Localization procedure and integrable solutions in non-local gravity models

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Introduction

Reliable astronomical data support the existence of four epochs of the Universe global evolution:

- inflation,
- a radiation dominated era,
- a matter dominated one
- the present dark energy epoch.

1. Initial inflation and dark energy domination are both characterized by an accelerated expansion of the Universe with almost constant Hubble parameter H.
2. The other epochs of the Universe evolution are described by power-law solutions with $H = J/t$, where J is a positive constant.
3. Power-law solutions with $H = J/t$ correspond to models with a perfect fluid whose EoS parameter reads $w_{\rm m} = -1 + 2/(3J)$.
   1. The radiation dominated epoch corresponds to solutions with $J = 1/2$,
   2. The matter dominated one corresponds to $J = 2/3$
4. To consider modified gravity models, it is therefore important to check for the existence of de Sitter and power-law solutions in the discussed models.

There are two basic motivations which lead cosmologists to modify gravity:

- The first one is an attempt to connect gravity with quantum physics, at least in a perturbative way, by including quantum correction terms to Einstein's equations.
- The second is an interest to describe the Universe evolution in a more natural way, without the dark energy and the dark matter components, which turn out to be avoidable in the modified models.

There are lots of ways to deviate from Einstein's gravity:

- $F(R)$ gravity
- Addition of higher-derivative terms to the Einstein–Hilbert action
- Non-local gravity

Most of the non-local cosmological models are inspired by string theory [1] or by quantum field theory [2]. Usually, nonlocal models include an analytic function of either the d'Alembertian operator $\Box$ or the inverse d'Alembertian operator $\Box^{-1}$. Note that models including $F(\Box R, \Box^2 R, \ldots, \Box^{-1} R, \Box^{-2} R, \ldots)$ have been investigated as well [3]

[1] J. C. Hwang and H. Noh, Phys.Rev. D 71, 063536 (2005)[gr-qc/0412126]

[2] S. Deser, R.P. Woodard,Phys. Rev. Lett. 99 (2007) 111301, [arXiv:0706.2151]

[3] S. Jhingan, S. Nojiri, S.D. Odintsov, M. Sami, I. Thongkool, and S. Zerbini, Phys. Lett. B 66 (2008) 424, [arXiv:0803.2613]; J. Kluson, J. High Energy Phys. 1109 (2011) 001, [arXiv:1105.6056]

Non-local models with inverse d'Alembertian

The following class of non-local gravity models has been proposed to explain current cosmic acceleration without dark energy:

$$S_2 = \int d^4x\sqrt{-g}\left\{\frac{{M_{\rm Pl}}^2}{2}\left[R\left(1+f(\Box^{-1}R)\right)-2\Lambda\right]+\mathcal{L}_{\rm m}\right\}, \qquad (1)$$

Here f is a differentiable function,
Λ is the cosmological constant,
$\mathcal{L}_{\rm m}$ is the matter Lagrangian,
$\Box$ is covariant d'Alembertian for a scalar field. The term $f(\Box^{-1}R)$ is can be understand as a prefactor for the Newtonian gravitational constant, and explain weakening of gravity at cosmological scales.

In the FLRW metric, the d'Alembert operator acting on a scalar $A(t)$ can be expressed as

$$\Box A \equiv \frac{1}{\sqrt{-g}}\,\partial_\rho\left(\sqrt{-g}\,g^{\rho\sigma}\partial_\sigma\right)A = -\frac{1}{a^3}\frac{d}{dt}\left(a^3\frac{dA}{dt}\right)\,,$$

while its inverse operator reduces to a double integration:

$$\Box^{-1}\left[A(t)\right] = -\int\limits_{\tilde{t}_0}^{t}\frac{d\tilde{t}}{a^3(\tilde{t})}\int\limits_{\eta_0}^{\tilde{t}} d\eta a^3(\eta)A(\eta)\,.$$

where $\tilde{t}_0$ and η_0 are two initial boundaries for the integrals.

The non-local action (1) can be rewritten in the "localized" form by introducing two scalar fields η and ξ:

$$\tilde{S}_2 = \int d^4x \frac{\sqrt{-g}}{16\pi G_N} \left\{ \left[R\left(1 + f(\eta)\right) + \xi \left(\Box\eta - R\right) - 2\Lambda \right] + \mathcal{L}_{\rm m} \right\}. \qquad (2)$$

By the variation over ξ, we obtain $\Box\eta = R$.
Substituting $\eta = \Box^{-1}R$ into (2), one reobtains action (1).[4]

[4]S. Nojiri, S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821, arXiv:0708.0924

The choice of the exponential $f(\eta)$

For the local formulation, a reconstruction procedure has been made in *T.S. Koivisto*, *Phys. Rev.* D **77** (2008) 123513, arXiv:0803.3399, *E. Elizalde, E.O. Pozdeeva, S.Yu. Vernov*, *Class. Quantum Grav.* **30** (2013) 035002, arXiv:1209.5957.

This procedure shows that the simplest choice of a such function $f(\eta)$ that the model has de Sitter and power-law solutions is

$$f(\eta) = f_0 e^{\alpha\eta}\,,$$

with f_0 and α nonzero real parameters.

The exponential $f(\Box^{-1}R)$ has been studied in many papers:
S. Nojiri, S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821; S. Jhingan, S. Nojiri, S.D. Odintsov, M. Sami, I. Thongkool, S. Zerbini, *Phys. Lett.* B **663** (2008) 424;T.S. Koivisto, *Phys. Rev.* D **77** (2008) 123513;S. Nojiri, S.D. Odintsov, M. Sasaki, Y.I. Zhang, *Phys. Lett.* B **696** (2011) 278; E. Elizalde, E.O. Pozdeeva, S.Yu. Vernov,*Phys. Rev.* D **85** (2012) 044002;E. Elizalde, E. O. Pozdeeva, S. Yu. Vernov and Y. I. Zhang, *J. Cosmol. Astropart. Phys.* **1307** (2013) 034.

De Sitter solutions

The de Sitter solutions with a constant nonzero $H = H_0$ can be presented in the form:

$$\rho_m = \rho_0 e^{-3(1+\omega_m)H_0 t}$$
$$\eta(t) = -4H_0(t-t_0),$$
$$\xi(t) = -\frac{3f_0}{3-4\alpha}e^{-4\alpha H_0(t-t_0)} + \frac{c_0}{3H_0}e^{-3H_0(t-t_0)} - \xi_0 \quad \text{at} \quad \alpha \neq 3/4,$$
$$\xi(t) = -f_0(c_0 + 3H_0(t-t_0))e^{-3H_0(t-t_0)} - \xi_0, \quad \text{at} \quad \alpha = 3/4$$

where c_0 and t are arbitrary constants,

$$\xi_0 = -1 - \frac{\Lambda}{3H_0^2}, \quad \rho_0 = \frac{6(1-2\alpha)H_0^2 f_0}{k^2}, \quad \omega_m = -1 + 4\alpha/3$$

Power-law solutions

At constant EoS parameter $\omega_m = P_m/\rho_m$ ($\omega_m \neq -1$) there are the following solutions:

$\eta(t) = \psi_1 t^{1-3n} - \frac{6n(2n-1)}{3n-1}\ln(t-t_0)$, where ψ_1, and t_0 are integration constants. Note that cases $n = 1/3$ and $n = 1/2$ are excluded from our analysis.

We specify $\psi_1 = 0$, so $f(t) = f_0\left(\frac{t}{t_0}\right)^m$ where $m = 6\alpha\frac{n(2n-1)}{3n-1}$,

$\xi(t) = \xi_0 + \xi_1(t-t_0)^{1-3n} + \frac{(3n-1)f_0}{3n+m-1}\left(\frac{t}{t_0}\right)^m$ for $m \neq 1-3n$, and

$\xi(t) = \xi_2 - mf_0\left(\frac{t}{t_0}\right)^m \ln\left(\frac{t}{t_1}\right), \quad \text{for} \quad m = 1-3n,$

where ξ_0, ξ_1, ξ_2 and t_1 are integration constants.

Special values of the power index n

In the case $n = 1/2$, $m = 0$ which corresponds $R = 0$.

$$\eta(t) = \psi_3 t^{-1/2} + \psi_4, \quad \xi(t) = \xi_3 t^{-1/2} + \xi_4,$$

where ψ_3, ψ_4, ψ_3, and ξ_4 are integral constants. In the case $\Lambda = 0$, conditions on the constants are following:

$$\psi_3 = 0, \quad \xi_4 = -1 - f_0 e^{\psi_4} + \frac{4}{3} k^2 \rho_0, \quad \omega = 1/3$$

while ρ_0, ψ_4, ξ_3 are to be determined by the initial conditions. In the case $n = 2/3$, $\alpha = \frac{3}{4} m$ we have:

$$\eta(t) = -\frac{4}{3} \ln(t - t_0),$$

$$\xi(t) = \xi_0 + \xi_1 (t - t_0)^{-1} + \frac{f_0}{1+m} \left(\frac{t}{t_0}\right)^m, \quad m \neq -1$$

$$\xi(t) = \xi_2 + f_0 \left(\frac{t}{t_0}\right)^{-1} \ln\left(\frac{t}{t}\right), \quad m = -1$$

Non-local models with the Gauss–Bonnet term

Non-local models with the Gauss-Bonnet term of a quite general form have been proposed in 2008[5], where accelerating cosmological solutions have been studied.
Also, a localization procedure that transforms a non-local model with the inverse d'Alambert operator acting on the Gauss-Bonnet term into a model of string-inspired scalar-Gauss-Bonnet gravity has been proposed in this paper.
We continue to investigate this class of non-local models, and check for the existence of de Sitter and power-law solutions.

[5] S. Capozziello, E. Elizalde, S. Nojiri, and S.D. Odintsov, Phys. Lett. B 671 (2009) 193, [arXiv:0809.1535]

Non-local models with the Gauss–Bonnet term and their localization

We consider the non-local model with the Gauss–Bonnet term $\mathcal{G}$:

$$S_{NL} = \int dx^4 \sqrt{-g} \left[\frac{M_{\mathrm{Pl}}^2}{16\pi} R + C \mathcal{G}^{n_1} \Box^{-n_2} \mathcal{G}^{n_3} - \Lambda \right], \tag{3}$$

where M_{Pl} is the Planck mass, C and Λ are constants, n_k are natural numbers, and the Gauss–Bonnet term

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}.$$

$R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar.
$\Box$ is the d'Alembertian operator in the metric $g_{\mu\nu}$ acting on a scalar.

To get local model corresponding to (3) we introduce scalar fields

$$S_L = \int dx^4 \sqrt{-g} \times$$

$$\left[\frac{M_{Pl}^2}{16\pi} R + C \mathcal{G}^{n_1} \phi_{n_2} + \xi_1 (\Box \phi_1 - \mathcal{G}^{n_3}) + \sum_{j=2}^{n_2} \xi_j (\Box \phi_j - \phi_{j-1}) - \Lambda \right]$$

Varying this action under ξ_j, we get

$$\Box \phi_1 = \mathcal{G}^{n_3}, \qquad j = 1,$$
$$\Box \phi_j = \phi_{j-1}, \qquad j = 2, \ldots, n_2.$$

This action can be presented in the form:

$$S_{L1} = \int dx^4 \sqrt{-g} \left[\frac{M_{Pl}^2}{16\pi} R + C\mathcal{G}^{n_1}\phi_{n_2} - \xi_1 \mathcal{G}^{n_3} + \sum_{j=1}^{n_2} \phi_j \Box \xi_j - \sum_{j=1}^{n_2-1} \xi_{j+1}\phi_j - \Lambda \right]$$

Thus, variation under ϕ_J leads to

$$\Box\xi_{n_2} = -C\mathcal{G}^{n_1}, \qquad j = n_2,$$
$$\Box\xi_j = \xi_{j+1}, \qquad j = 1, \ldots, n_2 - 1.$$

Action S_L can be linearized with respect to the Gauss-Bonnet term, by adding one more scalar field in the action. Let us consider the part of action S_L that includes the Gauss-Bonnet term:

$$S_{fGB} = \int dx^4 \sqrt{-g}\left[C\mathcal{G}^{n_1}\phi_{n_2} - \xi_1 \mathcal{G}^{n_3}\right].$$

To linearize this action with respect to $\mathcal{G}$ we introduce a scalar field σ and

$$f(\sigma) = C\sigma^{n_1}\phi_{n_2} - \xi_1\sigma^{n_3},$$

and get that the following equivalent action:

$$S_{GB\sigma} = \int dx^4 \sqrt{-g}\left[\frac{df}{d\sigma}(\mathcal{G}-\sigma) + f\right] =$$

$$= \int dx^4 \sqrt{-g}\left[\left(n_1 C\sigma^{n_1-1}\phi_{n_2} - n_3\xi_1\sigma^{n_3-1}\right)(\mathcal{G}-\sigma) + C\sigma^{n_1}\phi_{n_2} - \xi_1\sigma^{n_3}\right]$$

Varying over σ, one gets $\sigma = \mathcal{G}$ and the action S_{fGB}. Note that the scalar field σ is not dynamical, because it has no kinetic term.

So, the initial action S_{NL} can be written in the following scalar-tensor form:

$$S=\int dx^4\sqrt{-g}\left[\frac{M_{Pl}^2}{16\pi}R+F\mathcal{G}-V-\sum_{k=1}^{n_2}g^{\mu\nu}\partial_\mu\xi_k\partial_\nu\phi_k\right]$$

where we use the following redesignation

$$\begin{aligned} F &= n_1C\sigma^{n_1-1}\phi_{n_2}-n_3\xi_1\sigma^{n_3-1}\,, \\ V &= -C\sigma^{n_1}\phi_{n_2}(1-n_1)-\xi_1\sigma^{n_3}(n_3-1)+\sum_{k=1}^{n_2-1}\xi_{i+1}\phi_i+\Lambda\,. \end{aligned}$$

Varying the local action S thus obtained, we get the following equations

$$\left[\frac{M_{Pl}^2}{16\pi} - 4\Box F\right]\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right] + \frac{g^{\mu\nu}}{2}\left(\sum_{k=1}^{n_2}\partial_\sigma\phi_k\partial^\sigma\xi_k\right) \tag{4}$$

$$-\sum_{k=1}^{n_2}\partial^\mu\phi_k\partial^\nu\xi_k + \frac{g^{\mu\nu}}{2}V =$$

$$= F\left[\frac{1}{2}g^{\mu\nu}\mathcal{G} - 2RR^{\mu\nu} + 4R^\mu_\rho R^{\nu\rho} - 2R^{\mu\rho\sigma\tau}R^\nu_{\rho\sigma\tau} + 4R^{\mu\rho\sigma\nu}R_{\rho\sigma}\right]$$

$$+2R(D^\mu D^\nu F) - 4(D_\rho D^\mu F)R^{\nu\rho} - 4(D_\rho D^\nu F)R^{\mu\rho}$$

$$+4g^{\mu\nu}(D_\rho D_\sigma F)R^{\rho\sigma} - 4(D_\rho D_\sigma F)R^{\mu\rho\nu\sigma}.$$

Let us consider the trace of Eq. (4). After setting $R^{\mu\rho\sigma\nu} = R^{\nu\rho\mu\sigma}$ and $g_{\mu\nu}R^{\nu\rho\mu\sigma}R_{\rho\sigma} = R^{\rho\mu\sigma}_{\mu}R_{\rho\sigma} = R^{\rho\sigma}R_{\rho\sigma}$, we get

$$g_{\mu\nu}F\left[\frac{1}{2}g^{\mu\nu}\mathcal{G} - 2RR^{\mu\nu} + 4R^{\mu}_{\rho}R^{\nu\rho} - 2R^{\mu\rho\sigma\tau}R^{\nu}_{\rho\sigma\tau} + 4R^{\mu\rho\sigma\nu}R_{\rho\sigma}\right] = 0$$

Using $R^{\mu\nu}D_\mu D_\nu F = R\Box F$ and

$$-8(D_\rho D^\nu F)R^{\rho}_{\nu} + 16(D_\rho D_\sigma F)R^{\rho\sigma} - 4(D_\rho D_\sigma F)R^{\rho\nu\sigma}_{\nu} = 4(D_\rho D_\sigma F)R^{\rho\sigma},$$

we obtain the

- trace equation:

$$\frac{M_{Pl}^2}{16\pi}R - \left(\sum_{k=1}^{n_2}\partial_\sigma\phi_k\partial^\sigma\xi_k\right) - 2V - R(\Box F) = 0\,.$$

Friedmann equations

We consider the spatially flat FLRW universe with the interval

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right).$$

In this metric one gets $(i, j, m, l = 1, 2, 3)$:

$$R^{i0j0} = R^{0i0j} = -R^{0ij0} = -R^{i00j} = -\frac{(\dot{H} + H^2)}{a^2}\delta_{ij},$$

$$R^{ijml} = \frac{H^2}{a^4}(\delta_{im}\delta_{lj} - \delta_{il}\delta_{mj}),$$

$$\Gamma^0_{ij} = a^2 H \delta_{ij}, \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = H\delta^i_j,$$

$$R^{00} = -3(\dot{H} + H^2), \quad R^{ij} = \frac{(\dot{H} + 3H^2)}{a^2}\delta_{ij},$$

$$R = 6(\dot{H} + 2H^2), \quad \mathcal{G} = 24H^2(\dot{H} + H^2)$$

where the Hubble parameter $H = \dot{a}/a$ and dots mean the time derivatives.

We assume that all scalar fields depend on time only and get the following expressions including to components equations

$$\Box F = -3H\dot{F} - \ddot{F}$$

$$D_\rho D_\sigma F = \partial^2_{\rho\sigma} F - \Gamma^j_{\rho\sigma}\partial_j F$$

$$D^\mu D^\nu F = g^{\mu\sigma} g^{\nu\rho}(\partial^2_{\sigma\rho} F - \Gamma^k_{\sigma\rho}\partial_k F)$$

from where

$$D^i D^i F = -\frac{H}{a^2}\dot{F}, \quad D_i D_i F = -a^2 H\dot{F},$$

$$D^0 D^0 F = D_0 D_0 F = \ddot{F}, \qquad D^i D^0 F = D^0 D^i F = 0\,.$$

Field and Friedmann equations

Eqs. (4) in the FLRW metric read as follows

$$3H^2\frac{M_{Pl}^2}{16\pi}-\frac{1}{2}\sum_{k=1}^{n_2}\left(\dot{\phi}_k\dot{\xi}_k\right)-\frac{1}{2}V=-12H^3\dot{F},$$

$$\frac{\left(3H^2+2\dot{H}\right)M_{Pl}^2}{8\pi}+16H\left(H^2+\dot{H}\right)\dot{F}+8H^2\ddot{F}+\sum_{k=1}^{n_2}\dot{\phi}_k\dot{\xi}_k=V \quad (5)$$

From here we get

$$8H^3\dot{F}-4\Box FH^2+\frac{3M_{Pl}^2}{8\pi}H^2+8H\dot{F}\dot{H}+\frac{M_{Pl}^2}{8\pi}\dot{H}-V=0. \quad (6)$$

Note that Eqs. (5) and (6) are third order differential equations with respect to the Hubble parameter.

Search for de Sitter solutions

If the Hubble parameter is a constant: $H = H_0$, then the Gauss-Bonnet term reads $\mathcal{G} = 24H_0^4 \equiv \mathcal{G}_0$ and $\sigma = \mathcal{G}_0$. As a consequence, the corresponding field equations (4) get transformed into the following system of linear first order differential equations, with constant coefficients,

$$
\begin{aligned}
\dot{\phi}_1 &= \psi_1, \\
\dot{\psi}_1 &= -3H_0\psi_1 - \mathcal{G}_0^{n_3}, \\
\dot{\phi}_j &= \psi_j, \qquad j = 2, \ldots, n_2, \\
\dot{\psi}_j &= -3H_0\psi_j - \phi_{j-1}, \qquad j = 2, \ldots, n_2.
\end{aligned}
$$

The system (7) has the following solution

$$
\phi_j = P_j(t)e^{-3H_0 t} - \frac{\mathcal{G}_0^{n_3}}{j!(3H_0)^j}t^j + \tilde{P}_j(t),
$$

where $P_j(t)$ and $\tilde{P}_j(t)$ are $(j-1)$-degree polynomials of t with coefficients that include $2j$ arbitrary parameters.

Analogously, the system (4) acquires the following form

$$\ddot{\xi}_j + 3H_0\dot{\xi}_j + \xi_{j+1} = 0, \qquad j = 1, \dots, n_2 - 1,$$
$$\ddot{\xi}_{n_2} + 3H_0\dot{\xi}_{n_2} - C\mathcal{G}_0^{n_1} = 0,$$

and the solution reads

$$\xi_j = Q_j(t)e^{-3H_0 t} + \frac{C\mathcal{G}_0^{n_1}}{(n_2 - j + 1)!(3H_0)^{n_2-j+1}} t^{(n_2-j+1)} + \tilde{Q}_j(t),$$

where $Q_j(t)$ and $\tilde{Q}_j(t)$ are polynomials in t of degree $(n_2 - j) + 1$.
To check for the existence of de Sitter solutions, one must substitute the solutions of the field equations thus obtained into Eqs. (5) and (6).

In the case when $n_1 = n_2 = n_3 = 1$ the Sitter solutions have been found in [6]. To get de Sitter solution in the model is non-trivial problem. For example, we checked that de Sitter solution is absent in case $n_1 = 1$, $n_2 = 2$, $n_3 = 1$. We obtain de Sitter solution the case $n_2 = 2$.

[6] S. Capozziello, E. Elizalde, S. Nojiri, and S.D. Odintsov, *Phys. Lett.* B **671** (2009) 193, [arXiv:0809.1535]

case $n_2 = 2$

Let us consider the case $n_2 = 2$.
The field equations

$$-\ddot{\phi}_1 - 3H_0\dot{\phi}_1 = \mathcal{G}_0^{n_3}, \qquad -\ddot{\phi}_2 - 3H_0\dot{\phi}_2 = \phi_1$$

and

$$-\ddot{\xi}_2 - 3H_0\dot{\xi}_2 = -C\mathcal{G}_0^{n_1}, \qquad -\ddot{\xi}_1 - 3H_0\dot{\xi}_1 = \xi_2$$

have the following solutions:

$$\phi_1 = A_1 e^{-3H_0 t} - \frac{\mathcal{G}_0^{n_3}}{3H_0} t + B_1,$$

$$\phi_2 = \left(\frac{A_1}{3H_0} t + A_2\right) e^{-3H_0 t} + \frac{\mathcal{G}_0^{n_3}}{18H_0^2} t^2 - \left(\frac{\mathcal{G}_0^{n_3}}{27H_0^3} + \frac{B_1}{3H_0}\right) t + B_2$$

$$\xi_1 = \left(\frac{C_1}{3H_0} t + C_2\right) e^{-3H_0 t} - \frac{C\mathcal{G}_0^{n_1}}{18H_0^2} t^2 + C\left(\frac{\mathcal{G}_0^{n_1}}{27H_0^3} - \frac{D_1}{3H_0}\right) t + D_2,$$

$$\xi_2 = C_1 e^{-3H_0 t} + C\frac{\mathcal{G}_0^{n_1}}{3H_0} t + CD_1,$$

Substituting these expressions into Eq. (5),

$$3H_0^2\frac{M_{Pl}^2}{16\pi}-\frac{1}{2}\left(\sum_{k=1}^{n_2}\dot{\phi}_k\dot{\xi}_k\right)-\frac{1}{2}V=-12H_0^3\dot{F},$$

we see that this equation can be satisfied only if $n_1+n_3=4$.
Also, we get the following restriction to the integration constants

$A_1=0,\quad C_1=0,\quad C_2=-\frac{24^{2n_1}(2n_1-1)A_2CH_0^{8(n_1-2)}}{331776(2n_1-7)},$

$B_1=-\frac{331776(n_1-2)H_0^{16-8n_1}D_1+24^{n_1}442368H_0^{14-4n_1}}{(n_1-2)}.$

These restrictions are not valid for $n_1=2$. Λ is connect with parameters

$\Lambda=-\frac{3H_0^2M_{Pl}^2}{8\pi}-\frac{8192C(13n_1+4)H_0^{12}}{(n_1-2)}+24^{n_1}CB_2(n_1-1)H_0^{4n_1}+$
$24^{-n_1}331776D_2(n_1-3)H_0^{16-4n_1}-\frac{24^{-n_1}73728(5n_1-4)CD_1H_0^{14-4n_1}}{n_1-2}-$
$24^{-2n_1}331776CD_1^2H_0^{16-8n_1}.$

Consequently, the value of Λ fixes the value of one of the integration constants: B_2 for $n_1 = 3$ or D_2 for $n_1 = 1$.
Summing up, we do get explicitly de Sitter solutions for models with $n_1 = 1$, $n_2 = 2$, $n_3 = 3$ and $n_1 = 3$, $n_2 = 2$, $n_3 = 1$. And we have also discovered that models with $n_2 = 2$ and other values of n_1 and n_3 do not have de Sitter solutions[7].

[7] Straightforward substitution of the field expressions when $n_1 = n_3 = 2$ already proves the absence of the de Sitter solutions in this case.

Power-Law solutions

The search of power-law solutions with $H = J/t$ is more complicated. We consider the case when n_1 or n_3 is equal to 1. If $n_1 = 1$ and $n_3 = 1$, then

$$V = \xi_2\phi_1, \qquad F = C\phi_2 - \xi_1$$

with the following form for the field equations

$$\Box\phi_1 = \mathcal{G}, \qquad \Box\phi_2 = \phi_1,$$
$$\Box\xi_2 = -C\mathcal{G}, \qquad \Box\xi_1 = \xi_2,$$

where $\mathcal{G} = 24(J-1)J^3/t^4$.

Using these formulas, we immediately obtain the form of Eq. (6)

$$-\frac{\left(3H^2 + \dot{H}\right)M_{Pl}^2}{8\pi} - 8H\left(H^2 + \dot{H}\right)(C\dot{\phi}_2 - \dot{\xi}_1) + 4(C\phi_1 - \xi_2)H^2 + \xi_2\phi_1 = 0$$

The model with $n_1 = 1$ and $n_3 = 1$ yields power law solutions with $H = J/t$ at $J = 2/3$ and $J = 3$. The corresponding scalar fields admit two types of expressions.

The first type of solutions corresponds to

$$\phi_1 = -\frac{C_1\, t^{-3H_0+1}}{3J-1} + 4\,\frac{J^3}{t^2} - \frac{1}{2}\,\frac{K(3J+1)}{J\,C(J-1)}$$

$$\phi_2 = \frac{t^2 K}{4J\,C(J-1)} - \frac{K_3\, t^{-3J+1}}{C(3J-1)} - \frac{C_1\, t^{3-3J}}{6(3\,J^2-4\,J+1}) - \frac{4J^3\ln(t)}{3J-1} + C_4$$

$$\xi_1 = 4\,\frac{C J^3 \ln(t)}{3J-1} - \frac{K_3\, t^{-3J+1}}{3J-1} + K_4$$

$$\xi_2 = -4\,\frac{C J^3}{t^2},$$

where in the case $J = 2/3$, $C_1 = \frac{7168}{729\,C_3}$, while in the case $J = 3$, either $C_1 = 0$ or $C_3 = 0$.

Another type of solutions, with the same Hubble parameters, is given by

$$\phi_1 = 4\,\frac{J^3}{t^2},$$
$$\phi_2 = -\frac{C_3\,t^{-3J+1}}{3J-1} - 4\,\frac{J^3\ln(t)}{3J-1} + C_4,$$
$$\xi_1 = \frac{4CJ^3\ln(t)}{3J-1} - \frac{CC_3\,t^{-3J+1}}{3J-1} - \frac{t^2K}{4J(J-1)} - \frac{t^{3-3J}K_1}{6(3\,J^2-4\,J+1)} + K_4,$$
$$\xi_2 = -\frac{K_1\,t^{-3J+1}}{3J-1} - 4\,\frac{CJ^3}{t^2} + \frac{1}{2}\,\frac{K(3J+1)}{J\,(J-1)},$$

where in the case $J = 2/3$ we have the additional condition $K_1 = -\frac{7168\,C}{729\,C_3}$, while in the case $J = 3$, either $C_3 = 0$ or $K_1 = 0$. Note that the form of the solutions obtained excludes a few values of J, which must be checked separately.

Conclusions

We analyze two types of non-local gravity models: the non-local gravity model with function from inverse Dalambertian acting to Ricci scalar and the Gauss-Bonnet non-local gravity model.
In the first class of models with exponential type of modification function $f(\Box^{-1}R)$:

$$S_2 = \int d^4x\sqrt{-g}\left\{\frac{{M_{\rm Pl}}^2}{2}\left[R\left(1+f(\Box^{-1}R)\right)-2\Lambda\right]+\mathcal{L}_{\rm m}\right\}$$

we presented

- the de Sitter and power-law solutions
- including power-law solutions with $J=1/2$ and $J=2/3$, correspond to radiation dominated and matter-dominated phase epochs.

In the Gauss-Bonnet non-local gravity model:

$$S_{NL} = \int dx^4 \sqrt{-g} \left[\frac{M_{Pl}^2}{16\pi} R + C \mathcal{G}^{n_1} \Box^{-n_2} \mathcal{G}^{n_3} + \mathcal{L}_m \right].$$

and obtain

- in the specific case $n_2 = 2$, de Sitter solutions exist only in these two cases: for $n_1 = 1$ and $n_3 = 3$, or for $n_1 = 3$ and $n_3 = 1$. Both these models yield no power-law solutions;
- if $n_1 = 1$ and $n_3 > 1$ (or $n_1 > 1$ and $n_3 = 1$, respectively), then power-law solutions do not exist;
- in the case $n_1 = n_3 = 1$, power-law solutions with $H = J/t$ exist only for $J = 2/3$ and $J = 3$. Therefore, the model with $n_1 = 1$, $n_2 = 2$, and $n_3 = 1$, without additional matter, is suitable in order to describe the matter-dominated phase of the Universe evolution that corresponds to $J = 2/3$.

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Thank for your attention