{-y}dy\right]. \end{aligned}$$

Integrating per parts:

$$\Gamma(2) = \int ye^{-y}dy = -\int yd(e^{-y}) = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y}|_0^\infty = +1.$$

one finds:

$$N = \frac{1}{2iq}\left[\frac{1}{(a-iq)^2} - \frac{1}{(a+iq)^2}\right] = \frac{4aiq}{2iq(a^2+q^2)^2} = \frac{2a}{(a^2+q^2)^2}.$$

Finally:

$$F(q) = \frac{a^4}{(a^2+q^2)^2}.$$

Similarly one can verify all the results of Table 1.1.

Lecture I Exercise 1

Prove the relation $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$, in CMS or Lab system

From the four momentum conservation: $p_1 + p_2 = p_3 + p_4$ and the definition of four momenta as in Table 2.2. In Lab system:

$$s = (p_2 + p_4)^2 = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2E_1 m_2 \tag{4.11}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2E_4 m_2 \tag{4.12}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 = m_2^2 + m_3^2 - 2E_3 m_2 \tag{4.13}$$

Summing on both sides: $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2m_2(m_2 + E_1 - E_3 - E_4)$. From energy conservation the last term vanishes.

Lecture I Exercise 2

Prove the following relation between the electron scattering angles in the Lab system, θ_e and in the Breit system, θ_B:

$$\cot^2\frac{\theta_B}{2}=\frac{\cot^2\theta_e/2}{1+\tau}.$$

As the Breit system is moving along the z-axis, the x and y components of the particle three-momenta do not change after transformation from the Lab to the Breit system:

$$\begin{cases} k_{1B}^y = & k_{2B}^y = 0 \\ k_{1B}^x = & k_{1B}^x. \end{cases} \tag{4.14}$$

From $|\mathbf{k_1}^2| = k_1^{2x} + k_1^{2z}$ one can find:

$$\begin{aligned} k_1^{2x} &= \mathbf{k_1}^2 - \frac{(\mathbf{k_1}\cdot\mathbf{q})^2}{\mathbf{q}^2} = \frac{\mathbf{k_1}^2\mathbf{q}^2-(\mathbf{k_1}\cdot\mathbf{q})^2}{\mathbf{q}^2} \\ &= \frac{\mathbf{k_1}^2(\mathbf{k_1}-\mathbf{k_2})^2-[\mathbf{k_1}\cdot(\mathbf{k_1}-\mathbf{k_2})^2]}{\mathbf{q}^2} \\ &= \frac{(\mathbf{k_1}^2)^2+\mathbf{k_1}^2\mathbf{k_2}^2-2\mathbf{k_1}^2\mathbf{k_1}\cdot\mathbf{k_2}-(\mathbf{k_1}\cdot\mathbf{k_2})^2}{\mathbf{q}^2} \\ &= \frac{\mathbf{k_1}^2\mathbf{k_2}^2(\mathbf{k_1}\cdot\mathbf{k_2})^2}{\mathbf{q}^2} \\ &= \frac{\epsilon_1^2\epsilon_2^2\sin^2\theta_e}{\mathbf{q}^2}\frac{4\epsilon_1^2\epsilon_2^2}{\mathbf{q}^2}\sin^2\frac{\theta_e}{2}\cos^2\frac{\theta_e}{2}, \end{aligned} \tag{4.15}$$

where we replaced $\mathbf{q}=\mathbf{k_1}-\mathbf{k_2}$, $\mathbf{k_1}^2=\epsilon_1^2$, $\mathbf{k_2}^2=\epsilon_2^2$ after setting $m_e=0$. On the other hand we find for the square of the four-vector q^2, the following expression in the Lab system (in terms of the energies of the initial and final electron and of the electron scattering angle):

$$q^2=(k_1-k_2)^2=2m_e^2-2k_1\cdot k_2 \overset{m_e=0}{\simeq} -2\epsilon_1\epsilon_2+2\mathbf{k_1}\cdot\mathbf{k_2}=-2\epsilon_1\epsilon_2(1-\cos\theta_e)=$$

$$-4\epsilon_1\epsilon_2\sin^2\frac{\theta_e}{2}. \tag{4.16}$$

Comparing Eqs. (4.15) and (4.16), we find:

$$k_{1x}^2=\frac{(q^2)^2}{4\mathbf{q}^2}\cot^2\frac{\theta_e}{2}.$$

Using: $\mathbf{q}^2=\omega^2-q^2$ and $q^2+2q\cdot p_1+p_1^2=p_2^2$, we have, in the Lab system, $\omega=-\dfrac{q^2}{2m}$ and $\mathbf{q}^2=-q^2(1+\tau)$. Finally:

$$k_{1x}^2=-\frac{q^2}{4(1+\tau)}\cot^2\frac{\theta_e}{2}$$

So, from $k_{1x}^2 = (k_{1B}^x)^2$, we find the relation 2.5 between the electron scattering angle in the Lab system and in the Breit system.

Lecture I Exercise 3

Derive the relation 2.39:

$$\overline{u}(p_2)\frac{\sigma_{\mu\nu}q_\nu}{2m}u(p_1) = \overline{u}(p_2)\left[\gamma_\mu - \frac{(p_1+p_2)_\mu}{2m}\right]u(p_1).) \tag{4.17}$$

Using the definition for $\sigma_{\mu\nu}$, one can write:

$$\overline{u}(p_2)\frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{4m}q_\nu u(p_1) = \overline{u}(p_2)\frac{\gamma_\mu\hat{q} - \hat{q}\gamma_\mu}{4m}u(p_1).$$

Recalling that $q = p_2 - p_1$ with $\hat{a} = a_\mu\gamma_\mu$:

$$\overline{u}(p_2)\frac{\gamma_\mu(\hat{p_2} - \hat{p_1}) - (\hat{p_2} - \hat{p_1})\gamma_\mu}{4m}u(p_1).$$

Applying the Dirac equations 2.11 we find:

$$\overline{u}(p_2)\frac{\gamma_\mu(\hat{p_2} - m) - (m - \hat{p_1})\gamma_\mu}{4m}u(p_1) = -\frac{1}{2}\overline{u}(p_2)\gamma_\mu u(p_1) + \frac{1}{4m}\overline{u}(p_2)\left[\gamma_\mu\hat{p_2} + \hat{p_1}\gamma_\mu\right]u(p_1) \tag{4.18}$$

Using the properties: $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$, $\hat{a}\hat{b} + \hat{b}\hat{a} = 2ab$, $\hat{a}\gamma_\mu + \gamma_\mu\hat{a} = 2a_\mu$ we have $\hat{p_1}\gamma_\mu = -\gamma_\mu\hat{p_1} + 2p_{1\mu}$, so that:

$$\begin{aligned}
\frac{1}{4m}\overline{u}(p_2)\left[\gamma_\mu\hat{p_2} + \hat{p_1}\gamma_\mu\right]u(p_1) &= \frac{1}{4m}\overline{u}(p_2)\left[-\hat{p_2}\gamma_\mu + 2p_{2\mu} - \gamma_\mu\hat{p_1} + 2p_{1\mu}\right]u(p_1) \\
&= \frac{1}{4m}\overline{u}(p_2)\left[-2\gamma_\mu m + 2(p_{2\mu} + p_{1\mu})\right]u(p_1) \\
&= \frac{1}{2}\overline{u}(p_2)\left[-\gamma_\mu + \frac{(p_{2\mu} + p_{1\mu})}{m}\right]u(p_1).
\end{aligned} \tag{4.19}$$

Inserting in Eq. (4.19) in (4.18), we find Eq. (4.17).

Note in this respect, that the relation (2.74) is correct only for the case when both nucleons are on mass shell, i.e; they are described by the four-component spinors $u(p)$, satisfying the Dirac equation. It is not the case for the quasi-elastic scattering of electrons by atomic nuclei, $a + A \to e + p + x$, which corresponds to the scattering $e + p^* \to e + p$, where p^* is a virtual nucleon.

Lecture II Exercise 1

Let us prove the following relation

$$\cos^2\tilde{\theta} \to \frac{1+\epsilon}{1-\epsilon} = \frac{\cot^2\theta_e/2}{1+\tau} + 1, \tag{4.20}$$

where θ_e is the laboratory scattering angle of the electron in elastic ep scattering and $\tilde{theta}$ is the CMS angle of the antiproton produced in the annihilation: $e^- + e^+ \to \bar{p} + p$ with respect to the beam direction.

This kinematical relation shows clearly the physical link between the linear ϵ dependence of the Rosenbluth differential cross section for elastic ep-scattering in Lab system (or $\cot^2\theta_e/2$) and the even distribution in $\cos^2\tilde{\theta}$ for the differential annihilation cross section in $\bar{p}+p \leftrightarrow e^+ + e^-$.

Crossing symmetry allows to connect scattering and annihilation channels (change a particle into antiparticle, change sign to the momenta):

$$e^-(k_1)+p(p_1)\to e^-(k_2)+p(p_2),\ e^-(k_1)+e^+(-k_2)\to \bar{p}(-p_1)+p(p_2).$$

1. Let us calculate s and t in the scattering channel:

$$\begin{aligned} s &= (p_1+k_1)^2 = M_p^2+2\epsilon_1 M_p = M_p(M_p+2\epsilon_1)\to \epsilon_1=\frac{s-M_p^2}{2M_p}; &(4.21)\\ t &= (k_1-k_2)^2 = k_1^2+k_2^2-2\epsilon_1\epsilon_2+2|\mathbf{k}_1||\mathbf{k}_2|\cos\theta_e = -4\epsilon_1\epsilon_2\sin^2\frac{\theta_e}{2}. &(4.22)\end{aligned}$$

   where we assumed $m_e=0$ and we calculate t as function of the electron variables.

2. The energy and momentum conservation are: $\epsilon_1+M_p=\epsilon_2+E_2$; $\mathbf{k_1}=\mathbf{k_2}+\mathbf{p}_2$;

3. Let us express t from the hadron variables:

$$t=(p_2-p_1)^2=2M_p^2-2M_pE_2=2M_p^2-2M_p(\epsilon_1+M_p-\epsilon_2)=2M_p(\epsilon_2-\epsilon_1). \quad (4.23)$$

   From the equality of Eqs. (4.22) and (4.23):

$$t=2M_p(\epsilon_2-\epsilon_1)=-4\epsilon_1\epsilon_2\sin^2\frac{\theta_e}{2}. \quad (4.24)$$

   Hence

$$\epsilon_2=\frac{\epsilon_1}{1+2\dfrac{\epsilon_1}{M_p}\sin^2\dfrac{\theta_e}{2}}=\frac{M_p(s-M_p^2)}{2\left[M_p^2+(s-M_p^2)\sin^2\dfrac{\theta_e}{2}\right]}. \quad (4.25)$$

4. Inserting the expression of ϵ_1 and ϵ_2 as a functions of s in Eq. 4.23:

$$\frac{1}{t}=-\frac{M_p^2}{(s-M_p^2)^2\sin^2\dfrac{\theta_e}{2}}-\frac{1}{s-M_p^2}. \quad (4.26)$$

5. In the annihilation channel (CMS) one has $\tilde{\epsilon}_1=\tilde{\epsilon_2}=\tilde{E}_1=\tilde{E}_2=\tilde{\epsilon}$; $\mathbf{k}_1=-\widetilde{\mathbf{k_2}}=\mathbf{k}$, $\mathbf{p}_1=-\mathbf{p}_2=\mathbf{p}\neq\mathbf{k}$:

$$\begin{aligned} s &= (k_1-p_2)^2 = M_p^2-2\tilde{\epsilon_1}^2+2\tilde{\epsilon_2}\vec{p_2}\cos\tilde{\theta} &(4.27)\\ t &= (k_1+k_2)^2 = 2\tilde{\epsilon_1}^2-2\tilde{\epsilon_1}\tilde{\epsilon_2}\cos\widetilde{\widetilde{\mathbf{k_1}}\widetilde{\mathbf{k_2}}}=4\tilde{\epsilon_1}^2, &(4.28)\end{aligned}$$

   from where we find the expression of $\cos\tilde{\theta}$ as a function of the invariants s and t:

$$\cos\tilde{\theta}=\frac{s-M_p^2+2\tilde{\epsilon}^2}{2\tilde{\epsilon}\sqrt{\tilde{\epsilon}^2-M_p^2}} \ \to\ \cos^2\tilde{\theta}=\frac{(s-M_p^2)^2+ts}{t\left(\dfrac{t}{4}-M_p^2\right)}+1. \quad (4.29)$$

Reminding that $\tau = -t/(4M_p^2)$, one finds

$$t\left(\frac{t}{4} - M_p^2\right) = -M_p^2 t(\tau + 1). \tag{4.30}$$

Inserting the relation $\left(\sin^2\frac{\tilde{\theta}}{2}\right)^{-1} = \cot^2\frac{\tilde{\theta}}{2} + 1$ in Eq. (4.26), one finds

$$\cot^2\frac{\tilde{\theta}}{2} = \frac{(s - M_p^2)^2 + ts}{-M_p^2 t}. \tag{4.31}$$

6. Comparing Eqs. (4.31) and (4.29) with the help of (4.30) one verifies the relation Eq. (3.15).

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