intermezzo: SUSY, superspace & superfields

- 4D $\mathcal{N}=1$ SUSY algebra

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2M\cdot\delta_{\alpha\dot\beta}, \quad \{Q_\alpha, Q_\beta\} = 0$$

- can rewrite as commutators using <u>anti-commuting</u> Grassmann variables

ξ_α: $\quad \xi Q = \xi^\alpha Q_\alpha, \quad \bar\xi\bar Q = \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha}$

fermionic $\qquad \alpha = 1,2$

$$[\xi Q, \bar\xi\bar Q] = 2\cdot\xi\sigma^\mu\bar\xi\cdot P_\mu$$

$$[\xi Q, \xi Q] = [\bar\xi\bar Q, \bar\xi\bar Q] = [P^\mu, \xi Q] = [P^\mu, \bar\xi\bar Q] = 0$$

'super-Lie algebra'

$\rightsquigarrow$ group elements:

$$G(x^\mu, \theta, \bar\theta) = e^{i(-x^\mu P_\mu + \theta Q + \bar\theta\bar Q)}$$

multiplication also acts on parameters $x^\mu, \theta, \bar\theta$:

$$G(0, \xi, \bar\xi)\, G(x^\mu, \theta, \bar\theta) = G(x^\mu + i\theta\sigma^\mu\bar\xi - i\xi\sigma^\mu\bar\theta, \theta+\xi, \bar\theta+\bar\xi)$$

$\Downarrow$

$(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})$ coordinates on <u>superspace</u>

group action infinitesimally given by differential operators:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu}$$

$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

$\Downarrow$

notion of superfield – function on superspace:

$$F(x^\mu,\theta,\bar\theta) = f(x) + \theta^\alpha \varphi_\alpha(x) + \bar\theta^{\dot\alpha}\bar\chi_{\dot\alpha}(x)$$
$$+ \theta\theta \cdot m(x) + \bar\theta\bar\theta \cdot n(x)$$
$$+ \theta\sigma^\mu\bar\theta\, v_\mu(x)$$
$$+ \theta\theta \cdot \bar\theta\bar\lambda(x) + \bar\theta\bar\theta \cdot \theta\psi(x)$$
$$+ \theta\theta\bar\theta\bar\theta \cdot d(x)$$

$$(\theta^2)^n\big|_{n\geq 2} = (\bar\theta^2)^n\big|_{n\geq 2} = 0$$

by anticommutation

- chiral superfields Φ satisfy:

$$\bar D_{\dot\alpha}\Phi = 0$$

since $\bar D_{\dot\alpha}(x^\mu + i\theta\sigma^\mu\bar\theta) = 0$

$$\bar D_{\dot\alpha}\theta = 0$$

can use: $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar\theta$

$\Downarrow$ complex scalar $\downarrow$ Weyl fermion $\swarrow$ auxiliary $\downarrow$ field

$$\Phi = A(y) + \sqrt{2}\theta\cdot\psi(y) + \theta\theta\cdot F(y)$$

expand y^μ out

$$= A(x) + i\theta\sigma^\mu\bar\theta\,\partial_\mu A(x)$$
$$+ \frac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box A(x)$$
$$+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\cdot\partial_\mu\psi\sigma^\mu\bar\theta$$
$$+ \theta\theta\cdot F(x)$$

- Grassmann integration:
  - single Grassmann variable η:

    all functions on (x^μ,η) only

    $$f(x,\eta) = c(x) + \eta\cdot\Delta(x)$$

    rules: $\int d\eta = 0$, $\int d\eta\cdot\eta = 1$

    $$\eta^n\big|_{n\geq 2} = 0$$

$\Rightarrow \int d\eta \cdot f = \Delta, \int d\eta \cdot \eta \cdot f = c$

- now θ^α : $\alpha = 1, 2$

$$\int d^2\theta = 0, \int d^2\theta \cdot \theta^2 = 1$$

$\Downarrow$

- general effective action in superspace:

$$S = \int d^4x \cdot \left[\int d^2\theta \, d^2\bar\theta \cdot K(\Phi, \bar\Phi) + \left(\int d^2\theta \cdot W(\Phi) + c.c. \right) \right]$$

- example:

$$K = \bar\Phi \Phi$$

$$\underbrace{\int d^2\theta \, d^2\bar\theta \, \bar\Phi \Phi}_{\substack{\text{picks all} \\ \theta^2\bar\theta^2\text{-terms in } \bar\Phi\Phi}} = i \, \partial_\mu \psi \sigma^\mu \bar\psi + \partial_\mu A \partial^\mu \bar A + F\bar F$$

$$W = m \cdot \Phi^2$$

$$\int d^2\theta \, W = m \cdot A \cdot F + \frac{1}{2} m \cdot \psi\psi$$

$$\frac{\delta S}{\delta F} = 0 \Rightarrow \bar F = m \cdot A, \; V = \bar F F = m^2 A^2 = \left| \frac{\partial W}{\partial A} \right|^2$$

- can generalize to <u>curved</u> superspace for supergravity

$$S = \int d^4x \int d^2\theta \, 2\mathcal{E} \left[\frac{3}{8} (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) e^{-\frac{K}{3}} + W \right] + h.c.$$

$$V_F = e^K \cdot \left(K^{i\bar j} D_i W \overline{D_j W} - 3|W|^2 \right)$$

$$D_i W = \frac{\partial W}{\partial \phi^i} + W \cdot \frac{\partial K}{\partial \phi^i}$$