Preface

The course "Introduction to Supergravity" is devoted to some basic notions of so called $N=1$, $D=4$ supergravity. This course can be also entitled as "Elementary Introduction to Supergravity" or as "Supergravity for Beginners." It means that I am not going to consider a modern progress, current development or various applications in cosmology or string theory. My aim is purely pedagogical introduction. It is assumed that the audience never studied supergravity before and now you face this subject for the first time.

However, I ~~take into account~~ assume that the audience is familiar with main notions of general relativity, group theory, Yang-Mills theory, Dirac equation and other basic notions of classical field theory. Nevertheless in prosess of consideration I will try to remind all this.

General motivations

Supersymmetry in physics means a hipotetical symmetry of Nature unifying the bosons and fermions. In its essence, the supersymmetry is an extension of special relativity symmetry. On can say that the supersymmetry is a special relativity symmetry completed by a symmetry between bosons and fermions. From this point of view a supergravity is a supersymmetric theory of gravity.

Main idea of supersymmetry in field theory can be explained as follows. Let us consider a model of field theory. Any such a model is given in terms of action functional depending on a set of bosonic fields $b(x)$ and fermionic fields $f(x)$. The action is $S[b,f]$. Consider the infinitesimal transformations of the form

$$b \to b + \delta b, \quad \delta b \sim f$$
$$f \to f + \delta f, \quad \delta f \sim b$$

Let the action is invariant under these transformations, $\delta S[b,f]=0$. Then the model under consideration is called supersymmetric. The transformation $\delta b \sim f$, $\delta f \sim b$ are called the supersymmetry transformation or supertransformations.

Of course, the above considerations look like very schematic and naive while we have no answers the questions:

1. What is an explicit set of the fields b and f for a concrete model?
2. What is an explicit for of the transformations $\delta b \sim f$, $\delta f \sim b$?
3. What is an explicit form of the invariant action $S[b, f]$?

Let us try to discuss qualitatively how it will be possible to answer these questions for supergravity. We say that the supergravity is supersymmetric theory of gravity. Conventional gravity theory is Einstein general relativity. In this case the dynamical variable is the metric $g_{mn}(x)$ $(m, n = 0,1,2,3)$ and the action is $S_E[g_{mn}] \sim \frac{1}{\varkappa^2}\int d^4x \sqrt{|g|}\, R$, where $g = \det g_{mn}$ and R is a scalar curvature. If to write $g_{mn} = \eta_{mn} + \varkappa h_{mn}$ with η_{mn} is a Minkowski metric, then we can treat general relativity as a theory of massless field h_{mn} in flat space. Such field has a spin (to be more precise, as we will see later helicity) equal to 2.

the supersymmetry connects the fields with spins s and $s \pm 1/2$. As a result, the simplest field content of the supergravity includes the fields with spins 2 and 3/2.

Lagrangian formulation of spin 3/2 field in flat space has been constructed by W. Rarita and J. Schwinger in 1941 in term of vector spinor $\psi_m(x)$. Since the gravity is manifestation of space-time curvature, the spin 3/2 field should be formulated in curved space-time. Therefore one can expect the the supergravity action should have the form

$$S_{SUGRA} = S_{Einstein} + S_{Rarita-Schwinger} + \ldots$$

Here $S_{Einstein}$ is the action of general relativity and $S_{Rarita-Schwinger}$ is the action of spin 3/2 field in curved space-time. Now let us pay attention that the spinors are defined by their transformation law under the Lorentz transformations. Hence, we immediately face a question about definition of spinor in curved space-time. It is natural to consider that in curved space the $\psi_m(x)$ is a vector under general coordinate transformations and spinor under local

Lorentz transformation. It leads to question that partial derivative of spinor $\partial_n \psi$ is not a spinor and we need a covariant derivative of spinor. Besides, it is natural to assume that there are some relations between general coordinate transformations and local Lorentz transformations. To construct such relations one introduce the tetrad $e^a{}_m(x)$. The tetrad is a four vectors, numerated by index a ($a = 0, 1, 2, 3$) and satisfying the normalization condition $e^a{}_m(x) e^b{}_n(x) g^{mn}(x) = \eta^{ab}$ where η^{ab} is an inverse Minkowski metric. Namely the tetrad is a basic object to formulate spinor field theory in curved space-time. Since the spin 3/2 field is formulated in curved space ~~with~~ with help of tetrad, we should reformulate the conventional gravity theory in terms of tetrad. As a result, one can expect the supergravity action in the form

$$S_{SUGRA} = S_{Einstein} + S_{Rarita-Schwinger} + \ldots$$

Here $S_{Einstein}$ is the action of general relativity in tetrad formalism and $S_{Rarita-Schwinger}$ is the space action of spin 3/2 field in curved space-time.

As we pointed out, the supersymmetry relates the boson and fermion fields we should find the transformations

$$\delta e^a_m \sim \psi_m, \quad \delta \psi_m \sim e^a_m$$

and check that the action S_{SUGRA} is invariant under these transformations.

Now a little bit about history. Supersymmetry has been proposed in by Yu. A. Gol'fand and E. P. Likhtman from Lebedev Physical Institute in Moscow. They developed what is now called superalgebra and constructed first supersymmetric field model. Then supersymmetry has been proposed again in some another form in 1972 by D. V. Volkov and V. P. Akulov from Khar'kov Institute of Physics and Technology. In 1974 J. Wess and B. Zumino from Karlsruhe University and CERN respectively started a systematic construction and study the supersymmetric field models. After that, a number of papers on supersymmetry became to increase as a rolling snow ball. All of the above concerns to four-dimensional supersymmetry. Another type of supersymmetry has been proposed in context of string theory. It was two-dimensional supersymmetry discovered by P. Ramond, A. Neveu, J. Schwarz, J. Gervais, B. Sakita in 1971. As to the supergravity, the pioneer paper have been published by D. Z. Freedman, P. van Nieuwenhuizen, S. Ferrara and independently by S. Deser and B. Zumino in 1976. In the first of these papers the authors started with some action S_{SUGRA} and proposed the supersymmetry transformations $\delta e^a_m \sim \psi_m$, $\delta\psi_m \sim e^a_m$. The main problem was to prove that action is invariant. It was very tedious and technically complicated task, there should be cancellation of about 2000 terms. It was done. In the second paper the authors used the first order formalism and simplified the calculations. The most symmetric formulation of any supersymmetric theory is given, where it is possible, in terms of unconstrained superfields. In such a formulation, no need to check invariance of action under the supersymmetry transformations. The supersymmetry is manifest. There were several attempts to develop the superfield formulation for $N=1$, $D=4$ supergravity. First successful results in this direction have been obtained in 1979, 1980 independently by S. J. Gates, W. Siegel from one side and V. I. Ogievetsky and E. Sokatchev from another side. I did not plan to discuss this approach but I think it is important to say about this.

1. Lorentz and Poincare Groups

1.1 Basic definitions

Consider the four-dimensional Minkowski space with coordinates x^a $(a=0,1,2,3)$ and metric

$$ds^2 = \eta_{ab}\, dx^a dx^b \tag{1.1}$$

where $\eta_{ab} = diag(-1,1,1,1)$. It is easy to see that the form of metric (1.1) is invariant under the following linear inhomogeneous transformations

$$x'^a = \Lambda^a{}_b\, x^b + a^a \tag{1.2}$$

Here a^a is a constant four dimensional vector and $\Lambda = (\Lambda^a{}_b)$ is a constant matrix satisfying the condition

$$\eta_{ab} \Lambda^a{}_c \Lambda^b{}_d = \eta_{cd} \tag{1.3}$$

or

$$\Lambda^T \eta \Lambda = \eta$$

where Λ^T is a transpose matrix with elements $(\Lambda^T)_b{}^a = \Lambda^a{}_b$.

The transformations (1.2) with matrix Λ satisfying (1.3) are called the inhomogeneous Lorentz transformations. Further we denote the transformations (1.2) as (a, Λ).

A set of transformations (a, Λ) has two evident subsets

<u>a</u> The transformations (a, I), where I is unite 4×4 matrix with elements $\delta^a{}_b$. It corresponds to ~~space-time~~ coordinate transformation

$$x'^a = x^a + a^a \qquad (1.4)$$

These transformations are called the space-time translations.

<u>b</u> The transformations $(0, \Lambda)$ ~~realize the~~ correspond to coordinate transformations

$$x'^a = \Lambda^a{}_b\, x^b \qquad (1.5)$$

where Λ satisfies (1.3). These transformations are called the Lorentz rotations or homogeneous Lorentz transformations.

Let us consider two inhomogeneous transformations (a_1, Λ_1) and (a_2, Λ_2) one after another. It leads to relation

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (\Lambda_2 a_1 + a_2, \Lambda_2 \Lambda_1) \qquad (1.6)$$

Here $\Lambda_2 a_1 + a_2$ means the constant vector $\Lambda_2{}^a{}_b\, a_1{}^b + a_2^a$ and $\Lambda_2 \Lambda_1$ is the matrix with elements $\Lambda_2{}^a{}_c\, \Lambda_1{}^c{}_b$. One can prove, if the matrices Λ_1, Λ_2 satisfy the relation (1.3) their product $\Lambda_2 \Lambda_1$ satisfies this relation as well. Therefore $(\Lambda_2 a_1 + a_2, \Lambda_2 \Lambda_1)$ is again inhomogeneous Lorentz transformation.

It is easy to see that the set of transformations (a, Λ) forms a group where a multiplication law is given by (1.6). Here the unit group element is $(0, I)$ and the element inverse to element (a, Λ) is $(-\Lambda^{-1}a, \Lambda^{-1})$ with Λ^{-1} be the inverse matrix, satisfying the relation

$$\Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = I \qquad (1$$

The relation (1.3) garantees that the inverse matrix exists. This group is called the Poincare group. One can show that the subset $(0, \Lambda)$ forms a group which is called the Lorentz group. Subset of elements (a, I) also forms a group which is the translation group.

Further we will use only the infinitesimal forms of homogeneous and inhomogeneous Lorentz transformations. In this case

$$\Lambda = 1 + \omega$$

where $\omega = (\omega^a{}_b)$ is a matrix with infinitesimal elements. In this case the relation (1.3) gives us

$$\omega_{ab} = -\omega_{ba} \qquad (1.8)$$

where $\omega_{ab} = \eta_{ac}\omega^c{}_b$.

Therefore, the Lorentz group is six-parametric Lie group. Consider the Poincare group.

with elements (a, Λ) where a^a is an infinitesimal vector and $\Lambda = I + \omega$. It means that the Poincare group is ten-parametric Lie group.

1.2. Two-component spinors

Consider again the relation (1.3). It leads to relations

det $\Lambda = \pm 1$, sign $\Lambda^0{}_0 = \pm 1$

One can prove that subset $(0, \Lambda)$ with det $\Lambda = 1$, $\Lambda^0{}_0 > 0$ forms a group which is called the proper Lorentz group and denoted $L^\uparrow_+$. It is evident that proper Lorentz group is a subset of Lorentz group.

Further we will show that the proper Lorentz group allows us to introduce the specific objects which are called the two-component spinors.

First of all we consider a set of 2×2 complex matrices N with unit determinant, det $N = 1$. It is easy to see that the set of these matrices forms a group, where the multiplication law is defined as an ordinary matrix product. This group is called the two-dimensional ~~special~~ complex special linear group and is denoted as $SL(2|\mathbb{C})$. ~~[illegible]~~ One can show that for each matrix $N \in SL(2|\mathbb{C})$ there exists the matrix $\Lambda \in L_+^\uparrow$ such that

a $\Lambda(N_1 N_2) = \Lambda(N_1)\Lambda(N_2)$

b $\Lambda(N_1) = \Lambda(N_2)$ if and only if $N_1 = \pm N_2$.

1. Consider the linear space of Hermitean 2×2 matrices X, $X^+ = X$. The basis in this space can be taken as follows

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (1.9)$$

In this case, the arbitrary matrix X is written as

$$X = x^a \sigma_a \qquad (1.10)$$

where x^a are the real numbers. Also ~~~~ introduce the matrices $\tilde{\sigma}_a = (\sigma_0, -\vec{\sigma})$. Then one can show that

$$\mathrm{tr}\, \tilde{\sigma}^a \sigma_b = -2\, \delta^a{}_b \qquad (1.11)$$

Using the relations (1.10), (1.11) one obtains

$$x^a = -\frac{1}{2} \mathrm{tr} (\tilde{\sigma}^a X) \qquad (1.12)$$

2. Let $N \in SL(2|\mathbb{C})$. Consider the transformation

$$X' = N X N^+ \qquad (1.13)$$

Since $\det N = 1$, one gets

$$\det X' = \det X \qquad (1.14)$$

3. Using (1.12), (1.14) one obtains

$$x'^a = -\frac{1}{2} \mathrm{tr}\, \tilde{\sigma}^a X' = -\frac{1}{2} \mathrm{tr} (\tilde{\sigma}^a N X N^+) =$$

$$= -\frac{1}{2} \mathrm{tr} (\tilde{\sigma}^a N \sigma_b N^+) x^b = \Lambda^a{}_b(N) x^b \qquad (1.15)$$

where

$$\Lambda^a{}_b(N) = -\frac{1}{2} \mathrm{tr} (\tilde{\sigma}^a N \sigma_b N^+) \qquad (1.16)$$

4. Using the explicit form of the basic matrices (1.9) and relation (1.10) one can write

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} \qquad (1.17)$$

Therefore

$$\det X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = \eta_{ab}\, x^a x^b \qquad (1.18)$$

The same consideration leads to

$$\det X' = \eta_{ab}\, x'^a x'^b$$

Since $\det X' = \det X$, one gets

$$\eta_{ab}\, x'^a x'^b = \eta_{cd}\, x^c x^d$$

Substituting the relation (1.15) one obtains

$$\Lambda^T(N)\, \eta\, \Lambda(N) = \eta \qquad (1.19)$$

It means that matrices $\Lambda(N)$ (1.16) belong to Lorentz group

5. One can show that

$$\Lambda^0{}_0(N) > 0, \qquad \det \Lambda(N) = 1$$

Therefore, the matrices $\Lambda(N)$ (1.16) belong to proper Lorentz group.

As a result, the proper Lorentz group is associated with $SL(2,\mathbb{C})$ group and vice versa.

Let us return back to matrices $N \in SL(2|C)$. These matrices act in two-dimensional complex vector space. Denote the such vectors as φ_α $(\alpha = 1,2)$. Action of matrices N on φ_α looks like (1.20)

$$\varphi'_\alpha = N_\alpha{}^\beta \varphi_\beta$$

Since each matrix $N \in SL(2|C)$ is associated with some matrix $\Lambda \in L_+^\uparrow$ one can say that (1.20) is transformation law of the two-dimensional complex vectors under the Lorentz transformations. The vectors φ_α with the transformation law (1.20) are called the left Weyl spinors.

Let us consider the matrix $\varepsilon_{\alpha\beta}$ of the form

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (1.21)$$

and inverse matrix

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (1.22)$$

One can prove the following identities

$$N \varepsilon N^T = \varepsilon, \qquad N^T \varepsilon^{-1} N = \varepsilon^{-1} \qquad (1.23)$$

That means that the matrices ε, ε^{-1} are the invariant objects of the $SL(2|C)$ group. These matrices are used to rise and lower the spinor indices

$$\varphi^\alpha = \varepsilon^{\alpha\beta} \varphi_\beta, \qquad \varphi_\alpha = \varepsilon_{\alpha\beta} \varphi^\beta \qquad (1.24)$$

Moreover, one can show that the quantity $\varphi_1^{\alpha}\varphi_{2\alpha}$ is the Lorentz invariant

$$\varphi'^{\alpha}_1 \varphi'_{2\alpha} = \varphi^{\alpha}_1 \varphi_{2\alpha} \qquad (1.25)$$

where φ'_α is given by (1.20).

Let $N \in SL(2|\mathbb{C})$ and let N^* is the conjugate matrix. We denote the elements of this matrix as $(N^*)_{\dot\alpha}{}^{\dot\beta}$ $(\dot\alpha, \dot\beta = \dot 1, \dot 2)$. The matrix N^* acts in complex vector space of the two-dimensional vectors $\chi_{\dot\alpha}$ by the rule

$$\chi'_{\dot\alpha} = (N^*)_{\dot\alpha}{}^{\dot\beta} \chi_{\dot\beta} \qquad (1.26)$$

The vectors $\chi_{\dot\alpha}$ with the transformation law (1.26) are called the right Weyl spinors.

Analogously to previous discussion, one introduces the matrices $\varepsilon_{\dot\alpha\dot\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$ of the form

$$\varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (1.27)$$

these matrices are the invariant quantities of the $SL(2|\mathbb{C})$ group and used to rise and lower the dotted indices

$$\chi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\chi_{\dot\beta}, \qquad \chi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\chi^{\dot\beta} \qquad (1.28)$$

Besides, one can prove the relations

$$\chi'_{1\dot\alpha}\chi'^{\dot\alpha}_2 = \chi_{1\dot\alpha}\chi^{\dot\alpha}_2$$

wher $\chi'_{\dot\alpha}$ is given by (1.26). Hence, the quantity $\chi_{1\dot\alpha}\chi_2^{\dot\alpha}$ is the Lorentz invariant.

Since the dotted spinors transform with help of conjugate matrix N^* we can define a conjugation for the spinors as follows

$$(\varphi_\alpha)^* = \bar\varphi_{\dot\alpha} \qquad (1.29)$$

Consider the matrices σ_a (1.9). Their matrix elements are denoted as $(\sigma_a)_{\alpha\dot{\beta}}$. Also we introduce the matrix with upper indices

$$(\sigma_a)^{\alpha\dot{\alpha}} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma_a)_{\beta\dot{\beta}} \equiv (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} \qquad (1.30)$$

One can show that $\tilde{\sigma}_a$ are just the matrices $\tilde{\sigma}_a = (\sigma_0, -\vec{\sigma})$ which have been introduced before.

The matrices σ_a, $\tilde{\sigma}_a$ satisfy the many identities, for example

$$(\sigma_a\tilde{\sigma}_b + \sigma_b\tilde{\sigma}_a)_\alpha{}^\beta = -2\eta_{ab}\,\delta_\alpha{}^\beta$$
$$(\tilde{\sigma}_a\sigma_b + \tilde{\sigma}_b\sigma_a)^{\dot{\alpha}}{}_{\dot{\beta}} = -2\eta_{ab}\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} \qquad (1.31)$$

These identities are proved using the explicit forms of the matrices σ_a, $\tilde{\sigma}_a$. Also one can prove that the quantities

$$v^a = (\bar{\chi}_1\tilde{\sigma}^a\varphi_2) \equiv \bar{\chi}_{1\dot{\alpha}}(\tilde{\sigma}^a)^{\dot{\alpha}\alpha}\varphi_{2\alpha}$$

$$u_a = (\varphi_1\sigma_a\bar{\chi}_2) \equiv \varphi_1{}^\alpha(\sigma_a)_{\alpha\dot{\alpha}}\bar{\chi}_2{}^{\dot{\alpha}}$$

are the contravariant and covariant vectors under the Lorentz transformations respectively.

Consider the infinitesimal form of the transformations (1.20), (1.26). Let us write $N = E + T$ where E is 2×2 unit matrix and T is 2×2 matrix with infinitesimal elements. Since $\det N = 1$, then $\operatorname{tr} T = 0$. Therefore $T = \sum_{i=1}^{3} z_i\sigma_i$ with z_1, z_2, z_3 are complex coefficients. This relation can be identically rewritten as follows

$$T = -\frac{i}{2}\omega^{ab}(i\sigma_{ab})$$

where

$$\sigma_{ab} = \frac{1}{4}(\sigma_a\tilde{\sigma}_b - \tilde{\sigma}_b\sigma_a) \qquad (1.32)$$

and $\omega^{ab} = -\omega^{ba}$ are six real numbers.

Here

$$z_1 = -\frac{1}{2}(\omega^{01} + i\omega^{23})$$

$$z_2 = -\frac{1}{2}(\omega^{02} + i\omega^{31})$$

$$z_3 = -\frac{1}{2}(\omega^{03} + i\omega^{12})$$

We expressed three complex numbers z_1, z_2, z_3 through six real numbers ω^{ab}. Therefore

$$\delta\varphi_\alpha = -\frac{i}{2}\omega^{ab}(i\sigma_{ab})_\alpha{}^\beta\varphi_\beta \qquad (1.3$$

Analogously

$$\delta\chi^{\dot\alpha} = -\frac{i}{2}\omega^{ab}(i\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}\chi^{\dot\beta} \qquad (1.34$$

where

$$\tilde{\sigma}_{ab} = \frac{1}{4}(\tilde{\sigma}_a\sigma_b - \tilde{\sigma}_b\sigma_a) \qquad (1.3$$

The relations (1.33), (1.34) define the transformations of the two-component spinors under the infinitesimal Lorentz transformations. The matrices $i\sigma_{ab}$, $i\tilde{\sigma}_{ab}$ are called the generators of Lorentz transformations of ~~dotted~~ undotted and dotted spinors.

1.3 Four-component spinors

Taking into account the two-component spinors φ_α, $\chi^{\dot\alpha}$ we introduce the four-component quantity

$$\Psi = \begin{pmatrix}\varphi_\alpha \\ \chi^{\dot\alpha}\end{pmatrix} \qquad (1.$$

The components of Ψ will be ~~denoted~~ numerated by ~~ind~~ index $A = 1, 2, 3, 4$. The four-component column Ψ (1.36) is called the Dirac spinor.

Consider the transformation of Ψ under the Lorentz transformations. Using the relations (1.33), (1.34) one gets

$$\delta\Psi = -\frac{i}{2}\omega^{ab}\begin{pmatrix} i(\sigma_{ab})_{\alpha}{}^{\beta}\varphi_{\beta} \\ (i\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\chi^{\dot{\beta}} \end{pmatrix}$$

This relation can be written in the form

$$\delta\Psi = -\frac{i}{2}\omega^{ab}\Sigma_{ab}\Psi \qquad (1.37)$$

where

$$\Sigma_{ab} = \frac{i}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a) \qquad (1.38)$$

Here γ_a are the 4×4 matrices

$$\gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{pmatrix} \qquad (1.39)$$

It is easy to check, using the relations (1.31), that the matrices γ_a satisfy the relation

$$\gamma_a\gamma_b + \gamma_b\gamma_a = -2\eta_{ab}I \qquad (1.40)$$

where I is the unit 4×4 matrix. The matrices γ_a, satisfying the relation (1.40) are called the Dirac matrices.

Let Ψ is the Dirac spinor, one defines the Dirac conjugate spinor $\bar{\Psi}$ by the rule

$$\bar{\Psi} = \Psi^{+}\gamma_0 \qquad (1.41)$$

Using the relations (1.36), (1.39) one gets the expression for $\bar{\Psi}$ in terms of two-component spinors

$$\bar{\Psi} = (\bar{\chi}^{\alpha}, \bar{\varphi}_{\dot{\alpha}}) \qquad (1.41)$$

where $\bar{\varphi}_{\dot{\alpha}} = (\varphi_{\alpha})^{*}$, $\bar{\chi}^{\alpha} = (\chi^{\dot{\alpha}})^{*}$.

Consider the Dirac spinor Ψ (1.36). Here $\chi^{\dot{\alpha}}$ is completely independent on φ_{α}. Therefore the spinor Ψ (1.36) has 8 independent real components. However we can consider a special Dirac spinor where $\chi^{\dot{\alpha}} = \bar{\varphi}^{\dot{\alpha}} = (\varphi^{\alpha})^{*}$. In this case the spinor has 4 real independent components

$$\Psi = \begin{pmatrix} \varphi_{\alpha} \\ \bar{\varphi}^{\dot{\alpha}} \end{pmatrix} \qquad (1.42)$$

Such a spinor is called the Majoran spinor.

Let $\Psi(x)$ is the Dirac spinor field. This field satisfies some equation which is called the Dirac equation

$$i\gamma^{a}\partial_{a}\Psi + m\Psi = 0 \qquad (1.43)$$

where m is mass parameter. There

simple enough arguments leading to this equation. Equation for ψ can be constructed only from derivatives ∂_a and matrices σ^a, $\tilde{\sigma}^a$ related to Lorentz group. The simplest possibility looks like

$$i(\tilde{\sigma}^a)^{\dot{\alpha}\alpha}\partial_a\varphi_\alpha + m\,\chi^{\dot{\alpha}} = 0$$
$$i(\sigma^a)_{\alpha\dot{\alpha}}\partial_a\chi^{\dot{\alpha}} + m\,\varphi_\alpha = 0 \qquad (1.44)$$

If to express $\chi^{\dot{\alpha}}$ from first equation and substitute to second we obtain

$$\Box\varphi_\alpha - m^2\varphi_\alpha = 0, \qquad \Box\chi^{\dot{\alpha}} - m^2\chi^{\dot{\alpha}} = 0$$

Each component of φ_α and $\chi^{\dot{\alpha}}$ satisfies Klein-Gordon equation what consistent with relativistic equation $p^2 + m^2 = 0$ for particle 4-momentum. Now if to introduce the four-component spinor ψ and Dirac matrices, then the equations (1.44) are written in form of (1.43).

2. Notion on Irreducible Representations of The Poincare Group

2.1. Poincare algebra

First of all I would like to remind a definition of group representation. Let G is a group with elements $g_1, g_2, \ldots$. Representation of group G is a map ~~on~~ of this group into group of linear operators acting in some vector space under the following conditions

$$g \to R(g)$$
$$g_1 g_2 \to R(g_1 g_2) = R(g_1) R(g_2)$$
$$g^{-1} \to R(g^{-1}) = R^{-1}(g)$$

We will consider the unitary representations of Poincare group where the multiplication law is defined as

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (\Lambda_2 a_1 + a_2, \Lambda_2 \Lambda_1) \qquad (2.1)$$

In particular

$$(a, \Lambda) = (a, I)(0, \Lambda) \qquad (2.2)$$

Let $U(a, \Lambda)$ is the operator of the representation ~~[illegible]~~ According to definition of representation one can write

$$U(a_2, \Lambda_2) U(a_1, \Lambda_1) = U(\Lambda_2 a_1 + a_2, \Lambda_2 \Lambda_1) \qquad (2.3)$$

Taking into account (2.2) we gets

$$\mathcal{U}(a,\Lambda) = \mathcal{U}(a,I)\,\mathcal{U}(0,\Lambda) = \mathcal{U}(a)\,\mathcal{U}(\Lambda) \quad (2.4)$$

Since (a, I) is element of translation group, $\mathcal{U}(a)$ is a representation of translation group. Analogously $\mathcal{U}(\Lambda)$ is a representation of Lorentz group. In vicinity of unit element, the operators of representation are written in the form

$$\mathcal{U}(a) = e^{i a^{a} P_{a}}$$
$$\mathcal{U}(\Lambda) = e^{-\frac{i}{2}\omega^{ab} J_{ab}} \quad (2.5)$$

Here a^a, ω^{ab} are the parameters of infinitesimal translations and Lorentz rotations respectively. The operators P_a и J_{ab} are called the generators of translations and Lorentz rotations. Our aim is to find the commutators among these operators.

There is a general algebraic method to calculate such commutators. Using the relations (2.3), (2.4), (2.5) we gets

$$\mathcal{U}(a_2)\,\mathcal{U}(\Lambda_2)\,\mathcal{U}(a_1)\,\mathcal{U}(\Lambda_1) = \mathcal{U}(\Lambda_2 a_1 + a_2)\;\mathcal{U}(\Lambda_1\Lambda_2)$$

$$e^{i a_2^{a} P_{a}}\, e^{-\frac{i}{2}\omega_2^{ab} J_{ab}}\, e^{i a_1^{a} P_{a}}\, e^{-\frac{i}{2}\omega_1^{ab} J_{ab}} =$$
$$= e^{i a_3^{a} P_{a}}\, e^{-\frac{i}{2}\omega_3^{ab} J_{ab}} \quad (2.6)$$

where $a_3^{a} = \Lambda_2{}^{a}{}_{b} a_1{}^{b} + a_2{}^{a}$ and $\omega_3{}^{ab}$ are the parameters corresponding to $\Lambda_2 \Lambda_1$. Finding the commutators on the base of (2.6) is tedious and long ~~work~~ work and I am not going to do that. Instead, we consider one example giving true result.

Let $t^{a}(x)$ is a contravariant vector field under the transformation law

$$t'^{a}(x') = \frac{\partial x'^{a}}{\partial x^{b}} t^{b}(x)$$

where $x'^{a} = \Lambda^{a}{}_{b} x^{b} + a^{a} = x^{a} + \omega^{a}{}_{b} x^{b} + a$

where the a^{a} and $\omega^{a}{}_{b}$ are the infinitesim parameters of inhomogeneous Lorentz transformations. Then

$$t'^{m}(x + \omega x + a) = (\delta^{a}{}_{b} + \omega^{a}{}_{b}) t^{b}(x)$$

Or

$$t'^{a}(x) + a^{b} \partial_{b} t^{a}(x) + \omega^{b}{}_{c} x^{c} \partial_{b} t^{a}(x) =$$

$$= t^{a}(x) + \omega^{a}{}_{b} t^{b}(x)$$

Denote $\delta t^{a}(x) = t'^{a}(x) - t^{a}(x)$. Then

$$\delta t^{a}(x) = - a^{b} \partial_{b} t^{a} - \omega^{b}{}_{c} x^{c} \partial_{b} t^{a} + \omega^{a}{}_{b} t^{b}$$

This relation can be written as follows

$$\delta t^a(x) = i a^c (P_c)^a{}_b t^b - \frac{i}{2}\omega^{cd}(J_{cd})^a{}_b t^b \qquad (2.7)$$

where

$$(P_c)^a{}_b = \delta^a_b (i\partial_c) \qquad (2.8)$$

$$(J_{cd})^a{}_b = (L_{cd})^a{}_b + (S_{cd})^a{}_b \qquad (2.9)$$

$$(L_{cd})^a{}_b = x_c (P_d)^a{}_b - x_d (P_c)^a{}_b$$

$$(S_{cd})^a{}_b = i(\delta^a_c \eta_{db} - \delta^a_d \eta_{cb})$$

The operators P_c, J_{cd} are called the generators of translations and Lorentz rotations in contravariant vector representation.

Now we can calculate the commutators among the operators P_c, J_{cd}

$$[P_c, P_d] = 0$$

$$[J_{bc}, P_a] = i(\eta_{ac} P_b - \eta_{ab} P_c) \qquad (2.10)$$

$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac})$$

The relations (2.10) define the Lie algebra of the Poincare group. Usually this algebra is called the Poincare algebra. The last line in (2.10) defines the Lie algebra of the Lorentz group.

We derived the Poincare algebra on the basis of concrete example. Actually the same result will be obtained ~~[illegible]~~ on the based of general relation (2.6) without use of concrete form of generators.

2.2. Brief survey of irreducible representations of the Poincare group.

Consider the Poincare algebra (2.10). One can show that this algebra has two operators commuting with all generators P_a, J_{ab}. Such operators are called the Casimir operators. In the case under consideration they have the following form

$$C_1 = -p^a p_a \qquad (2.11)$$

$$C_2 = W^a W_a$$

with

$$W_a = \frac{1}{2} \varepsilon_{abcd} J^{bc} p^d \qquad (2.12)$$

Basis of space of irreducible representation can be constructed as the eigen vectors of the Casimir operators. Among the different irreducible representations of the Poincare group there are two physically accepted representations. They are called the massive and massless respectively. The basis vectors of massive representations $|p, m, s\rangle$ are defined by the equations

$$P_a|p,m,s\rangle = p_a|p,m,s\rangle$$
$$P^a P_a|p,m,s\rangle = -m^2|p,m,s\rangle \qquad (2.13)$$
$$W^a W_a|p,m,s\rangle = m^2 s(s+1)|p,m,s\rangle$$

Here $s = 0, \frac{1}{2}, 1, 3/2, \ldots$. The parameter s is called spin. The parameter m plays a role of mass and must be positive from physical point of view. For each fixed value of s, ~~different~~ dimension of representation is 2s+1.

The basis vectors of massless irreducible representations $|p, \lambda\rangle$ are defined by the following equations

$$P_a|p,\lambda\rangle = p_a|p,\lambda\rangle \qquad (2.14)$$
$$P^a P_a|p,\lambda\rangle = 0$$
$$W^a W_a|p,\lambda\rangle = 0$$

One can prove that $W_a = \lambda P_a$. The parameter λ takes the values $\lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$. This parameter is called helicity. Sometimes the value $|\lambda|$ is called the spin of massless particle.

One can show that the Poincare algebra can be realized in linear space of tensor fields $\varphi_{a_1 a_2 \ldots a_n}(x)$ or tensor spinor fields $\psi_{a_1 \ldots a_n}(x)$.

Such fields are called the relativistic fields with given mass and spin. In case of even spin $s=n$ the relativistic fields are defined by the equations

$$\begin{aligned} &\varphi_{\alpha_1 \dots \alpha_n}(x) = \varphi_{(\alpha_1 \dots \alpha_n)}(x) \\ &(\Box - m^2)\, \varphi_{\alpha_1 \dots \alpha_n}(x) = 0 \\ &\partial^{\alpha_1} \varphi_{\alpha_1 \alpha_2 \dots \alpha_n}(x) = 0 \\ &\eta^{\alpha_1 \alpha_2} \varphi_{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n}(x) = 0 \end{aligned} \qquad (2.15)$$

In case of half integer spin $s=n+\frac{1}{2}$ the relativist fields are defined by the equation

$$\begin{aligned} &\psi_{\alpha_1 \dots \alpha_n}(x) = \psi_{(\alpha_1 \dots \alpha_n)}(x) \\ &(i\gamma^{\alpha} \partial_{\alpha} + m)\, \psi_{\alpha_1 \dots \alpha_n}(x) = 0 \\ &\partial^{\alpha_1} \psi_{\alpha_1 \alpha_2 \dots \alpha_n}(x) = 0 \\ &\gamma^{\alpha_1} \psi_{\alpha_1 \alpha_2 \dots \alpha_n}(x) = 0 \end{aligned} \qquad (2.16)$$

The massless relativistic fields are defined by the equations (2.15), (2.16) where the mass parameter $m=0$.

2.3. Rarita - Schwinger equation

According to relations (2.16) the spin $\frac{3}{2}$ field is described by vector spinor $\psi_{\alpha}(x)$

satisfying the relations

$$(i\gamma^{b}\partial_{b} + m)\psi_{a}(x) = 0$$
$$\partial^{a}\psi_{a}(x) = 0 \qquad (2.17)$$
$$\gamma^{a}\psi_{a}(x) = 0$$

Our aim to derive one linear equation which has the relations (2.17) as its consequences.

Most general relativistic covariant first order equation for field ψ_m is written as follows

$$(i\gamma^{b}\partial_{b} + m)\psi_{a} + a\,\gamma_{a}(i\partial^{b}\psi_{b}) +$$
$$+ b\,i\partial_{a}(\gamma^{b}\psi_{b}) + c\,\gamma_{a}(i\gamma^{b}\partial_{b})(\gamma^{c}\psi_{c}) +$$
$$+ d\,m\,\gamma_{a}(\gamma^{b}\psi_{b}) = 0 \qquad (2.18)$$

Here a, b, c, d are dimensionless real numerical coefficient. The problem is to find these coefficients from the condition that the eq. (2.18) leads to (2.17).

To fix the coefficients one acts by γ^{a} and $i\partial^{a}$ on the equation (2.18) and impose the conditions

$$(1+a)(b-4c-1) - 2(1+2a)(b-c) = 0 \qquad (2.19)$$
$$(1+a)(4d-1) - 2(1+2a)\,d = 0$$

After that will obtain

$$(1+2a)\,m\,\partial^{a}\psi_{a} = 0$$
$$(4d-1)\,m\,(\gamma^{a}\psi_{a}) = 0$$

It gives us $\gamma^a \psi_a = 0$, $\gamma^a \psi_a = 0$.
Substituting these relations into eq. (2.18) one get $(i\gamma^b \partial_b + m)\psi_a = 0$. As a result we reproduced all conditions for irreducible representation of the Poincaré group with spin $s = 3/2$. Solving the equations (2.15) ~~with~~ we obtain the eq. (2.18) in the form

$$(i\gamma^b \partial_b + m)\psi_a + a\gamma_a (i\partial^b \psi_b) + \frac{1}{3} i\partial_a (\gamma^b \psi_b) + \frac{1}{3}\gamma_a (i\gamma^b \partial_b)(\gamma^b \psi_b) + \frac{a+1}{2} m \gamma_a (\gamma^b \psi_b) = 0 \quad (2.20)$$

The coefficient a is still ~~undefined~~ undefined and can be taken arbitrary real number. If for example to put $a = -\frac{1}{3}$ we obtain the equation for spin $\frac{3}{2}$ field in the form given in pioneer paper by Rarita and Schwinger
the equation (2.20) can be ~~rewritten~~ rewritten in differents forms. For example, let us make field redefinition

$$\psi_a \to \psi_a + f\gamma_a(\gamma^b \psi_b) \quad (2.21)$$

with arbitrary real coefficient f. ~~...~~ ~~instead of ...~~ Using transformation (2.21) in eq (2.20) and selecting some appropriate value for f we can rewrite the equation (2.20) in the form

$$(i\gamma^b \partial_b + m)\psi_a - i\partial_a(\gamma^b \psi_b) - \gamma_a(i\partial^b \psi_b) + \gamma_a(m - \gamma^b i\partial_b)(\gamma^c \psi_c) = 0 \quad (2.22)$$

Consider the massless limit of this equation. Then the equation under consideration can be rewritten as follows

$$\gamma^{[a b c]} \, i \partial_b \Psi_c = 0 \qquad (2.23)$$

Here $\gamma^{[abc]} = \gamma^{[a}\gamma^{b}\gamma^{c]}$ where the square brackets mean total antisymmetrization. Now it is easily seen that the equation (2.23) is invariant under the gauge transformation

$$\Psi'_a = \Psi_a + \partial_a \varepsilon(x)$$

where $\varepsilon(x)$ is spinor gauge parameter.