An analytical fluid dark matter model

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Introduction

- The ordinary baryonic matter is not the dominant form of material in the universe. It is accepted that most of the mass appears to be in some as-yet-undiscovered strange new form which is non-luminous known as Dark Matter (DM).
- It is believed that the dark matter is the main component, responsible for the large-scale structure formation in the universe.
- Observational evidence indicating the existence of the dark matter (i) velocity scattered of stars in the galactic plane by Oort(J. Oort, *Bull. Astron. Ins. Nether.* **VI**, 249, 1932) (ii) Zwicky estimated the velocity dispersion in the coma cluster(F. Zwicky, *Helvet. Phys. Acta* **6**, 110, 1933) (iii) rotational curves of galaxies Rubin (V. C. Rubin and W. K. J. Ford, *The Astrophysical Journal* **159**, 379, 1970) (iv) bullet cluster ($1E0657-558$) (M. Markevitch, A. H. Gonzalez, D. Clowe, A. Vikhlinin, L. David, W. Forman, C. Jones, S. Murray and W. Tucker, *Astrophys. J.* **606**, 819, 2004).
- Dark matter particle could be Hot Dark Matter (HDM), Cold Dark Matter (CDM), weakly interacting massive particles (WIMPs).

- Dark matter could be described by a fluid with non-zero effective pressure (S. Bharadwaj and S. Kar, *Phys. Rev. D* **68**, 023516, 2003; K.-Y. Su and P. Chen, *Phys. Rev. D* **79**, 128301, 2009).
- Rahaman et al. (F. Rahaman, K. K. Nandi, A. Bhadra, M. Kalam and K. Chakraborty, *Phys. Lett. B* **694**, 10, 2010) considered dark matter as perfect fluid in their work.
- Dark matter may be modeled as a mixture of two non-interacting perfect fluids as was shown by Harko and Lobo (Tiberiu Harko and Francisco S. N. Lobo, arXiv:1106.2642v1 [gr-qc]).

Objective

- We have proposed a new model by considering the stellar object consists of core and envelope regions.
- A particular EoS to describe isotropic fluid dark matter in the core region which provides constant density throughout the interior. The outer envelope region is considered as anisotropic in nature and satisfying a linear pressure-density relation.
- In the core boundary, we have assume de-sitter metric as the exterior while Schwarzschild solution is assumed to describe the exterior boundary of the stellar object. Accordingly, the matching conditions are used in addition to setting radial pressure zero at the exterior boundary.
- Energy conditions and stability has also been discussed for the developed model.

Interior metric

We consider the line element in *Schwarzschild* co-ordinate system to describe the interior of a static and spherically symmetric stellar configuration as :

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (1)$$

where $e^{\nu(r)}$ and $e^{\lambda(r)}$ are known as the metric potential functions, where $\nu(r)$ and $\lambda(r)$ are functions of the radial coordinate 'r' only.

Energy momentum tensor

For an anisotropic matter distribution, the energy momentum tensor can be written as:

$$T_{\mu\nu} = \{\rho(r) + p_t(r)\}U_\mu U_\nu - p_t(r)g_{\mu\nu} + \{p_r(r) - p_t(r)\}\chi_\mu\chi_\nu, \quad (2)$$

where $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure and $p_t(r)$ is the tangential pressure of the of the fluid configuration. χ^μ is an unit 4-vector along the radial direction and U^μ is the 4-velocity. The quantities obey the following relation: $\chi_\mu\chi^\mu = 1$, $\chi_\mu U^\mu = 0$.

Einstein field equations

The Einstein field equations can be written as:

$$T_{\mu\nu} = \frac{1}{8\pi}\left\{R_{\mu\nu} - \frac{1}{2}R\ g_{\mu\nu}\right\}, \tag{3}$$

where $T_{\mu\nu}$, $R_{\mu\nu}$, $g_{\mu\nu}$ and R are the stress energy tensor, Ricci tensor, metric tensor and Ricci scalar, respectively.
The Einstein field equations (3) read as the following form for the metric (1) along with the energy tensor (2):

$$\rho(r) = \frac{1}{8\pi}\left\{\frac{1-e^{-\lambda}}{r^2} + \frac{e^{-\lambda}\lambda'}{r}\right\}, \tag{4}$$

$$p_r(r) = \frac{1}{8\pi}\left\{\frac{e^{-\lambda}-1}{r^2} + \frac{e^{-\lambda}\nu'}{r}\right\}, \tag{5}$$

$$p_t(r) = \frac{e^{-\lambda}}{8\pi}\left\{\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu'-\lambda'}{2r}\right\}. \tag{6}$$

Anisotropy

In a compact object, the pressures in radial and transverse directions may not be equal and the difference of radial pressure (p_r) and tangential pressure (p_t) produces **anisotropy**.

Identified factors assuming the compact star as anisotropic:

- In the **high density** regime of compact stars where the nuclear interactions must be treated relativistically, there may develop anisotropy inside the stellar objects as shown by Ruderman, Canuto (*Annu. Rev. Astron. Astrophys.* **10**, 427, 1972, *Ann. Rev. Astron. Astrophys.* **12**, 167, 1974).
- In relativistic stars anisotropy might occur due to the existence of a solid core or type **3A superfluid** as pointed out by Kippenhahn and Weigert (*Springer-Verlag, Berlin*, 1990).
- **Strong magnetic fields** can also regard as a source of anisotropic pressure inside a compact object as discussed by Weber (*IOP Publishing, Bristol*, 1999).

- Anisotropy may also develop due to the **slow rotation** of fluids (*Astrophys. J.* **438**, 308, 1995).
- A **mixture of perfect and a null fluid** may also be formally described as an anisotropic fluid (*Phys. Rev. D* **22**, 807, 1980).
- The existence of anisotropy in astrophysical objects may arise due to **viscosity**.
- Different kinds of **phase transitions** (*JETP* **79**, 1137, 1980).
- **Pion condensation** (*Phys. Rev. Lett.* **29**, 382, 1972).
- The presence of **strong electromagnetic field** (*Phys. Rev. D* **70**, 067301, 2004).
- Self-bound systems composed of scalar fields, the so-called **boson stars** are naturally anisotropy (*Class. Quantum Grav.* **20**, R301, 2003).
- **Wormholes** (*Am. J. Phys.* **56**, 395, 1988) and **gravastars** (*Class. Quantum Grav.* **22**, 4189, 2005) are also considered as anisotropic as well.

Generalized Tolman-Oppenheimer-Volkoff equation

From Eqs. (4) and (5), we obtain

$$\rho(r) + p_r(r) = \frac{\lambda' + \nu'}{8\pi r} e^{-\lambda}. \tag{7}$$

Again from Eq. (5), we get

$$\frac{dp_r(r)}{dr} = \frac{1}{8\pi} \left[e^{-\lambda} \left\{ \frac{\nu''}{r} - \frac{\nu'\lambda'}{r} - \frac{\nu' + \lambda'}{r^2} \right\} + \frac{2(1 - e^{-\lambda})}{r^3} \right]. \tag{8}$$

Then, by using Eqs. (4)-(8), we can write

$$\frac{\nu'\{\rho(r) + p_r(r)\}}{2} + \frac{dp_r(r)}{dr} = \frac{2\{p_t(r) - p_r(r)\}}{r}. \tag{9}$$

This is the generalized Tolman-Oppenheimer-Volkoff (TOV) equation for anisotropic fluid distribution.

The interior Solutions

To solve the system of equations, we assume a density profile of the dark matter as:

$$\rho(r) = \frac{k}{r(1 + \frac{r}{b})}, \tag{10}$$

where, b is the scale radius and k is a constant. We assume that the interior region of star is divided into two regions:
(i) The core, $0 \leq r \leq b$,
and
(ii) The outer region, $b < r \leq R$ to avoid the singularity at the center of stellar configuration.

Interior Solution: Solution in the core region, $0 \leq r \leq b$

The core of stellar object is isotropic in nature and satisfy the following equation of state (EoS):

$$p(r) = p_r(r) = p_t(r) = -\rho(r).$$

Using Eqs. (9) and (10), we obtain

$$\rho(r) = \text{constant} = \rho_c = \frac{k}{2b}. \tag{11}$$

The mass of the core can obtain as:

$$m(r) = 4\pi \int_0^r r'^2 \rho_c dr' = \frac{2r^2 k\pi}{3}. \tag{12}$$

The compactness parameter is

$$u(r) = \frac{2m(r)}{r} = \frac{4rk\pi}{3}, \tag{13}$$

The surface redshift is

$$z(r) = \{1 - u(r)\}^{-\frac{1}{2}} - 1 = \left(1 - \frac{4rk\pi}{3}\right)^{-\frac{1}{2}} - 1. \tag{14}$$

In Schwarzschild coordinate, we can write

$$e^{-\lambda} = 1 - \frac{2m(r)}{r} = 1 - \frac{4rk\pi}{3}. \tag{15}$$

For the outer region, we assure the density profile

$$\rho(r) = \frac{k}{r(1+\frac{r}{b})}, \tag{16}$$

and an EoS in the following linear form:

$$p_r(r) = \alpha\rho - \beta, \tag{17}$$

where $\alpha, \ \beta$ are constants.
The mass of the stellar object can be obtained as:

$$m(r) = 4\pi \left\{ \int_0^b \xi^2 \rho_c d\xi + \int_b^r \xi^2 \rho(\xi) d\xi \right\} \tag{18}$$

$$= \frac{2bk\pi}{3} \left\{ 6r - 5b - 6bf_1 \right\}, \tag{19}$$

where $f_1 = \log\left(\frac{r+b}{2b}\right)$.

The compactness parameter:

$$u(r) = \frac{2m(r)}{r} = \frac{4bk\pi}{3r}\left[6r - 5b - 6bf_1\right]. \quad (20)$$

Surface redshift:

$$\begin{aligned} z(r) &= \{1 - u(r)\}^{-\frac{1}{2}} - 1 \\ &= \left(1 - \frac{4bk\pi}{3r}\left[6r - 5b - 6bf_1\right]\right)^{-\frac{1}{2}} - 1. \end{aligned} \quad (21)$$

In Schwarzschild coordinate, we can write

$$\begin{aligned} e^{-\lambda} &= 1 - \frac{2m(r)}{r}, \\ &= 1 - \frac{4bk\pi}{3r}\left[6r - 5b - 6bf_1\right]. \end{aligned} \quad (22)$$

Using Eqs. (10) and (17), we get the expression of radial pressure

$$p_r(r) = \frac{\alpha k}{r(1+\frac{r}{b})} - \beta. \quad (23)$$

On imposing Eqs. (22)-(23) in Eq. (5) we get

$$\nu'(r) = \frac{4\pi}{r(r+b)f_2}\Big[bk\{5b^2 - br - br^2(\alpha+1)\} + 6r^3\beta(r+b) + 6b^2k(r+b)f_1\Big], \quad (24)$$

whereas

$$f_2 = \Big[4bk\pi(6r-5b) - 24b^2k\pi f_1 - 3r\Big]. \quad (25)$$

The transverse pressure and anisotropic factor are obtained as:

$$p_t(r) = \Big[6r\beta(r+b)f_3 - 12\pi r^4\beta^2(r+b)^2 + 12\pi k b^2 f_1 f_4 - b^2 k f_5\Big]\Big[2r(r+b)^2 f_2\Big]^{-1}, \qquad (26)$$

$$\Delta(r) = \beta - \frac{\alpha k}{r(1+\frac{r}{b})} + \Big[2r(r+b)^2 f_2\Big]^{-1} \times \Big[6r\beta(r+b)f_3 - 24\pi r^4\beta^2(r+b)^2 - b^2 k f_5 + 12\pi k b^2 f_1 f_4\Big], \qquad (27)$$

where

$$f_3 = 5\pi k b^3 - b^2 k\pi r + r^2 + br\{1 + 4k\pi r(\alpha - 1)\},$$
$$f_4 = bk\{b(\alpha-1) - r(\alpha+1)\} - 3r\beta(r+b)^2,$$
$$f_5 = 2bk\pi r(1 - 11\pi) + 10b^2 k\pi(\alpha - 1) + 3r\{\alpha + 4k\pi r(\alpha+1)^2\}. \qquad (28)$$

Boundary conditions

To determine the values of involved constants within solutions, we have matched our solutions with the *de-sitter* metric at the core boundary $r = b$ and the exterior *Schwarzschild* solution at the surface boundary $r = R\ (> 2M)$. The *de-sitter* and exterior *Schwarzschild* metric are given by

$$ds^2 = \left(1 - \frac{r^2}{d^2}\right) dt^2 - \left(1 - \frac{r^2}{d^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (29)$$

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (30)$$

Continue....

$$e^{-\lambda}|_{r=b} = e^{\nu}|_{r=b} = \left(1 - \frac{b^2}{d^2}\right) = 1 - \frac{4\pi kb}{3}, \tag{31}$$

$$\begin{aligned} e^{-\lambda}|_{r=R} &= e^{\nu}|_{r=R} = \left(1 - \frac{2M}{R}\right) \\ &= 1 - \frac{4bk\pi}{3R}\left[6R - 5b - 6bf_1(R)\right], \end{aligned} \tag{32}$$

$$p_r(R) = 0. \tag{33}$$

Using these boundary conditions (31)-(33), we have $k = \frac{3M}{2\pi b(6R-5b-6bf_1(R))}$; $d = \sqrt{\frac{3b}{4\pi k}}$; $\beta = \frac{\alpha bk}{R(R+b)}$. where b is a free parameter, which will be the measurement of the core radius of the stellar fluid configuration. For our model, we consider $b = 2\ km$.

Here $A = EXO\ 1785 - 248$, $B = Vela\ X - 1$, $C = 4U\ 1538 - 52$.

Star	$M(M_\odot)$	$R(km)$	$\alpha(km^{-2})$	$\beta(km^{-2})$	$b(km)$	k	d(km
A	1.3	8.8	0.3	0.00006	2	0.010	6.8
			0.4	0.00008	2	0.010	6.8
			0.5	0.00011	2	0.010	6.8
			0.6	0.00012	2	0.010	6.8

Star	$M(M_\odot)$	$R(km)$	$\alpha(km^{-2})$	$\beta(km^{-2})$	$b(km)$	k	d(km
B	1.77	9.56	0.3	0.00007	2	0.012	6.2
			0.4	0.00009	2	0.012	6.2
			0.5	0.00011	2	0.012	6.2
			0.6	0.00013	2	0.012	6.2

Star	$M(M_\odot)$	$R(km)$	$\alpha(km^{-2})$	$\beta(km^{-2})$	$b(km)$	k	$d(km$
C	0.87	7.87	0.3	0.000061	2	0.007	7.8
			0.4	0.000081	2	0.007	7.8
			0.5	0.000101	2	0.007	7.8
			0.6	0.000122	2	0.007	7.8

Graphical Presentation: Density

Compact star EXO 1785-248 corresponding to $\alpha = 0.3$, 0.4, 0.5 and 0.6.

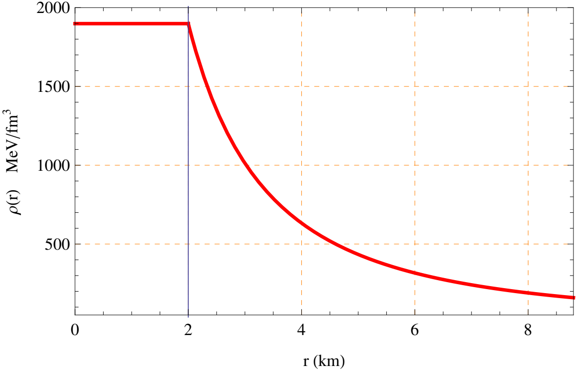

Figure 1: Behavior of the density with respect to the radial coordinate r for the compact star EXO 1785-248 corresponding to the numerical value of constants given in Table-1

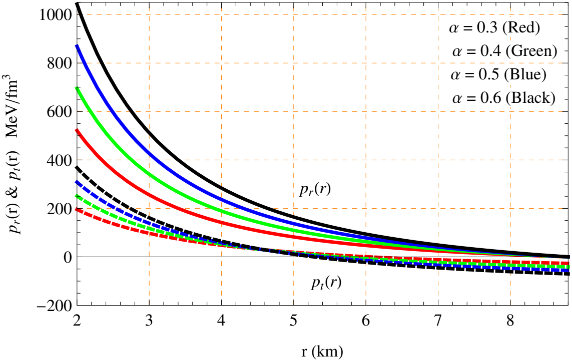

Figure 2: Behaviors of the radial and transverse pressures with respect to the radial coordinate r for the compact star EXO 1785-248 corresponding to the numerical value of constants given in Table- 1

Mass function and compactness parameter: $m(r)$ and $u(r)$ are finite, zero at the centre and then monotonically increasing toward the outer layer surface of the fluid configuration.

Moreover, the value of compactness parameter is more that the mass in the region $0 < r < b$ and coincide at the core boundary $r = b = 2\ km$.

According to Buchdahl (H. A. Buchdahl, *Phys. Rev.* **116**, 1027 (1959)), at the outer layer surface of compact star, $u_s = u(R) = \frac{2M}{R} < \frac{8}{9}$. Compactness parameter are given in Table-2, ensure that our solutions satisfied the Buchdahl limit.

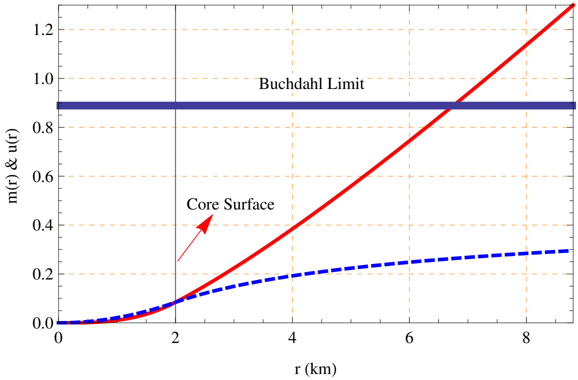

Figure 3: Behaviors of the mass and compactness parameter with respect to the radial coordinate r for the compact star EXO 1785-248 corresponding to the numerical value of constants given in Table-1

The physical conditions regarding the compact star for our present solutions.

- Energy Conditions:

$$\begin{aligned} NEC_r &: \rho(r)+p_r(r)\geq 0, \quad NEC_t : \rho(r)+p_t(r)\geq 0. \\ WEC_r &: \rho(r)\geq 0, \quad \rho(r)+p_r(r)\geq 0. \\ WEC_t &: \rho(r)\geq 0, \quad \rho(r)+p_t(r)\geq 0. \\ SEC &: \rho(r)+p_r(r)+2p_t(r)\geq 0. \end{aligned} \tag{34}$$

Moreover, the energy density is positive and $\rho(r)+p(r)=0$ within the core region i.e. NEC and WEC are satisfied there. Consequently, the solutions represent a physical matter distribution.

Graphical Presentation: Energy

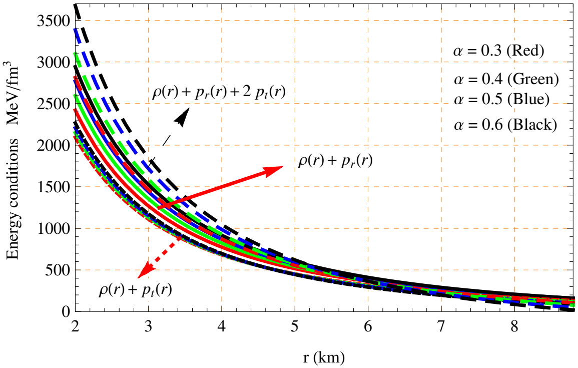

Figure 4: Behaviors of the energy conditions with respect to the radial coordinate r for the compact star EXO 1785-248 corresponding to the numerical value of constants given in Table-1

Equilibrium condition: TOV equation

The equilibrium condition represents by the Tolman-Oppenheimer-Volkoff (TOV) equation which has the generalized form for the anisotropic fluid distribution can be written as:

$$-\frac{M_g\{\rho(r)+p_r(r)\}}{r^2}\, e^{(\lambda-\nu)/2}-\frac{dp_r(r)}{dr}+\frac{2\Delta(r)}{r}=0, \qquad (35)$$

where $M_g(r)$ is the effective gravitational mass, which can obtain with the help of Tolman-Whittaker mass formula as:

$$M_g(r)=\frac{r^2}{2}\nu' e^{(\nu-\lambda)/2}. \qquad (36)$$

Therefore, Eq. (35) reduces to

$$-\frac{\nu'\{\rho(r)+p_r(r)\}}{2}-\frac{dp_r(r)}{dr}+\frac{2\Delta(r)}{r}=0. \qquad (37)$$

Eq. (37) also can be written as

$$F_g(r)+F_h(r)+F_a(r)=0. \qquad (38)$$

Gravitational force, $F_g(r) = -\frac{\nu'\{\rho(r)+p_r(r)\}}{2}$;
Hydrostatic force, $F_h(r) = -\frac{dp_r(r)}{dr}$;
Anisotropic force, $F_a(r) = \frac{2\Delta(r)}{r}$.

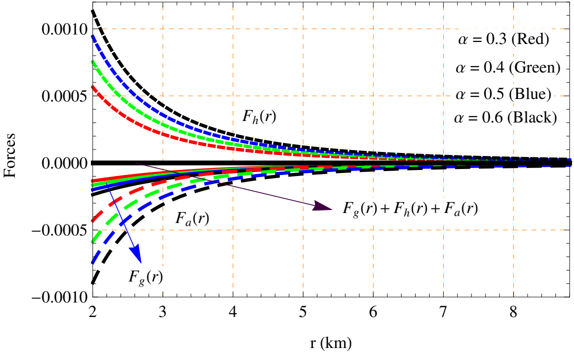

Figure 5: Behaviors of the forces with respect to the radial coordinate r for the compact star EXO 1785-248.

Sound velocity: $v_r(r) = \sqrt{\frac{dp_r(r)}{d\rho(r)}}$, $v_t(r) = \sqrt{\frac{dp_t(r)}{d\rho(r)}}$. Causality condition: $0 \le v_r(r), v_t(r) < 1$.

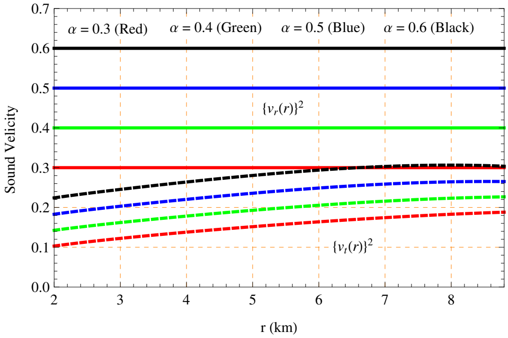

Figure 6: Behaviors of the radial and transverse velocities of sound with respect to the radial coordinate r for the compact star EXO 1785-248 corresponding to the numerical value of constants given in Table-1

Stability analysis: Adiabatic index

The relativistic adiabatic index is defined as: $\Gamma_r(r) = \frac{\rho(r)+p_r(r)}{p_r(r)} \frac{dp_r(r)}{d\rho(r)}$ For our solutions, the value of adiabatic index $\Gamma_r(r)$ is more than $4/3$.

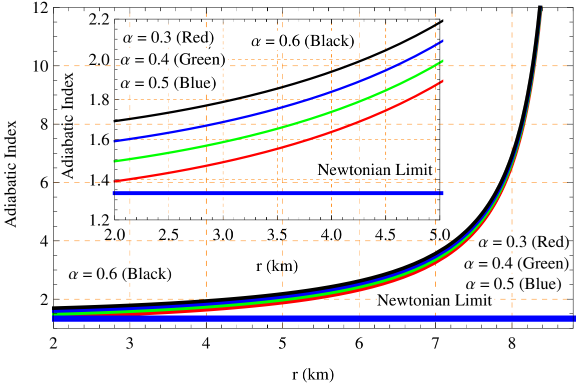

Figure 7: Behavior of the adiabatic index with respect to r for the compact star EXO 1785-248 corresponding to the numerical value of constants given in Table-1

Stbility analysis: Harrison-Zeldovich-Novikov criterion

Harrison-Zeldovich-Novikov criterion: $dM(\rho_c)/d\rho_c > 0$

$$M(\rho_c) = \frac{4\pi b^2 \rho_c}{3}\left[6R - 5b - 6bf_1(R)\right], \tag{39}$$

$$\frac{\partial M(\rho_c)}{\partial \rho_c} = \frac{4\pi b^2}{3}\left[6R - 5b - 6bf_1(R)\right] > 0. \tag{40}$$

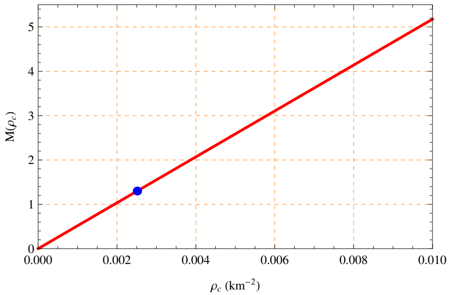

Figure 8: Mass vs core density ρ_c for the compact star EXO 1785-248.

Anisotropic factor $\Delta(r) = 0$ in inner part. In the outer region nature of anisotropy indicates that the force due to anisotropy is inward-directed i.e. compact star becomes less stable.

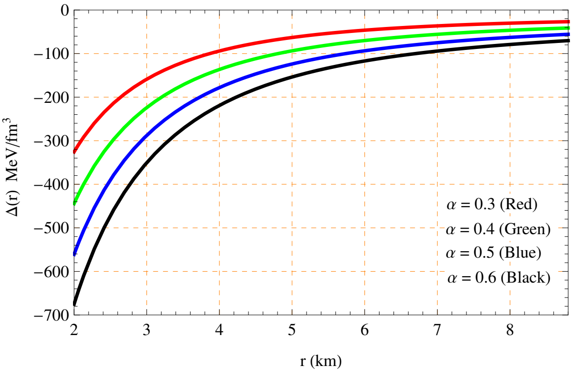

Figure 9: Behavior of the anisotropic factor with respect to the radial coordinate r for the compact star EXO 1785-248.

Equation of state parameters

The EoS, $p_r(r) = -\rho(r)$ shows that the EoS parmeter $\omega(r) = -1$. For the real feasible matter distribution, $0 < \omega_r(r) < 1$ (F. Rahaman, S. Ray, A. K. Jafry, K. chakraborty, *Phys. Rev. D* **82**, 104055 (2010)). The obtained solution satisfy $0 < \omega_r(r) < 1$, provided in outer part of the fluid configuration and hence the DM becomes as real feasible matter within outer region. Thus, the celestial compact stars, which are formed by DM distributed in two parts: (i) Inner part, formed by unfeasible DM, (ii) Outer part, formed by real feasible DM.

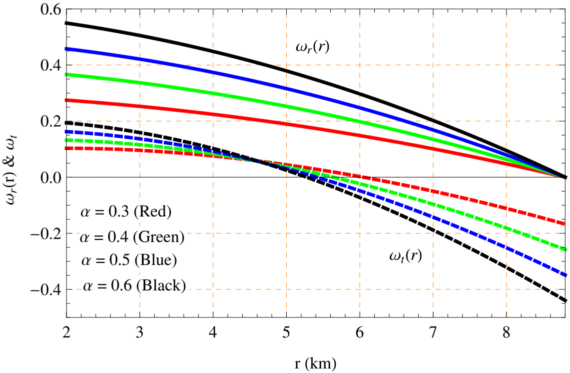

Figure 10: Behaviors of the equation of state parameters with respect to the radial coordinate r for the compact star EXO 1785-248 corresponding to the numerical value of constants given in Table-1

The stability factor $\{v_r(r)\}^2 - \{v_r(t)\}^2$ negative refers to a potentially stable configuration.

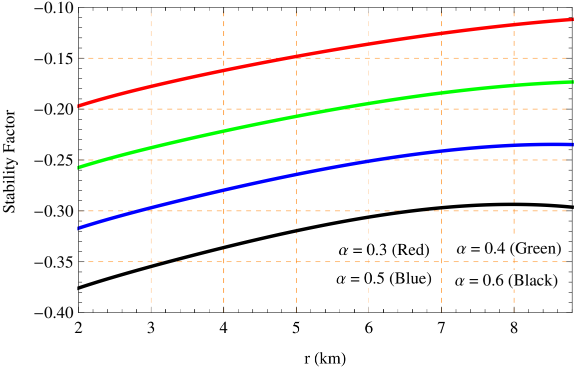

Figure 11: Behavior of the stability factor with respect to the radial coordinate r for the compact star EXO 1785-248.

Numerical values of the physical parameters for three well-known celestial compact stars corresponding to the values of constant given in Table-1.

We have calculated the numerical values of all physical parameters for the compact stars EXO 1785 − 248 along with more well-known compact stars *Vela X* − 1 and 4*U* 1538 − 52 in tabular form to make our solutions more feasible.

Compact Star	ρ_c (10^{15}) $(g.cm^{-3})$	ρ_s (10^{14}) $(g.cm^{-3})$	p_{nc} (10^{35}) $(d.cm^{-2})$	z_s	u_s	Buchdahl Limit
A	3.39	2.84	8.35	0.19	0.29	$< \frac{8}{9}$
	3.39	2.84	11.1	0.19	0.29	$< \frac{8}{9}$
	3.39	2.84	13.9	0.19	0.29	$< \frac{8}{9}$
	3.39	2.84	16.7	0.19	0.29	$< \frac{8}{9}$

Compact Star	ρ_c (10^{15}) $(g.cm^{-3})$	ρ_s (10^{14}) $(g.cm^{-3})$	p_{nc} (10^{35}) $(d.cm^{-2})$	z_s	u_s	Buchdahl Limit
B	4.11	2.97	10.2	0.26	0.37	$< \frac{8}{9}$
	4.10	2.97	13.7	0.26	0.37	$< \frac{8}{9}$
	4.10	2.97	17.1	0.26	0.37	$< \frac{8}{9}$
	4.10	2.97	20.5	0.26	0.37	$< \frac{8}{9}$

Compact Star	ρ_c (10^{15}) $(g.cm^{-3})$	ρ_s (10^{14}) $(g.cm^{-3})$	p_{nc} (10^{35}) $(d.cm^{-2})$	z_s	u_s	Buchdah Limit
C	2.65	2.73	6.4	0.13	0.22	$< \frac{8}{9}$
	2.65	2.73	8.5	0.13	0.22	$< \frac{8}{9}$
	2.65	2.73	10.6	0.13	0.22	$< \frac{8}{9}$
	2.65	2.73	12.8	0.13	0.22	$< \frac{8}{9}$

Here $A = EXO\ 1785 - 248$, $B = Vela\ X - 1$, $C = 4U\ 1538 - 52$.

Discussions and conclusion

Finally, all the salient key features of our solutions ensure that our solutions are well-behaved and physically acceptable to represent the physical DM fluid configuration containing two parts: the isotropic inner part of unfeasible DM with constant density and anisotropic outer part filled with feasible DM.

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Thank you!