Information aspects of holographic models

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In collaboration with R. C. Rashkov and H. Dimov:

- Holographic systems as higher-order PU oscillators
  – Nucl. Phys. B 918 (2017) 317-336
- Strings on plane wave background
  – Phys. Rev. D 96, 126004 (2017)
- 3d warped black hole dual to 2d WCFTs
  – Phys. Rev. D 99 (2019) 126007

Outline

- Black hole thermodynamics
- Thermodynamic information geometry
- Applications to warped 3d TMG dual to warped 2d CFT

Black hole thermodynamics

Area law (Bekenstein-Hawking 1973-75):

$$S = k_B \frac{A}{4L_p^2} + corrections \tag{1}$$

The first law of thermodynamics:

$$d\Phi = TdS + \Xi_i dQ^i = I_a dE^a, \quad I_a = \frac{\delta\Phi}{\delta E^a}. \tag{2}$$

Thermal stability (P. Davies 1977):

$$C = T\frac{\partial S}{\partial T} \begin{cases} > 0, & stable, \\ < 0, & radiating\ (unstable), \\ = 0, & phase\ transitions, \\ \to \infty, & phase\ transitions \end{cases} . \tag{3}$$

Equilibrium state space

The space of extensive (intensive) parameters $\mathcal{E} := \{\Phi, E^a\}$ ($\mathcal{I} := \{\Phi, I^a\}$) becomes an equilibrium manifold if equipped with a Riemannian metric.

1. "Hessian thermodynamic metrics"
   (F. Weinhold 1975, G. Ruppeiner 1979)

2. "Legendre invariant metrics"
   (H. Quevedo 2006)

3. "The method of conjugate potentials"
   (B. Mirza, A. Mansoori 2014 & 2019)

Thermodynamic fluctuation theory

Consider an open finite volume system A enclosed by a larger thermal reservour. The system A exchanges energy through fluctuations. The microcanonical ensemble asserts that all accessible microstates of A occur with equal probability. Therefore, the probability of finding the internal energy $u = U/V$ per volume of A between u and $u + du$ is proportional to the number of microstates of A corresponding to this range:

$$P(u, V)du = C\Omega(u, V)du, \tag{4}$$

where Ω is the density of states, and C is a normalization factor. Boltzmanns expression for the entropy $S = k_B \ln \Omega$, yields Einsteins famous relation ($k_B = 1$):

$$P(u, V)du = Ce^{S(u,V)}du, \tag{5}$$

Thermodynamic fluctuation theory

More fluctuating variables $\vec{E} = (E^1, E^2, \ldots, E^n)$ lead to

$$P(\vec{E})d^nE = Ce^{S(\vec{E})}d^nE \tag{6}$$

Now, expand the entropy S up to quadratic terms in E^a:

$$S(E^a) - S_0 = \frac{V}{2}\frac{\partial^2 S}{\partial E^a \partial E^b}\Delta E^a \Delta E^b + \cdots$$

where $S_0 = S(\langle E^a \rangle)$ and $\Delta E^a = E^a - \langle E^a \rangle$. At equilibrium $\partial_a S = 0$ and S is maximized, thus $\partial_a \partial_b S < 0$. Define the quantity

$$g_{ab}^{(R)} = -\frac{\partial^2 S}{\partial E^a \partial E^b} = -\mathrm{Hess} S(\vec{E}) \tag{7}$$

Therefore one arrives at the Gaussian approximation:

$$P(\vec{E})d^nE = \frac{1}{2\pi}\exp\left(-\frac{V}{2}g_{ab}^{(R)}\Delta E^a \Delta E^b\right)\sqrt{|g|}d^nE \tag{8}$$

Hessian thermodynamic metrics

One can also calculate the averages:

$$\langle \Delta E^a \rangle = \int \Delta E^a P(\vec{E}) d^n E = 0 \tag{9}$$

$$\left\langle \Delta E^a \Delta E^b \right\rangle = \int \Delta E^a \Delta E^b P(\vec{E}) d^n E = \frac{g^{ab}}{V} \tag{10}$$

Ruppeiner information metric (G. Ruppeiner 1979):

$$g^{(R)}_{ab} = -\frac{\partial^2 S}{\partial E^a \partial E^b} = -\mathrm{Hess} S(\vec{E}) \tag{11}$$

Weinhold information metric (F. Weinhold 1975):

$$g^{(W)}_{ab} = \frac{\partial^2 U}{\partial E^a \partial E^b} = \mathrm{Hess} U(\vec{E}) \tag{12}$$

Validity of equilibrium description

Breakdown of the Gaussian approximation:

$$V < |R|, \quad R \sim \xi^d, \tag{13}$$

where V is the volume of the system, R is the thermodynamic scalar curvature, ξ is the corelation length, and d is the dimension of the system. For condensed matter systems at finite temperature it is known that:

1. Quasi-static processes = geodesics on $\mathcal{E}$.
2. The strength of interactions in the underlying statistical theory $\propto |R|$.
3. Type of inter-particle interactions is defined by $\mathrm{sign}(R)$.
4. Phase transitions = divergencies of R.

Legendre invariant metrics

- Consider $(2n+1)$ TD phase space $\mathcal{T}$ with coordinates $Z^A = (\Phi, I^a, E^a)$, $a = 1, \ldots, n$, where Φ is a TD potential.

- Select on $\mathcal{T}$ a non-degenerate Legendre invariant metric $G = G(Z^A)$ and a Gibbs 1-form $\Theta(Z^A)$, namely

$$G^{GTD} = \Theta^2 + (\xi_{ab} E^a I^b)(\eta_{cd} dE^c dI^d), \quad \Theta = d\Phi - \delta_{ab} I^a dE^b,$$

where δ_{ab} is the identity matrix, η_{ab} is the Minkowski metric, and ξ_{ab} is some constant tensor.

- Take the pullback $\phi^* : \mathcal{T} \to \mathcal{E}$ to find (H. Quevedo '17):

$$ds^2 = \left(\delta_{ac} \xi^{cb} E^a \frac{\partial \Phi}{\partial E^b} \right) \left(\eta^d_e \frac{\partial^2 \Phi}{\partial E^d \partial E^f} dE^e dE^f \right) \tag{14}$$

Conjugate thermodynamic potentials

For general black holes with $(m+1)$ TD variables, (S, Ξ_i), and Enthalpy potential, $\bar{M} = M - \Xi_i Q_i$, one can define the metric (B. Mirza, A. Mansoori '19):

$$\hat{g} = \text{blockdiag}\left(\frac{1}{T}\frac{\partial^2 M}{\partial S^2}, -\frac{1}{T}\frac{\partial^2 M}{\partial Y^i \partial Y^j}\right), \tag{15}$$

where $Y^i = (Q_1, \ldots, Q_m)$. Consider a configurational probability distribution given by the Gibbs distribution:

$$p(y|\lambda) = \frac{1}{Z} e^{-\beta H(y,\lambda)} = \frac{1}{Z} e^{-\lambda^i(t) X_i(y)} \tag{16}$$

One ca relate (15) to the covariance matrix in quantum thermodynamics,

$$\mathcal{G}_{ij} = \frac{\partial^2 \psi}{\partial \lambda^i \partial \lambda^j} = \langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \rangle \tag{17}$$

via the partition function Z in the corresponding ensemble:

Topological Massive Gravity (TMG)

Topological Massive Gravity (TMG):

$$I_{TMG} = \frac{1}{16\,\pi\,G}\int_{\mathcal{M}} d^3x\,\sqrt{-g}\,\left(R + \frac{2}{L^2}\right) + \frac{1}{\mu}\,I_{CS} + \int_{\partial\mathcal{M}} B$$

The $WAdS_3$ metric (D. Annios, W. Li, M. Padi, W. Song and A. Strominger 2009):

$$ds^2 = L^2(dt^2 + 2M(r)dtd\theta + N(r)d\theta^2 + D(r)dr^2) \qquad (19)$$

where

$$M(r) = \nu r - \frac{1}{2}\sqrt{r_+ r_-(\nu^2+3)} \qquad (20)$$

$$D(r) = \frac{1}{(\nu^2+3)(r-r_+)(r-r_-)} \qquad (21)$$

$$N(r) = M^2(r) - \frac{1}{4D(r)} \qquad (22)$$

Black hole charges and first law

$$dM = T\,dS + \Omega\,dJ \tag{23}$$

$$M = \frac{(\nu^2+3)}{24\,G}\,\left(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+\,r_-\,(\nu^2+3)}\right) \tag{24}$$

$$S = \frac{\pi\,L}{24\,\nu\,G}\,\left((9\,\nu^2+3)\,r_+ - (\nu^2+3)\,r_- - 4\,\nu\,\sqrt{(\nu^2+3)\,r_+\,r_-}\right) \tag{25}$$

$$T = \frac{\left(\nu^2+3\right)\,(r_+ - r_-)}{4\,\pi\,L\left(2\,\nu\,r_+ - \sqrt{(\nu^2+3)\,r_+\,r_-}\right)} \tag{26}$$

$$\Omega = \frac{2}{L\,\left(2\,\nu\,r_+ - \sqrt{(\nu^2+3)\,r_+\,r_-}\right)} \tag{27}$$

$$J = \frac{\nu\,L\,(\nu^2+3)}{96\,G}\,\left[\left(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+\,r_-\,(\nu^2+3)}\right)^2 + \cdots\right] \tag{28}$$

$\mathrm{WCFT}_2/\mathrm{WAdS}_3$ duality

In terms of the dual CFT temperatures and charges the entropy takes the Cardy form:

$$S = \frac{\pi^2 L}{3} \left(c_L T_L + c_R T_R \right) , \tag{29}$$

where c_L and c_R are the central charges given by

$$c_R = \frac{(5\,\nu^2 + 3)\, L}{\nu\,(\nu^2 + 3)} , \qquad c_L = \frac{4\,\nu\, L}{(\nu^2 + 3)}, \tag{30}$$

and T_L and T_R are the left and right temperatures

$$T_R = \frac{(\nu^2 + 3)\,(r_+ - r_-)}{8\,\pi\, L}, \tag{31}$$

$$T_L = \frac{(\nu^2 + 3)}{8\,\pi\, L} \left(r_+ + r_- - \frac{\sqrt{(\nu^2 + 3)\, r_+ r_-}}{\nu} \right) . \tag{32}$$

Stability constraints I

Direct (assuming unitarity):

$$0 < c_L < L, \quad L < c_R < 2\,L, \quad \frac{1}{2} < \frac{c_L}{c_R} < \frac{4}{5} \tag{33}$$

Non-extremality ($T > 0$):

$$c_L > \frac{2\,J}{3\,M^2} = \frac{2\,a}{3}, \quad a < \frac{3\,L}{2} \tag{34}$$

Local TD stabilty ($0 < C = T\partial S/\partial T$):

$$c_L < \frac{3\,S^2}{2\,\pi^2\,J} \tag{35}$$

Global TD stability (concavity of Gibbs free energy):

$$T_c = \frac{1}{\pi\,(c_L + \sqrt{c_L\,c_R})} \tag{36}$$

Stability constraints II

Thermodynamic geometry:

$$T < T_c \tag{37}$$

Coimplexity growth:

$$M \geq \frac{(2\,c_L - c_R)\,\sqrt{c_L}}{3\,\sqrt{12\,c_R - 15\,c_L}} \tag{38}$$

$$M \geq \frac{L}{3}\left(1 - \frac{4}{3+\nu^2}\right) \tag{39}$$

Logarithmic corrections ($S' = S + \alpha\,\log(C\,T^2)$):

$$J > \frac{27\,\left(1 - c_L\,\pi\,T\,\left(2 + (c_R - c_L)\,\pi\,T\right)\right)\alpha^2}{2\,c_L\,c_R^2\,\pi^4\,T^2}\,. \tag{40}$$

Further Aspects

- Non-equilibrium case
- Quantum entanglement
- Fisher information (quantum monotones)
- Quantum chaos
- Bulk reconstruction
- Complexity

Credits

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