On deformations of classical mechanics due to Planck-scale physics

O.I. Chashchina, A. Sen, Z.K. Silagadze

Heisenberg uncertainty principle

$$(\Delta q)^2(\Delta p)^2 \geq \left(\frac{\hbar}{2}\right)^2$$

Stability of hydrogen atom.

$$E = -\frac{e^2}{r} + \frac{p^2}{2m} \sim -\frac{e^2}{r} + \frac{\hbar^2}{2mr^2}$$

F. Rioux, J. Chem. Educ., 50 (1973), 550 (The stability of the hydrogen atom).

V.F. Weisskopf, Am. J. Phys. 53 (1985), 206 (Search for Simplicity: Quantum mechanics of atoms).

Heisenberg inequality

$$\left(\int_{\mathbb{R}^3} r^2 |\Psi(\vec{r})|^2 dV\right)\left(\int_{\mathbb{R}^3} |\nabla\Psi(\vec{r})|^2 dV\right) \geq \frac{9}{4}$$

Hardy inequality

$$\int_{\mathbb{R}^3} |\nabla\Psi(\vec{r})|^2 dV \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{1}{r^2}|\Psi(\vec{r})|^2 dV$$

E.H. Lieb, Bull. Amer. Math. Soc. 22 (1990), 1 (The stability of matter: from atoms to stars).

- String theory possesses a fundamental length scale l_s which determines the typical space-time extension of fundamental string. $l_s = \sqrt{\alpha'}$ where α is regge slope. Quantum theory is unaware of presence of such a scale.
- Natural question: Can we extend the theory in such a way so as to incorporate this minimal length which is equivalent to incorporating gravity?
- $\Delta x \approx \frac{\hbar}{\Delta p} + \frac{l_s^2}{\hbar}\Delta p$
- $l_s \approx l_p$ is understood to be the minimal length below which spacetime distances cannot be resolved. $\delta s = l_p \approx l_s$

$$\Delta x \approx \frac{\hbar}{p} + G\frac{(pc)/c^2}{l^2}\left(\frac{l}{c}\right)^2 = \frac{\hbar}{p} + \frac{G}{c^3}p$$

$$\Delta x \approx \frac{\hbar}{\Delta p} + \frac{L_p^2}{\hbar}\Delta p, \quad L_p^2 = \frac{\hbar G}{c^3}$$

Momentum inversion symmetry

$$\Delta p_1 \Delta p_2 = \frac{\hbar^2}{L_p^2} \rightarrow \Delta x_1 = \Delta x_2, \quad \Delta x_{min} = 2L_p$$

R.J. Adler, Am. J. Phys. 78 (2010), 925 (Six easy roads to the Planck scale)

Modified commutation relations

$$[q_i, p_j] = i\hbar\delta_{ij}(1+\beta\vec{p}^{\,2}), \quad \beta \sim \frac{L_p^2}{\hbar^2}$$

$$[q_i, q_j] = 2i\hbar\beta(p_i q_j - p_j q_i)$$

$$[q_i, p_j] = i\hbar\left[(1+\beta\vec{p}^{\,2})\delta_{ij} + \beta' p_i p_j\right]$$

$$[q_i, q_j] = i\hbar\frac{2\beta - \beta' + \beta(2\beta+\beta')\vec{p}^{\,2}}{1+\beta\vec{p}^{\,2}}(p_i q_j - p_j q_i)$$

G. Wataghin, Nature 142 (1938), 393 (Quantum Theory and Relativity).

A. Hagar, Stud. Hist. Phil. Sci. 46B (2014), 217 (Squaring the circle: Gleb Wataghin and the prehistory of quantum gravity)

Planck scale modified quantum mechanics

$$[x,p] = i\hbar(1+\beta p^2) \to \Delta x \Delta p \geq \frac{\hbar}{2}\left(1+\beta(\Delta p)^2 + \beta < p >^2\right)$$

$$\Delta x_{min} = \hbar\sqrt{\beta}\,\sqrt{1+\beta < p >^2}$$

Hilbert space representation

$$\hat{p}\Psi(p) = p\,\Psi(p), \quad \hat{x}\Psi(p) = i\hbar(1+\beta p^2)\partial_p\,\Psi(p)$$

$$< \Psi|\Phi > = \int\limits_{-\infty}^{\infty} \frac{dp}{1+\beta p^2}\,\Psi^*(p)\Phi(p)$$

A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. D52 (1995), 1108
(Hilbert Space Representation of the Minimal Length Uncertainty Relation)

Canonical commutation relations (CCR)

$$[q, p] = i\hbar.$$

- q and p are unbounded

$$e^{i\tau p} q e^{-i\tau p} = q + \hbar\tau$$

- everywhere defined self-adjoint operator is bounded (Hellinger-Toeplitz theorem) — a source of a large part of the mathematical subtleties of quantum mechanics
- q and p are not everywhere defined in $\mathcal{H}$ — at best on a dense subset of $\mathcal{H}$
- difficulties of precise mathematical formulation

Weyl's form of CCR

$$U(\alpha) = e^{i\alpha q/\hbar}, \quad V(\beta) = e^{i\beta p/\hbar}$$

Weyl commutation relations

$$U(\alpha)V(\beta) = V(\beta)U(\alpha)e^{-i\alpha\beta/\hbar}$$

$$U(\alpha)U(\beta) = U(\alpha+\beta), \;\; V(\alpha)V(\beta) = V(\alpha+\beta)$$

$$U^*(\alpha) = U(-\alpha), \; V^*(\alpha) = V(-\alpha)$$

Weyl C^*-algebra

Another option: rigged Hilbert space

R. de la Madrid, Eur. J. Phys. 26 (2005) 287 (The role of the rigged Hilbert space in Quantum Mechanics)

$$\frac{1}{i\hbar}[\hat{A},\hat{B}] \to \{A,B\}$$

S. Benczik et al., arXiv:hep-th/0209119 (Classical Implications of the Minimal Length Uncertainty Relation).

$$[X_i,P_j] = i\hbar(\delta_{ij} + \beta\vec{P}^{\,2}\delta_{ij} + 2\beta P_iP_j)$$

$$X_i = x_i, \quad P_i = p_i\left[1+\beta\vec{p}^{\,2}\right], \quad [x_i,p_j] = i\hbar\delta_{ij}$$

$$H = \frac{\vec{P}^2}{2m} + V(X) \to \frac{\vec{p}^2}{2m} + V(x) + \frac{\beta}{m}(\vec{p}^2)^2$$

S. Das, E.C. Vagenas, Phys. Rev. Lett. 101 (2008), 221301 (Universality of Quantum Gravity Corrections).

- Translation of classical mechanics into the language of Hilbert spaces B.O. Koopman, Proc. Nat. Acad. Sci. 17 (1931), 315 (Hamiltonian Systems and Transformations in Hilbert Space), J. von Neumann, Annals Math. 33 (1932), 587 (Zur Operatorenmethode In Der Klassischen Mechanik)
- Classical mechanics as a hidden variable quantum theory! E.C.G. Sudarshan, Pramana 6 (1976), 117 (Interaction between classical and quantum systems and the measurement of quantum observables)
- "If we assume that not all quantum dynamical variables are actually observable, and if we set rules for distinguishing measurable from nonmeasurable operators, it is then possible to define a classical system as a special type of quantum system for which all measurable operators commute" A. Peres, D.R. Terno, Phys. Rev. A 63 (2001), 022101 (Hybrid classical-quantum dynamics)

Koopman-von Neumann mechanics (continued)

"The determinism of classical physics turns out to be an illusion, created by overrating mathematico-logical concepts. It is an idol, not an ideal in scientific research" (M. Born, 1954 Nobel lecture).

$$i\frac{\partial \rho}{\partial t}=\hat{L}\rho=i\{H,\rho\}=i\left(\frac{\partial H}{\partial q}\,\frac{\partial \rho}{\partial p}-\frac{\partial H}{\partial p}\,\frac{\partial \rho}{\partial q}\right).$$

The Liouville operator

$$\hat{L}=i\left(\frac{\partial H}{\partial q}\,\frac{\partial}{\partial p}-\frac{\partial H}{\partial p}\,\frac{\partial}{\partial q}\right)=\frac{1}{\hbar}\left(\frac{\partial H}{\partial q}\,\hat{Q}+\frac{\partial H}{\partial p}\,\hat{P}\right),\;\hat{P}=-i\hbar\frac{\partial}{\partial q},\;\hat{Q}=i\hbar\frac{\partial}{\partial p}$$

is linear in derivatives $\rightarrow \psi(q,p,t)=\sqrt{\rho(q,p,t)}$ obeys the same Liouville equation

$$i\frac{\partial \psi(q,p,t)}{\partial t}=\hat{L}\psi(q,p,t).$$

$$[\hat{q},\hat{P}]=i\hbar,\;[\hat{Q},\hat{p}]=i\hbar,\;i\hbar\frac{\partial \psi}{\partial t}=\hat{\mathcal{H}}\psi,\;\;\hat{\mathcal{H}}=\frac{\partial H}{\partial q}\,\hat{Q}+\frac{\partial H}{\partial p}\,\hat{P}.$$

The main idea:

modification of the commutation relations in the encompassing (in the Sudarshan sense) quantum system will alter classical dynamics in the classical subspace.

One-dimensional classical harmonic oscillator $H=\frac{1}{2m}(p^2+m^2\omega^2\,q^2)$. The quantum Hamiltonian

$$\mathcal{H}=\frac{1}{m}\left(\hat{p}\hat{P}+m^2\omega^2\hat{q}\hat{Q}\right)$$

In the Sudarshan-encompassing two-dimensional quantum system we can identify $x_1=q$, $x_2=Q$, $p_1=P$, $p_2=p$. Kempf et al. modification of the commutation relations

$$[\hat{q},\hat{P}]=i\hbar[1+\beta(\hat{p}^2+3\hat{P}^2)],\quad [\hat{q},\,\hat{p}]=[\hat{Q},\,\hat{P}]=i\hbar\,2\beta\hat{p}\hat{P},\quad [\hat{Q},\,\hat{p}]=i\hbar[1+\beta(3\hat{p}^2+\hat{P}^2)],\quad [\hat{q},\,\hat{Q}]=0,\quad [\hat{p},\,\hat{P}]=0.$$

Equations of motion (Heisenberg picture)

$$\frac{d\hat{q}}{dt}=\frac{\hat{p}}{m}\left[1+\beta\left(\hat{p}^2+5\hat{P}^2\right)\right],\quad \frac{d\hat{p}}{dt}=-m\omega^2\left\{\hat{q}\left[1+\beta(3\hat{p}^2+\hat{P}^2)\right]+2\beta\,\hat{p}\,\hat{P}\hat{Q}\right\},$$

$$\frac{d\hat{Q}}{dt}=\left[1+\beta\left(5\hat{p}^2+\hat{P}^2\right)\right]\frac{\hat{P}}{m},\quad \frac{d\hat{P}}{dt}=-m\omega^2\left\{\left[1+\beta\left(\hat{p}^2+3\hat{P}^2\right)\right]\hat{Q}+2\beta\,\hat{q}\hat{p}\hat{P}\right\}.$$

Hidden variables P and Q do appear in the equations of motions of the "classical" sector due to the Plank scale modification of the commutation relations!

If the effects of the hidden variables P and Q can be approximately discarded:

$$\dot{q}=\left(1+\beta p^2\right)\frac{p}{m},\quad \dot{p}=-m\omega^2\left(1+3\beta p^2\right)q.$$

$$\ddot{p}-\frac{6\beta p}{1+3\beta p^2}\,\dot{p}^2+\omega^2\left(1+\beta p^2\right)\left(1+3\beta p^2\right)p=0.$$

The quadratic Liénard type equation.

Variable mass Lagrangian system (in the p-space)

$$\mathcal{L} = \frac{1}{2}\mu(p)\dot{p}^2 - V(p)$$

$$\mu(p) = \frac{m}{(1+3\beta p^2)^2}, \quad V(p) = \frac{m\omega^2}{6}\left[p^2 + \frac{2}{3\beta}\ln\left(1+3\beta p^2\right)\right]$$

The corresponding conserved "energy" $\frac{1}{2}\mu(p)\dot{p}^2 + V(p)$ gives a first integral

$$\frac{1}{2}\frac{m}{(1+3\beta p^2)^2}\dot{p}^2 + \frac{m\omega^2}{6}\left[p^2 + \frac{2}{3\beta}\ln\left(1+3\beta p^2\right)\right] = m^2\omega^2 E,$$

where E is some constant.

Period of oscillations

$$T = 4\int_0^{p_0} \frac{dp}{\sqrt{\frac{2}{\mu}(m^2\omega^2 E - V(p))}} = \frac{4}{\omega}\int_0^{p_0} \frac{dp}{(1+3\beta p^2)\sqrt{2mE - \frac{1}{3}\left[p^2 + \frac{2}{3\beta}\ln(1+3\beta p^2)\right]}}.$$

At the first order in β, and assuming $\dot{p}(0) = 0$, $p(0) = p_0$ initial conditions, we have $E = \frac{p_0^2}{2m}(1-\beta p_0^2)$ and

$$T \approx \frac{4}{\omega}\int_0^{p_0} \frac{dp}{(1+3\beta p^2)\sqrt{(p_0^2-p^2)\left[1-\beta(p_0^2+p^2)\right]}} \approx \frac{4}{\omega}\int_0^{p_0} \frac{dp}{\sqrt{p_0^2-p^2}}\left[1-\frac{\beta}{2}(5p^2-p_0^2)\right] = \frac{2\pi}{\omega}\left(1-\frac{3\beta p_0^2}{4}\right).$$

At the final step, we have used elementary integrals

$$\int_0^{p_0} \frac{dp}{\sqrt{p_0^2-p^2}} = \frac{\pi}{2}, \quad \int_0^{p_0} \sqrt{p_0^2-p^2}\,dp = \frac{\pi p_0^2}{4}.$$

A canonical transformation

$$\hat{q} = \frac{1}{\sqrt{2}}(\hat{q}_1 - \hat{q}_2), \quad \hat{Q} = \frac{1}{\sqrt{2}}(\hat{q}_1 + \hat{q}_2), \quad \hat{p} = \frac{1}{\sqrt{2}}(\hat{p}_1 + \hat{p}_2), \quad \hat{P} = \frac{1}{\sqrt{2}}(\hat{p}_1 - \hat{p}_2).$$

Kempf et al. commutation relations (to first order in β):

$$[\hat{q}_i,\, \hat{q}_j] = 0, \quad [\hat{q}_i,\, \hat{p}_j] = i\hbar\left[1 + \beta(\hat{p}_1^2 + \hat{p}_2^2)\delta_{ij} + 2\beta\,\hat{p}_i\hat{p}_j\right], \quad [\hat{p}_i,\, \hat{p}_j] = 0.$$

An auxiliary "low energy momentum" operators $\hat{\pi}_i$ with the canonical commutation relations $[\hat{q}_i,\, \hat{\pi}_j] = i\hbar\delta_{ij}$, $[\hat{\pi}_i,\, \hat{\pi}_j] = 0$. "high energy momentum" operators:

$$\hat{p}_i = \hat{\pi}_i\left[1 + \beta(\hat{\pi}_1^2 + \hat{\pi}_2^2)\right].$$

The quantum Hamiltonian

$$\mathcal{H} = \mathcal{H}_1 - \mathcal{H}_2, \quad \mathcal{H}_i = \frac{1}{2m}\left[\hat{\pi}_i^2 + m^2\omega^2\hat{q}_i^2 + 2\beta\hat{\pi}_i^4\right].$$

Perturbed quantum oscillator

$$\hat{h} = \frac{1}{2}(\hat{\eta}^2 + \hat{\xi}^2) + \alpha\hat{\eta}^4 = \hat{h}_0 + \alpha\hat{\eta}^4,$$

in dimensionless variables

$$\hat{\xi} = \sqrt{\frac{m\omega}{\hbar}}\,\hat{q}, \quad \hat{\eta} = \sqrt{\frac{1}{\hbar m\omega}}\,\hat{\pi}, \quad \hat{h} = \frac{1}{\hbar\omega}\left(\frac{\hat{\pi}^2}{2m} + \frac{\beta}{m}\hat{\pi}^4 + \frac{1}{2}m\omega^2\hat{q}^2\right), \quad \alpha = \beta m\hbar\omega.$$

Unperturbed quantities are well known:

$$\psi_n^{(0)}(\xi) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\,e^{-\xi^2/2}H_n(\xi), \quad \epsilon_n^{(0)} = n + \frac{1}{2}$$

Corrections can be found by using the usual perturbation theory:

$$\epsilon_n^{(1)} = \frac{\alpha}{4}(6n^2 + 6n + 3)$$

$$|n^{(1)}> = \frac{\alpha}{4}\left[-\frac{1}{4}\sqrt{\frac{(n+4)!}{n!}}\,|(n+4)^{(0)}> + (2n+3)\sqrt{\frac{(n+2)!}{n!}}\,|(n+2)^{(0)}> - (2n-1)\sqrt{\frac{n!}{(n-2)!}}\,|(n-2)^{(0)}> + \frac{1}{4}\sqrt{\frac{n!}{(n-4)!}}\,|(n-4)^{(0)}>\right].$$

$$\epsilon_{n_1,n_2} = \epsilon_{n_1} - \epsilon_{n_2} = (n_1 - n_2)\left[1 + \frac{3\alpha}{2}(n_1 + n_2 + 1)\right], \quad |n_1, n_2 >= |n_1 > \otimes |n_2 >,$$

where $\epsilon_n = \epsilon_n^{(0)} + \epsilon_n^{(1)}$ and $|n >= |n^{(0)} > + |n^{(1)} >$. The general time-dependent solution of the corresponding Schrödinger equation is

$$\psi(\xi_1, \xi_2, t) = \sum_{n_1,\, n_2=0}^{\infty} b_{n_1 n_2}\, e^{-i\epsilon_{n_1,n_2}\omega t}\, \psi_{n_1 n_2}(\xi_1, \xi_2),$$

The expansion coefficients $b_{n_1 n_2}$ are determined by the initial wave function $\psi(\xi_1, \xi_2, 0)$:

$$b_{n_1 n_2} = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \psi_{n_1 n_2}(\xi_1, \xi_2)\, \psi(\xi_1, \xi_2, 0)\, d\xi_1 d\xi_2.$$

$$\psi(\xi_1,\xi_2,0)=\psi_{00}(\xi_1-\xi_0,\xi_2+\xi_0,0)=\psi_0(\xi_1-\xi_0)\psi_0(\xi_2+\xi_0),$$

with

$$\psi_0(\xi\pm\xi_0)=\frac{e^{-\frac{(\xi\pm\xi_0)^2}{2}}}{\pi^{1/4}}\left[1+\frac{\alpha}{64}\Big(24\,H_2(\xi\pm\xi_0)-H_4(\xi\pm\xi_0)\Big)\right].$$

The expansion coefficients:

$$b_{n_1 n_2}=\int\limits_{-\infty}^{\infty}\psi_{n_1}(\xi_1)\psi_0(\xi_1-\xi_0)d\xi_1\int\limits_{-\infty}^{\infty}\psi_{n_2}(\xi_2)\psi_0(\xi_2+\xi_0)d\xi_2\equiv b_{n_1}(\xi_0)b_{n_2}(-\xi_0),$$

where

$$\psi_n(\xi)=\frac{e^{-\xi^2/2}}{\sqrt{2^n n!\sqrt{\pi}}}\left[H_n(\xi)+\frac{\alpha}{4}\left(-\frac{1}{16}H_{n+4}(\xi)+\frac{2n+3}{2}H_{n+2}(\xi)-\frac{2(2n-1)n!}{(n-2)!}H_{n-2}(\xi)+\frac{n!}{(n-4)!}H_{n-4}(\xi)\right)\right].$$

$$\int\limits_{-\infty}^{\infty}e^{-\frac{\xi^2+(\xi-\xi_0)^2}{2}}H_n(\xi)H_m(\xi-\xi_0)\,d\xi=\sqrt{\pi}e^{-\frac{\xi_0^2}{4}}\sum_{k=0}^{\min(n,m)}\frac{2^k n!m!}{k!(n-k)!(m-k)!}\xi_0^{n-k}(-\xi_0)^{m-k}.$$

Koopman-von Neuman wave function

$$\psi(\xi_1,\xi_2,t)=\psi_{(\xi_0)}(\xi,t)\psi_{(-\xi_0)}(\xi,t)$$

$$b_n(\xi_0)=\frac{e^{-\xi_0^2/4}\xi_0^n}{\sqrt{2^n n!}}\left[1+\frac{\alpha}{32}\left(-\xi_0^4+12(n+2)\xi_0^2-12n(n+3)-16n(n-1)(n-2)\xi_0^{-2}\right)\right]\equiv b_n^{(0)}(\xi_0)+\alpha b_n^{(1)}(\xi_0).$$

$$\psi_{(\xi_0)}(\xi,t)=\sum_{n=0}^{\infty}b_n(\xi_0)e^{-in\omega t\left[1+\frac{3\alpha}{2}(n+1)\right]}\psi_n(\xi)\approx\psi^{(c)}_{(\xi_0)}(\xi,t)+\alpha\psi^{(nc)}_{(\xi_0)}(\xi,t)$$

$\psi^{(c)}_{(\xi_0)}(\xi,t)$ is a (generalized) coherent state:

$$\hat{a}|\xi_0(c)>=\frac{\xi_0}{\sqrt{2}}\,|\xi_0(c)>,\quad |\xi_0(c)>=\sum_{n=0}^{\infty}\frac{e^{-\xi_0^2/4}\xi_0^n}{\sqrt{2^n n!}}\,|n>.$$

$$\hat{a}=\hat{a}^{(0)}+\frac{\alpha}{4}\left(2\hat{a}^{(0)\,3}-6\hat{N}^{(0)}\hat{a}^{(0)\,+}+\hat{a}^{(0)\,+\,3}\right),\quad \hat{a}^{+}=\hat{a}^{(0)\,+}+\frac{\alpha}{4}\left(2\hat{a}^{(0)\,+\,3}-6\hat{a}^{(0)}\hat{N}^{(0)}+\hat{a}^{(0)\,3}\right),$$

P. Bosso, S. Das and R. B. Mann, Phys. Rev. D **96** (2017), 066008. P. Bosso and S. Das, Annals Phys. **396** (2018), 254.

$$<\xi> \approx <\xi_0(c)|\hat{\xi}|\xi_0(c)> + \alpha \left(<\xi_0(c)|\hat{\xi}|\xi_0(nc)> + <\xi_0(nc)|\hat{\xi}|\xi_0(c)> \right).$$

The first term conveniently can be calculated in the Heisenberg picture.

$$\hat{a}(t) = e^{i\omega t\hat{h}}\hat{a}(0)e^{-i\omega t\hat{h}} = \hat{a}(0) + i\omega t[\hat{h},\ \hat{a}(0)] + \frac{(i\omega t)^2}{2!}[\hat{h},\ [\hat{h},\ \hat{a}(0)]] + \frac{(i\omega t)^3}{3!}[\hat{h},\ [\hat{h},\ [\hat{h},\ \hat{a}(0)]]] + \cdots$$

$$\hat{h} = \hat{N} + \frac{1}{2} + \frac{3\alpha}{4}\left(2\hat{N}^2 + 2\hat{N} + 1\right)$$

$$\hat{a}(t) = \hat{a}(0)\, e^{-i\omega t\left(1+3\alpha\hat{N}\right)} = e^{-i\omega t\left[1+3\alpha\left(\hat{N}+1\right)\right]}\, \hat{a}(0).$$

For classical motion, we can neglect the difference between $\hat{N}$ and $\hat{N}+1$ operators ($\xi_0^2 \gg 1$) and freely commute $\hat{a}(0)$ and $e^{-i\omega t\left(1+3\alpha\hat{N}\right)}$, as well as $\hat{a}(0)$ (or $\hat{a}^+(0)$) and $\hat{N}$.

$$\hat{\xi} = \frac{1}{\sqrt{2}}\left(\hat{a}^{(0)} + \hat{a}^{(0)+}\right) = \frac{1}{\sqrt{2}}\left[\hat{a} + \hat{a}^+ - \frac{3\alpha}{4}\left(\hat{a}^3 - 2(\hat{N}\hat{a}^+ + \hat{a}\hat{N}) + \hat{a}^{+\,3}\right)\right],$$

$$<\xi_0(c)|\hat{\xi}|\xi_0(c)> = \xi_0\left(1 + \frac{3\alpha}{4}\,\xi_0^2\right)\cos\left[\omega t\left(1 + \frac{3\alpha}{2}\,\xi_0^2\right)\right] - \frac{3\alpha}{8}\,\xi_0^3\,\cos\left[3\,\omega t\left(1 + \frac{3\alpha}{2}\,\xi_0^2\right)\right].$$

We can assume $\alpha = 0$ in the $\psi^{(c)}_{(\xi_0)}(\xi, t)$ wave function:

$$\psi^{(c)}_{(\xi_0)}(\xi, t)\Big|_{\alpha=0} = \frac{e^{-(\xi_0^2-\xi_0^2(t))/4}}{\pi^{1/4}}\, e^{-\frac{1}{2}(\xi-\xi_0(t))^2},$$

$$\psi^{(nc)}_{(\xi_0)}(\xi, t) = \frac{e^{-(\xi_0^2-\xi_0^2(t))/4}}{32\,\pi^{1/4}}\, e^{-\frac{1}{2}(\xi-\xi_0(t))^2}\left(A_1\xi^4 + A_2\xi^3 + A_3\xi^2 + A_4\xi\right),$$

$$A_1 = -16\xi_0 e^{-3i\omega t}, \quad A_2 = 12\xi_0^2 e^{-2i\omega t}\left(2e^{-2i\omega t} - 1\right),$$
$$A_3 = 12\xi_0 e^{-i\omega t}\left[\xi_0^2\left(1 + e^{-2i\omega t} - e^{-4i\omega t}\right) + 2\left(e^{-2i\omega t} - 2\right)\right],$$
$$A_4 = -\xi_0^4\left(1 + 6e^{-2i\omega t} + 3e^{-4i\omega t} - 2e^{-6i\omega t}\right) + 2\xi_0^2\left(12 + 15e^{-2i\omega t} - 6e^{-4i\omega t}\right).$$

$$\alpha(<\xi_0(c)|\hat{\xi}|\xi_0(nc)> + \text{c.c.}) = \frac{-3\alpha}{8}\xi_0(4+\xi_0^2)\cos\omega t \approx -\frac{3\alpha}{8}\xi_0^3\cos\omega t \approx -\frac{3\alpha}{8}\xi_0^3\cos[\omega t(1+\frac{3\alpha}{2}\xi_0^2)]$$

Mean values - final results

$< \xi_2 >= - < \xi_1 >.$

$$< q >= \sqrt{\frac{2\hbar}{m\omega}}\, \xi_0 \left\{ \left(1 + \frac{3\alpha}{8}\, \xi_0^2\right) \cos\left[\omega t \left(1 + \frac{3\alpha}{2}\, \xi_0^2\right)\right] - \frac{3\alpha}{8}\, \xi_0^2 \cos\left[3\, \omega t \left(1 + \frac{3\alpha}{2}\, \xi_0^2\right)\right]\right\}.$$

$$< Q >= 0.$$

The effect of the Planck scale physics on the mean value of the classical variable q is twofold:

- a small admixture of the third-harmonic is excited.
- The period of oscillations is modified. $T = \frac{2\pi}{\omega}\left(1 - \frac{3\alpha}{2}\, \xi_0^2\right).$

The amplitude of the oscillations $q_m \approx \sqrt{\frac{2\hbar}{m\omega}}\, \xi_0$, the oscillator energy $E = \frac{1}{2}\, m\omega^2 q_m^2 \approx \hbar\omega\xi_0^2$ (thus the condition of classicality $\xi_0^2 \gg 1$ is the same as $E \gg \hbar\omega$), and maximum momentum

$$p_m^2 = 2mE = 2m\hbar\omega\xi_0^2 = \frac{2\alpha}{\beta}\, \xi_0^2.$$

Conclusions

- KvN mechanics may provide an innovative road to the Planck-scale deformed classical mechanics.
- From this perspective, Planck scale quantum gravity effects destroy classicality.
- This breakdown is controlled by a small dynamical parameter $\frac{p^2}{p_P^2}$ and can be neglected for all practical purposes.
- It may happen that the interrelations between quantum mechanics, classical mechanics and gravity are much more tight and intimate than anticipated.
- We believe the Koopman-von Neumann formulation of classical mechanics might be useful in investigating a twilight zone between quantum and classical mechanics. "It deserves to be better known among physicists, because it gives a new perspective on the conceptual foundations of quantum theory, and it may suggest new kinds of approximations and even new kinds of theories" (Frank Wilczek).