3. Tetrad Formulation of Gravity

3.1. Tetrad covariant derivative

Consider the Riemann space with metric $g_{mn}(x)$, $(m,n=0,1,2,3)$. In each point $P(x)$ of this space one introduces the four basis ~~ve~~vectors $e_a{}^m(x)$ $(a=0,1,2,3)$ under the normalization condition

$$g_{mn}(x)\, e_a{}^m(x)\, e_b{}^n(x) = \eta_{ab}, \tag{3.1}$$

where η_{ab} is the Minkowski space metric. Hereinafter a,b,c,d are the Minkowski space indices (flat indices) and m,n,k,l, are the curved space indices (curved indices). These four vectors $e_a{}^m$ are called tetrad. Index m is vectorial and the index a numerates the vectors.

~~It is clear that the relation (3.1) is invariant under local Lorentz rotations~~

~~ea~~ Let us define also the inverse tetrad e^{am} by the relation

$$e^{am} = \eta^{ab} e_b{}^m \tag{3.2}$$

Then the following relations take ~~place~~ place

$$e^{am} e^{bn} \eta_{ab} = g^{mn}, \qquad e^a{}_m e^b{}_n \eta_{ab} = g_{mn}.$$

$$e^{am} e_{bm} = \delta^a{}_b, \tag{3.3}$$

$$e^{am} e_{an} = \delta^m{}_n, \qquad \det e^a{}_m = \sqrt{|g|}.$$

It is clear that the relation $e^a{}_m e^b{}_n \eta_{ab} = g_{mn}$ is invariant under the local Lorentz transformation of tetrad $e'^a{}_m = \Lambda^a{}_b(x) e^b{}_m$ where $\Lambda^T(x)\eta\Lambda(x) = \eta$. In infinitesimal form $\Lambda^a{}_b(x) = \eta^a{}_b + \omega^a{}_b(x)$, where $\omega_{ab} = -\omega_{ba}$. Therefore infinitesimal local Lorentz transformation of tetrad is

$$\delta e^a{}_m = \omega^a{}_b e^b{}_m \tag{3.4}$$

The tetrad is used ~~to~~ for conversion of the space-time indices into local Lorentz indices. For example, a vector T^m ~~[illegible]~~ is transformed into local Lorentz vector T^a as follows $T^a = e^a{}_m T^m$.

Now we ~~[illegible]~~ discuss a definition of covariant derivatives in the tetrad formalism. To do that ~~[illegible]~~ one remembers some basic notions of Yang-Mills theory. Let φ^I is a field defined up to ~~[illegible]~~ global ~~[illegible]~~ transformations $\varphi'^I = h^I{}_J \varphi^J$ with $h^I{}_J$ is an element of internal symmetry group. It is evident that $\partial_m \varphi'^I = h^I{}_J \partial_m \varphi^J$, that is the derivative transforms like φ^I. However, if $h^I{}_J$ is not a constant, the transformation law for derivative is violated. To preserve the above transformation law ~~[illegible]~~ for derivative one introduces the covariant derivative

$$\nabla_m \varphi^I = \partial_m \varphi^I - i A_m^r (t^r)^I{}_J \varphi^J, \qquad (3.5)$$

where $(t^r)^I{}_J$ are the generators of Lie group with elements $h^I{}_J$ and A_m^r are the gauge field. Number of gauge fields is equal to a number of generators.

Let us apply the construction of covariant derivative to the case where a role of local group is played by Lorentz group. Taking into account eq (3.5) one gets

$$\nabla_m T^a = \partial_m T^a - \frac{i}{2} \omega_m{}^{cd} (S_{cd})^a{}_b T^b \qquad (3.6)$$

Here $(S_{cd})^a{}_b$ are the generators of Lorentz group in contravariant vector representation and $\omega_m{}^{cd}$ are the corresponding gauge field which is usually called the spin connection. The generators S_{cd} have been calculated on one of the previous lectures and have the form

$$(S_{cd})^a{}_b = i(\delta_c^a \eta_{bd} - \delta_d^a \eta_{cb}) \qquad (3.7)$$

Substituting (3.7) into (3.6) one gets

$$\nabla_m T^a = \partial_m T^a + \omega_m{}^a{}_b T^b \qquad (3.8)$$

where $\omega_m{}^a{}_b = \omega_m{}^{ac}\eta_{cb}$. It is evident that $\omega_m{}^{ab} = -\omega_m{}^{ba}$ by construction

~~...~~ Consider the relation

$\nabla_m T^n = \cancel{\ldots}$

$= \nabla_m (e_a{}^n T^a) = \partial_m (e_a{}^n T^a) + \Gamma^n{}_{mk} e_a{}^k T^a$

Or

$(\nabla_m e_a{}^n) T^a + e_a{}^n \nabla_m T^a =$

$= (\partial_m e_a{}^n) T^a + e_a{}^n \partial_m T^a + \Gamma^n{}_{mk} e_a{}^k T^a$

Or

$(\nabla_m e_a{}^n) T^a + e_a{}^n \partial_m T^a + e_a{}^n \omega_m{}^a{}_b T^b =$

$= (\partial_m e_a{}^n) T^a + e_a{}^n \partial_m T^a + \Gamma^n{}_{mk} e_a{}^k T^a$

It gives us the expression for covariant derivative of tetrad in the form

$$\nabla_m e_a{}^n = \partial_m e_a{}^n + \Gamma^n{}_{mk} e_a{}^k - e_b{}^n \omega_m{}^b{}_a \qquad (3.9)$$

The right hand side includes two contributions one relatated with vector index n and second related with local index a. Note that the tetrad $e_a{}^m$ and spin connection $\omega_m{}^{ab}$ are independent geometrical objects.

Now we take into account that in Riemann geometry there is a metricity condition $\nabla_k g_{mn} = 0$. Using $g^{mn} = e^{am} e_a{}^n$ we impose the condition $\nabla_m e_a{}^n = 0$. Then the relation (3.9) allows us to express the spin connection in the form

$$\omega_m{}^{ab} = e^a{}_k e^{bn} \Gamma^k{}_{mn} - e^{bn} \partial_m e^a{}_n \qquad (3.10)$$

It is possible to show that $\omega_m{}^{ab} = -\omega_m{}^{ba}$.

3.2. Covariant derivative of spinor

In flat space the spinor is defined by transformation law under the Lorentz transformation. In infinitesimal form such a transformation looks like

$$\delta \psi = -\frac{i}{2} \omega^{ab} \Sigma_{ab} \psi$$

where ω^{ab} are the parameters of Lorentz translations and Σ_{ab} are the Lorentz group generators in spinor representation

$$\Sigma_{ab} = \frac{i}{4} (\gamma_a \gamma_b - \gamma_b \gamma_a) \qquad (3.11)$$

γ_a are the Dirac matrices satisfying the relation

$$\gamma_a \gamma_b + \gamma_b \gamma_a = -2 \eta_{ab} \qquad (3.12)$$

We define the spinor field $\psi(x)$ in curved space-time as a scalar under general coordinate transformations and a spinor under local Lorentz rotations.

$$\delta\psi(x) = -\frac{i}{2}\,\omega^{ab}(x)\,\Sigma_{ab}\,\psi(x) \qquad (3.13)$$

where ~~ω~~ $\omega^{ab}(x)$ are now the space-time dependent parameters.

The spinor field covariant derivatives is constructed ~~on~~ on the base of general Yang-Mills covariant derivative. It leads to

$$\nabla_m\psi = \partial_m\psi - \frac{i}{2}\,\omega_m{}^{ab}\,\Sigma_{ab}\psi \qquad (3.14)$$

Here $\omega_m{}^{ab}$ is the corresponding gauge field. Actually $\omega_m{}^{ab}$ is the same spin connection introduced in previous subsection.

To write the Dirac equation in curved space one defines the space-time dependent Dirac matrices

$$\gamma_m(x) = e^a{}_m(x)\,\gamma_a \qquad (3.15)$$

where $e^a{}_m(x)$ is a tetrad. Since $\nabla_n e^a{}_m = 0$, we have ~~∇_n~~

$$\nabla_n\gamma_m(x) = 0 \qquad (3.16)$$

$$\gamma_m(x)\gamma_n(x) + \gamma_n(x)\gamma_m(x) = -2g_{mn}(x)$$

The curved space γ-matrices are covariantly constant. As a result, the Dirac equation in curved space-time is written in the form

$$i\gamma^m(x)\,\nabla_m\psi(x) + m\,\psi(x) = 0 \qquad (3.17)$$

~~The~~ Let left hand side is a scalar under general coordinate transformations.

Now one considers the Rarita-Schwinger field in curved space. Such a field $\Psi_m(x)$ is defined as the vector under the general coordinate transformation and the spinor under the local Lorentz rotations. It allows to write down the corresponding covariant derivative in the form

$$\nabla_n \Psi_m(x) = \partial_n \Psi_m(x) - \Gamma^k{}_{nm} \Psi_k(x) - \\ - \frac{i}{2} \omega_n{}^{ab} \Sigma_{ab} \Psi_m(x) \qquad (3.18)$$

Here the term $-\Gamma^k{}_{nm}\Psi_k$ is due to the Ψ_m is ~~vector~~ a covariant vector and second one is due to that Ψ_m is a spinor.

Curved space-time generalization of ~~the~~ massless Rarita-Schwinger equation ~~...~~ has the form

$$\gamma^{[m}(x)\gamma^n(x)\gamma^{k]}(x) \nabla_n \Psi_k(x) = 0 \qquad (3.19)$$

It can be rewritten as

$$\gamma^{[m}\gamma^n\gamma^{k]} \mathcal{D}_n \Psi_k = 0$$

where $\mathcal{D}_n\Psi_k = \partial_n \Psi_k - \frac{i}{2}\omega_n{}^{ab}\Sigma_{ab}\Psi_k$.

the term $\Gamma^k{}_{mn}$ in (3.18) drops out since $\Gamma^k{}_{mn}$ is symmetric in m,n and $\gamma^{[m}\gamma^n\gamma^{k]}$ is antisymmetric in all indices.

3.3. Gauge treatment of gravity

We will discuss here an general approach to gravity allowing to introduce the tetrad and spin connection from the very beginning without using the metric. In this approach the metric is a secondary notion. The gauge transformations are obtained automatically.

As well known the Yang-Mills strength $G_{\mu\nu}$ is constructed as a commutator of covariant derivatives. Let us apply the same procedure to theory with local Poincare algebra. We proved at one of the lectures that Poincare algebra for the generators P_a, J_{ab} has the form

$$
\begin{aligned}
&[P_a, P_b] = 0 \\
&[J_{ab}, P_c] = i(\eta_{bc} P_a - \eta_{ac} P_b) \\
&[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac})
\end{aligned}
\qquad (3.20)
$$

The corresponding covariant derivative is written as follows

$$\nabla_m = \partial_m - i e^a{}_m P_a - \frac{i}{2} \omega_m{}^{ab} J_{ab} \qquad (3.21)$$

where $e^a{}_m$ and $\omega_m{}^{ab}$ are the gauge fields for generators P_a and J_{ab} respectively.

Direct calculation of the commutator $[\nabla_m, \nabla_n]$ ~~[illegible]~~ with using (3.20) leads to

$$[\nabla_m, \nabla_n] = -i R^a{}_{mn} P_a - \frac{i}{2} R^{ab}{}_{mn} J_{ab} \qquad (3.22)$$

where

$$R^a{}_{mn} = \partial_m e^a{}_n - \partial_n e^a{}_m + e^b{}_m \omega_{nb}{}^a - e^b{}_n \omega_{mb}{}^a$$

$$R^{ab}{}_{mn} = \partial_m \omega_n{}^{ab} - \partial_n \omega_m{}^{ab} + \omega_m{}^{ac} \omega_{nc}{}^b - \omega_n{}^{ac} \omega_{mc}{}^b \qquad (3.23)$$

the quantities $R^a{}_{mn}$, $R^{ab}{}_{mn}$ are called the strength or curvatures corresponding to the generators P_a and J_{ab} respectively. The curvatures are given in terms of independent gauge fields e^a_m and $\omega_m{}^{ab}$. As we will see they can be associated with tentrad and spin connection respectively.

Now ~~we~~ one finds the gauge transformations. To do that one uses the Yang-Mills construction. Let φ is some field and $\varphi' = h\varphi$ is transformed field with h be an element of gauge group. If ∇_m is a covariant derivative then $\nabla'_m \varphi' = h \nabla_m \varphi$. Or $\nabla'_m h\varphi = h \nabla_m \varphi$. Since φ is arbitrary, one gets

$$\nabla'_m = h \nabla_m h^{-1} \qquad (3.24)$$

For infinitesimal transformations $h = 1 + iT$ where T is a linear combination of generators with local parameters related to given gauge group.

The parameters of local non-homogeneous Lorentz transformations are $a^a(x)$, $\omega^{ab}(x)$. Therefore in the case under consideration

$$T = a^a P_a + \frac{1}{2}\omega^{ab} J_{ab} \qquad (3.25)$$

Therefore

$$\nabla'_m = (1+iT)\nabla_m(1-iT) = \nabla_m - i[\nabla_m, T] \qquad (3.26)$$

Let us write

$$\nabla_m = \partial_m - i\Gamma_m$$

where

$$\Gamma_m = e^a{}_m P_a + \frac{i}{2}\omega_m^{ab} J_{ab} \qquad (3.27)$$

Then the relation (3.26) leads to

$$\Gamma'_m = \Gamma_m + \partial_m T - i[\Gamma_m, T] \qquad (3.28)$$

Substituting (3.25) and (3.27) one gets

$$\delta e^a{}_m P_a + \frac{1}{2}\delta\omega_m^{ab} J_{ab} =$$

$$= \partial_m a^a P_a + \frac{1}{2}\partial_m \omega^{ab} J_{ab} -$$

$$- i[e^a{}_m P_a + \frac{1}{2}\omega_m{}^{ab} J_{ab}, a^c P_c - \frac{1}{2}\omega^{cd} J_{cd}]$$

Direc calculation of commutators on the base of Poincare algebra leads to

см. ?

$$\delta e^a{}_m = \partial_m a^a + \omega^{ab} e_{bm} + \omega_m{}^{ab} a_b \qquad (3.25)$$

$$\delta \omega_m{}^{ab} = \partial_m \omega^{ab} + \omega_m{}^{ac} \omega_c{}^b - \omega_m{}^{bc} \omega_c{}^a$$

As a result we obtain ~~the~~ ~~...~~ a gauge theory in terms of two independent gauge fields $e^a{}_m(x)$, $\omega_m{}^{ab}(x)$ defined up to gauge transformations (3.25). The transformations contain two independent parameters $a^a(x)$, $\omega^{ab}(x)$. Such a gauge theory is equivalent to gravity with independent torsion

In conventional gravity theory the torsion is ~~a~~ ~~...~~ iliminated with help of metricity condition $\nabla_m g_{nk} = 0$. To find the analogous formulation with less ~~less~~ number of fields in our approach we ~~also~~ also should impose some constraint. To preserve gauge invariance such a constraint should be ~~...~~ written in terms of curvatures. The simplest constraint has the form

$$R^a{}_{mn} = 0 \qquad (3.26)$$

or

$$\partial_m e^a{}_n - \partial_n e^a{}_m + e^b{}_m \omega_{nb}{}^a - e^b{}_n \omega_{mb}{}^a = 0 \quad (3.27)$$

Solution to this equation is written as follows

$$\omega_{nkm} = \frac{1}{2}[\, e_{ak}(\partial_m e^a{}_n - \partial_n e^a{}_m) + e_{am}(\partial_n e^a{}_k - \partial_k e^a{}_n) - e_{an}(\partial_k e^a{}_m - \partial_m e^a{}_k) \quad (3.28)$$

Here

$$\omega_{nkm} = \omega_n{}^{ab}\, e_{ak} e_{bm}$$

The relation (3.28) is analogous to expression of Christoffel symbols in terms of metric and its derivatives.

To see a relation of above theory to general relativity let us introduce the field

$$g_{mn} = e^a{}_m e^b{}_n \eta_{ab} \quad (3.29)$$

where $e^a{}_m$ is the gauge field corresponding to generator P_a. The gauge transformations for g_{mn} are dictated by the transformations (3.25) for $e^a{}_m$. The result looks like

$$\delta g_{mn} = e_{an}\partial_m a^a + e_{am}\partial_n a^a + \qquad (3.30)$$
$$+ \omega_m{}^{ab} e_{an} a_b + \omega_n{}^{ab} e_{am} a_b$$

Pay attention that the parameter of local Lorentz rotation is absent. ~~Here~~.

Now one defines the ~~vector~~ field

$$\xi^k = e_a{}^k a^a \qquad (3.31)$$

Then the relation (3.30) takes the form

$$\delta g_{mn} = g_{nk}\partial_m \xi^k + g_{mk}\partial_n \xi^k +$$
$$+ [e_{an}\partial_m e^a{}_k + e_{am}\partial_n e^a{}_k + \qquad (3.32)$$
$$+ \omega_m{}^{ab} e_{an} e_{bk} + \omega_n{}^{ab} e_{am} e_{bk}]\xi^k$$

Substituting the relation (3.28) for ω_n^{ab} to (3.22) one ~~obtains~~ gets finally

$$\delta g_{mn} = g_{nk}\partial_m \xi^k + g_{mk}\partial_n \xi^k + \xi^k \partial_k g_{mn} \qquad (3.33)$$

We have obtained the very well known transformation of metric under infinitesimal general coordinate transformation with parameter $\xi^k(x)$. Therefore the field g_{mn} (3.28) can be identified with metric and the gauge field $e^a{}_m$ can be identified with tetrad. The relation (3.28) means expression of ~~the~~ spin connection in terms of tetrad and its derivatives.

Consider the curvature $R^{ab}{}_{mn}$ (3.23) and substitude there the $\omega_m{}^{ab}$ in terms of tetrad (3.28). Then construct the quantity $R = R^{ab}{}_{mn} e_a{}^m e_b{}^n$. One can prove that the quantity R ~~depend~~ is expressed only in terms of metric g_{mn} and its derivatives. Actually this R coinsides with scalar curvature in Riemann geometry.

As a result we shown how the general relativity can be reformulated in tetrad formalism. ~~Thus the~~ The gauge approach under consideration allows us to derive both the curvatures and the gauge transformations. Further we will use this approach to construct the supergravity.

4. Gauge Approach to Supergravity

In this section we will consider a derivation of supergravity action and corresponding gauge transformations.

4.1 Supersymmetry algebra

As we noted in very beginning, the supersymmetry is extension of special relativity symmetry. From mathematical point of view the symmetry of special relativity is formulated in terms of generators P_a and J_{ab} satisfying the Poincaré algebra. Therefore, extension of special relativity symmetry means extension of Poincare algebra. A general idea is based on use of such an extension of spinor generators Q_α^I, $\bar{Q}_{\dot{\alpha}}^I$ where $I = 1, 2, \ldots, N$. Integer N numerates the spinor generators.

It is postulated that the generators Q and $\bar{Q}$ possesses a fermionic nature so that their commutation relations are given in terms of anticommutators. Extension of Poincaré algebra assumes that we preserve all relations among the generators P_m and J_{mn} and add the following commutators and anticommutators

$$[P_m, Q_\alpha{}^I], \quad [P_m, \bar{Q}_{\dot\alpha}{}^I]$$
$$[J_{mn}, Q_\alpha{}^I], \quad [J_{mn}, \bar{Q}_{\dot\alpha}{}^I] \qquad (4.1)$$
$$\{Q_\alpha{}^I, Q_\beta{}^J\}, \quad \{\bar{Q}_{\dot\alpha}{}^I, \bar{Q}^J{}_{\dot\beta}\}$$
$$\{Q_\alpha^I, \bar{Q}_{\dot\alpha}^J\}$$

Here $\{A, B\}$ means the anticommutator. We assume that right hand sides of above commutators and anticommutators should be the linear combinations of all generators P_m, J_{mn}, Q_α^I, $\bar{Q}_{\dot\alpha}^I$ with some coefficients. The Lorentz covariance imposes the conditions on these coefficients, they can be constructed from the invariant quantities of the Lorentz group η_{mn}, $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot\alpha\dot\beta}$, $(\sigma_m)_{\alpha\dot\alpha}$, $(\tilde\sigma_m)_{\dot\alpha\alpha}$. Therefore, the most general form for the commutators and anticommutators (4.1) are written as follows

$$[P_m, Q_\alpha{}^I] = c_1 (\sigma_m)_{\alpha\dot\alpha} \bar{Q}^{\dot\alpha I}, \quad [P_m, \bar{Q}_{\dot\alpha}{}^I] = c_2 (\tilde\sigma_m)_{\dot\alpha\alpha} Q^{\alpha I}$$
$$[J_{mn}, Q_\alpha{}^I] = c_3 (\sigma_{mn})_\alpha{}^\beta Q_\beta{}^I, \quad [J_{mn}, \bar{Q}_{\dot\alpha}{}^I] = c_4 (\tilde\sigma_{mn})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}{}^I$$
$$\{Q_\alpha{}^I, Q_\beta{}^J\} = c_5 \varepsilon_{\alpha\beta} Z^{IJ} + \tilde{c}_5 (\sigma^{mn})_{\alpha\beta} J_{mn} X^{IJ} \qquad (4.2)$$
$$\{\bar{Q}_{\dot\alpha}{}^I, \bar{Q}_{\dot\beta}{}^J\} = c_6 \varepsilon_{\dot\alpha\dot\beta} \bar{Z}^{IJ} + \tilde{c}_6 (\tilde\sigma^{mn})_{\dot\alpha\dot\beta} J_{mn} \bar{X}^{IJ}$$
$$\{Q_\alpha{}^I, \bar{Q}_{\dot\alpha}{}^J\} = c_7\, 2 (\sigma^m)_{\alpha\dot\alpha} P_m \delta^{IJ}$$

Here $c_1, c_2, \dots, c_7$ are some numerical coefficients; $Z^{IJ}=-Z^{JI}$, $\bar{Z}^{IJ}=-\bar{Z}^{JI}$, $X^{IJ}=X^{JI}$, $\bar{X}^{IJ}=\bar{X}^{JI}$ are some matrices. The matrices σ_{mn}, $\tilde{\sigma}_{mn}$ have been introduced before

$$\sigma_{mn}=\frac{1}{4}(\sigma_m\tilde{\sigma}_n-\sigma_n\tilde{\sigma}_m)$$
$$\tilde{\sigma}_{mn}=\frac{1}{4}(\tilde{\sigma}_m\sigma_n-\tilde{\sigma}_n\sigma_m) \qquad (4.3)$$

To fix the coefficients in algebra (4.2) one uses the Jacobi identities written in terms of double commutators and anticommutators. After some transformations one gets

$$[P_m, Q_\alpha^I]=0, \qquad [P_m, \bar{Q}_{\dot{\alpha}}^I]=0$$
$$[J_{mn}, Q_\alpha^I]= i(\sigma_{mn})_\alpha{}^\beta Q_\beta^I$$
$$[J_{mn}, \bar{Q}^{\dot{\alpha}I}]= i(\tilde{\sigma}_{mn})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}I} \qquad (4.4)$$
$$\{Q_\alpha^I, Q_\beta^J\}=\varepsilon_{\alpha\beta}Z^{IJ}, \qquad \{\bar{Q}_{\dot{\alpha}}^I, \bar{Q}_{\dot{\beta}}^J\}=\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{Z}^{IJ}$$
$$\{Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J\}= 2(\sigma^m)_{\alpha\dot{\alpha}}P_m\delta^{IJ}$$

The relations (4.4), together with Poincare algebra for P_m, J_{mn}, form the Poincare superalgebra. The fermionic generators Q_α^I, $\bar{Q}_{\dot{\alpha}}^I$ are called the supercharges. The matrices Z^{IJ}, $\bar{Z}^{IJ}$ commute with all P_m, J_{mn}, Q_α^I, $\bar{Q}_{\dot{\alpha}}^I$, they are called the central charges. If $N=1$, the supersymmetry is called simple or $N=1$ supersymmetry. In this case the central charges are absent. If $N>1$, the supersymmetry is called extended or ~~N~~ N-extended supersymmetry. Statement that the relations (4.4) are the most general extension of Poincare algebra by fermionic generators is called the Haag, Lopuszanski, Sohnius theorem which was proved in 1975.

Further we will consider only $N=1$ supersymmetry.

4.2 Representations of $N=1$ superalgebra

We will show that the simplest massless representation of superalgebra with helicity 2 must include the helicity 3/2.

As we pointed out that the massless representations of the Poincare algebra are described by momenta ~~...~~ p_m and helicities λ, where $p^2=0$ and $\lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \ldots$

At given p_m the representations are one-dimensional. Parity transformation relates the representations with helicities λ and $-\lambda$. Therefore, if the theory is invariant under parity transformations it obligatory contains together the helicities λ and $-\lambda$

Since $P^2 = 0$ there exists such a reference system where the generator P_m has the form $P_m = (-E, 0, 0, E)$, $E > 0$. In such a reference system the condition $W_m = \lambda P_m$ leads to the following equation for helicity

$$J_{12} |\lambda\rangle = \lambda |\lambda\rangle \qquad (4.5)$$

Here we use the quantum mechanical notation $|\lambda\rangle$ for eigen vector of the operator J_{12} although the problem under considera- tion is purely classical.

Consider the relation $[J_{mn}, Q_\alpha] = i(\sigma_{mn})_\alpha{}^\beta Q_\beta$.

It leads to

$$[J_{12}, Q_\alpha] = +i(\sigma_{12})_\alpha{}^\beta Q_\beta =$$

$$= +\frac{i}{4}(\sigma_1\tilde\sigma_2 - \sigma_2\tilde\sigma_1)_\alpha{}^\beta Q_\beta = +\frac{i}{2}(\sigma_1\tilde\sigma_2)_\alpha{}^\beta Q_\beta$$

$$\sigma_1\tilde\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore

$$[J_{12}, Q_\alpha] = +\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_\alpha{}^\beta Q_\beta$$

Hence

$$[J_{12}, Q_1] = +\frac{1}{2} Q_1 \qquad (4.6)$$
$$[J_{12}, Q_2] = -\frac{1}{2} Q_2$$

~~Analogued~~

Analogously

$$[J_{12}, \bar{Q}_{\dot 1}] = -\frac{1}{2}\bar{Q}_{\dot 1} \qquad (4.7)$$

$$[J_{12}, \bar{Q}_{\dot 2}] = +\frac{1}{2}\bar{Q}_{\dot 2}$$

Consider how the operators $Q_1, \bar{Q}_{\dot 1}$ act on the vector $|\lambda\rangle$. We have

$$J_{12}Q_1|\lambda\rangle = [J_{12}, Q_1]|\lambda\rangle + Q_1 J_{12}|\lambda\rangle =$$

$$= +\frac{1}{2}Q_1|\lambda\rangle + \lambda Q_1|\lambda\rangle = \left(\lambda+\frac{1}{2}\right)Q_1|\lambda\rangle$$

It means that the vector $Q_1|\lambda\rangle$ corresponds to helicity $\lambda+\frac{1}{2}$. The operator Q_1 increases ~~decreases~~ the helicity to $\frac{1}{2}$. Analogously

$$J_{12}\bar{Q}_{\dot 1}|\lambda\rangle = [J_{12}, \bar{Q}_{\dot 1}]|\lambda\rangle + \bar{Q}_{\dot 1} J_{12}|\lambda\rangle =$$

$$= \left(\lambda-\frac{1}{2}\right)\bar{Q}_{\dot 1}|\lambda\rangle$$

The vector $\bar{Q}_{\dot 1}|\lambda\rangle$ corresponds to helicity $\lambda-\frac{1}{2}$. Operator $\bar{Q}_{\dot 1}$ decreases ~~increases~~ the helicity to $\frac{1}{2}$.

Consider now the relation $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^m_{\alpha\dot\alpha}P_m$. In the above reference system one gets

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2(-(\sigma_0)_{\alpha\dot\alpha}P_0 + (\sigma_3)_{\alpha\dot\alpha}P_3) =$$

$$= 2E(\sigma_0+\sigma_3)_{\alpha\dot\alpha} =$$

$$= 2E\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]_{\alpha\dot{\alpha}} = 4E\begin{pmatrix}1&0\\0&0\end{pmatrix}_{\alpha\dot{\alpha}}$$

Therefore

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 4E \qquad (4.8)$$

$$\{Q_2, \bar{Q}_{\dot{2}}\} = 0 \quad , \quad \{Q_1, \bar{Q}_{\dot{2}}\} = 0$$

Using also the relations $\{Q_2^*, \bar{Q}_2\} = 0$, $\{\bar{Q}_{\dot{2}}, \bar{Q}_{\dot{2}}\} = 0$ togther with $\{Q_2, \bar{Q}_{\dot{2}}\} = 0$, we can consider that $\{Q_1, \bar{Q}_{\dot{2}}\} = 0$ we can consider that $Q_2 = \bar{Q}_{\dot{2}} = 0$.

Consider the first equation in (4.8) and put

$$a^+ = \frac{1}{2\sqrt{E}} Q_1, \quad a^* = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}} \qquad (4.9)$$

then ~~to~~ one gets

$$\{a, a^+\} = 1 \qquad (4.10)$$

We obtained the anticommutatector for fermionic annihilation and creation operators. ~~As~~

As a result, a problem of constructing a basis of irreducible massless representation of the superalgebra is reduced to problem of basis for fermionic oscillator.

Let $|\lambda\rangle$ is a vector with helicity λ and let by definition

$$a|\lambda\rangle = 0 \qquad (4.11)$$

Then the basis vectors $a|\lambda\rangle$ and $a^+|\lambda\rangle$. Since $a^+ \sim \bar{Q}_{\dot{1}}$, the operator a^+ acting on ~~the~~ vector $|\lambda\rangle$ increases the helicity to $\frac{1}{2}$. Therefore, the basis of irreducible massless representations of the superalgebra include the helicities λ, $\lambda+\frac{1}{2}$. The value of λ is arbitrary so far. If we put $\lambda = \frac{3}{2}$ we obtain the supermultiplet with helicities $\frac{3}{2}$ and 2. Helicity 2 corresponds to metric field, that is to gravity. ~~Therefore~~ Therefore, the supergravity is a theory of interacting spin 2 and spin $\frac{3}{2}$ field.

4.3. Superalgebra in terms of four-components supercharges

Consider the relation

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^m)_{\alpha\dot{\alpha}} P_m$$

Introduce the four component spinor supercharge

$$S_A = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix} \qquad (4.12)$$

Then

$$\{S_A, S_B\} = \begin{pmatrix} \{Q_\alpha, Q_\beta\} & \{Q_\alpha, \bar{Q}^{\dot\beta}\} \\ \{\bar{Q}^{\dot\alpha}, Q_\beta\} & \{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2(\sigma^m)_\alpha{}^{\dot\beta} P_m \\ 2(\tilde\sigma^m)^{\dot\alpha}{}_\beta P_m & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & (\sigma^m)_\alpha{}^{\dot\beta} \\ (\tilde\sigma^m)^{\dot\alpha}{}_\beta & 0 \end{pmatrix} P_m$$

Introduce the matrices

$$C = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot\alpha\dot\beta} \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} \qquad (4.13)$$

and consider

$$\gamma^m C^{-1} = \begin{pmatrix} 0 & \sigma^m \\ \tilde\sigma^m & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & (\sigma^m)_{\alpha\dot\alpha}\,\varepsilon^{\dot\alpha\dot\beta} \\ (\tilde\sigma^m)^{\dot\alpha\alpha}\,\varepsilon_{\alpha\beta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\sigma^m)_\alpha{}^{\dot\beta} \\ (\tilde\sigma^m)^{\dot\alpha}{}_\beta & 0 \end{pmatrix}$$

therefore

$$\{S_A, S_B\} = 2(\gamma^m C^{-1})_{AB} P_m \qquad (4.14)$$

This is the anticommutation relation for four-component supercharges

One can prove that the matrix C satisfies the relation

$$C\gamma^m C^{-1} = -(\gamma^m)^T \tag{4.15}$$

It means that C is a charge conjugation matrix. One can check that $C = i\gamma^0\gamma^2$.

Consider now the relations

$$[J_{mn}, Q_\alpha] = i(\sigma_{mn})_\alpha{}^\beta Q_\beta$$

$$[J_{mn}, \bar{Q}^{\dot\alpha}] = i(\tilde{\sigma}_{mn})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}$$

It can be rewritten as follows

$$[J_{mn}, S_A] = i(\Sigma_{mn})_{AB} S_B \tag{4.16}$$

where

$$\Sigma_{mn} = \frac{1}{4}(\gamma_m\gamma_n - \gamma_n\gamma_m) \tag{4.17}$$

We see that superalgebra is written completely through the four-component supercharge S_A

4.4. N = 1, D = 4 supergravity

Construction of the supergravity is based on gauging the superalgebra. Let us write again the algebra of ~~supercharges~~ in four dimensional form to

$$[P_a, P_b] = 0$$
$$[J_{ab}, P_c] = i(\eta_{bc} P_a - \eta_{ac} P_b)$$
$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}) \quad (4.18)$$
$$[J_{ab}, S_A] = i(\Sigma_{ab})_{AB} S_B$$
$$\{S_A, S_B\} = \cancel{2}(\gamma^a C^{-1}) P_a$$

with

$$C^{-1} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix}$$

We change the generators
$S_A \to \sqrt{2}\, S_A$

(4.19)

$$\Sigma_{ab} = \frac{1}{4}(\gamma_a \gamma_b - \gamma_b \gamma_a)$$

First of all let us calculate the curvatures or strengths corresponding to algebra (4.19). Introduce the covariant derivative

$$\nabla_m = \partial_m - i e^a{}_m P_a - \frac{i}{2}\omega_m{}^{ab} J_{ab} - i \bar\Psi_{mA} S_A = \quad (4.20)$$
$$= \nabla_m^{(0)} - i \bar\Psi_{mA} S_A$$

Here $\nabla_m^{(0)}$ is the covariant derivative corresponding to pure Poincaré algebra. This derivative was introduced in section 3. Ψ_{mA} is additional gauge field associated with supercharge S_A. This ~~[illegible]~~ is spin 3/2 ~~[illegible]~~ ~~field~~ vector Majorana spinor fields, i.e

$$\bar\Psi_{mA} = C_{AB} \Psi_B^T \quad (4.21)$$

where C_{AB} is charge conjugate matrix.

Consider

$$[\nabla_m, \nabla_n] = [\nabla_m^{(0)} - i\bar\Psi_{mA} S_A, \nabla_n^{(0)} - i\bar\Psi_{nB} S_B] =$$

$$= [\nabla_m^{(0)}, \nabla_n^{(0)}] - i[\nabla_m^{(0)}, \bar\Psi_{nA} S_A] -$$

$$- i[\bar\Psi_{mA} S_A, \nabla_n^{(0)}] - \bar\Psi_{mA}\bar\Psi_{nB}\{S_A, S_B\} \qquad (4.22)$$

Note, to get the anticommutator we should assume that the fields $\bar\Psi_{mA}$ are anticommuting,

$$\bar\Psi_{mA}\bar\Psi_{nB} = -\bar\Psi_{nB}\bar\Psi_{mA} \qquad (4.23)$$

Using the commutators of generators (4.20) we obtain

$$[\nabla_m, \nabla_n] = -iR^a{}_{mn}P_a - \frac{i}{2}R^{ab}{}_{mn}J_{ab} - \qquad (4.24)$$

$$-iR_{mnA}S_A$$

where

$$R^a{}_{mn} = \mathcal{D}_m e^a{}_n - \mathcal{D}_n e^a{}_m$$

$$R^{ab}{}_{mn} = \partial_m\omega_n{}^{ab} + \omega_m{}^{ac}\omega_{nc}{}^b - \partial_n\omega_m{}^{ab} - \omega_n{}^{ac}\omega_{mc}{}^b =$$

$$= \mathcal{D}_m\omega_n{}^{ab} - \mathcal{D}_n\omega_m{}^{ab} = R^{ab}{}_{mn}(\omega) \qquad (4.25)$$

$$R_{mnA} = \mathcal{D}_m\bar\Psi_{nA} - \mathcal{D}_n\bar\Psi_{mA}$$

Here

$$D_m e^a{}_n = \partial_m e^a{}_n + e^b{}_m \omega_{nb}{}^a - \frac{i}{2}\bar{\Psi}_m \gamma^a \Psi_n \qquad (4.26)$$

$$D_m \bar{\Psi}_{nA} = \partial_m \bar{\Psi}_{nA} + \frac{1}{2}\omega_m{}^{ab}\bar{\Psi}_{nB}(\Sigma_{ab})_{BA}$$

$$D_m \omega_n{}^{ab} = \partial_m \omega_n{}^{ab} + \omega_m{}^{ac}\omega_{nc}{}^b$$

pay attention that the curvature $R^{ab}{}_{mn}$ has the same form as in the conventional gravity considered in subsection ~~3.3~~ 3.3. It depends only on $\omega_m{}^{ab}$.

In principle, like in conventional gravity we can impose the constraint

$$R^a{}_{mn} = 0 \qquad (4.27)$$

this constraints is solved the same way like in subsection 3.3 and allows us to express the spin connection $\omega_m{}^{ab}$ in terms of $e_m{}^a$ and Ψ_{mA}

$$\omega_{nkm} = \frac{1}{2}\Big[e_{ak}\big(\partial_m e^a{}_n - \frac{i}{2}\bar{\Psi}_m \gamma^a \Psi_n - \partial_n e^a{}_m + \frac{i}{2}\bar{\Psi}_n \gamma^a \Psi_m\big) +$$

$$+ e_{am}\big(\partial_n e^a{}_k - \frac{i}{2}\bar{\Psi}_n \gamma^a \Psi_k - \partial_k e^a{}_n + \frac{i}{2}\bar{\Psi}_k \gamma^a \Psi_m\big) -$$

$$- e_{an}\big(\partial_k e^a{}_m - \frac{i}{2}\bar{\Psi}_k \gamma^a \Psi_m - \partial_m e^a{}_k + \frac{i}{2}\bar{\Psi}_m \gamma^a \Psi_k\big)\Big] \qquad (4.28)$$

It means that

$$\omega_{nkm} = \bar{\omega}_{nkm}(e, \Psi) \qquad (4.29)$$

$$\omega_{nkm} = \omega_n{}^{ab} e_{ak} e_{bm}$$

Now let us find the gauge transformations. Write

$$\nabla_m = \partial_m - i\Gamma_m$$

Then

$$\delta\Gamma_m = \partial_m T - i[\Gamma_m, T] \qquad (4.30)$$

where

$$\Gamma_m = e^a{}_m P_a + \frac{1}{2}\omega_m{}^{ab} J_{ab} + \bar{\Psi}_{mA} S_A = \Gamma_m^{(0)} + \bar{\Psi}_{mA} S_A \qquad (4.31)$$

$$T = a^a P_a - \frac{1}{2}\omega^{ab} J_{ab} + \bar{\epsilon}_A S_A \qquad (4.32)$$

Here $\Gamma_m^{(0)}$ is a connection corresponding to pure gravity, a^a, ω^{ab}, $\bar{\epsilon}_A$ are the parameters of local translations, local Lorentz rotations and supersymmetry transformations respectively.

We write

$$T = T^{(0)} + \bar{\epsilon}_A S_A$$

Then the relation (4.24) gives

$$\delta\Gamma^{(0)}_m + \delta\bar\Psi_{mA} S_A = \partial_m T^{(0)} + \partial_m \bar\epsilon_A S_A -$$

$$- i\,[\Gamma_m^{(0)} + \bar\Psi_{mA} S_A,\ T^{(0)} + \bar\epsilon_B S_B] = \partial_m T^{(0)} + \partial_m \bar\epsilon_A S_A$$

$$\cancel{+}(-i)[\Gamma_m^{(0)}, T^{(0)}] - i\,[\bar\Psi_{mA} S_A, \epsilon_B S_B] -$$

$$- i\,[\Gamma_m^{(0)}, \bar\epsilon_A S_A] - i\,[\bar\Psi_{mA} S_A, T^{(0)}] \qquad (4.33)$$

All commutators are calculated on the base of algebra (4.18). Pay attention to commutator

$$\bar\Psi_{mA} S_A\, \epsilon_B S_B - \bar\epsilon_B S_B \bar\Psi_{mA} S_A =$$

$$= \bar\Psi_{mA} \epsilon_B\, S_A S_B - \bar\epsilon_B \bar\Psi_{mA}\, S_B S_A$$

To use the algebra (4.18) we should get the anticommutator of supercharges. It is possible only the fields Ψ_{mA} and the parameters $\bar\epsilon_B$ are anticommuting, i.e.

$$\bar\Psi_{mA} \epsilon_B = -\bar\epsilon_B \bar\Psi_{mA}. \qquad (4.34)$$

Writing $\delta\Gamma^{(0)}_m = \delta e^a{}_m P_a + \frac{1}{2}\delta\omega_m^{ab} J_{ab} +$ $+ \delta\bar\Psi_{mA} S_A$, $\partial_m T = \partial_m a^a P_a - \frac{1}{2}\partial_m \omega^{ab} J_{ab} +$ $+ \partial_m \bar\epsilon_A S_A$ and calculating the commutators in (4.32) one gets

$$\delta e^a{}_m = D_m a^a + \cancel{\omega^{ab} e_{bm}} + i(\bar\epsilon \gamma^a \psi_m) \qquad (4.35)$$

$$\delta \psi_{mA} = D_m \epsilon_A + \frac{1}{2}\omega^{ab}(\Sigma_{ab})_{AB}\psi_B$$

Gauge transformation for $\omega_m{}^{ab}$ has the same form as in conventional gravity.

As a result we derived both the strengths and the gauge transformations. Further we will focus only on supersymmetry transformations with parameter ϵ_A

$$\delta e^a{}_m = i(\bar\epsilon \gamma^a \psi_m) \qquad (4.36)$$

$$\delta \psi_{mA} = D_m \epsilon_A$$

The transformations with parameters a^a, ω^{ab} are responsible for local translations and local Lorentz transformations. One can prove that general coordinate transformations with parameter $\xi^m(x) = e_a{}^m a^a(x)$ is a linear combination of the ~~transformations~~ all above transformations with field dependent parameters plus $R^a{}_{mn}$. Therefore if $R^a{}_{mn} = 0$, that is $\omega_m{}^{ab} = \omega_m{}^{ab}(e, \psi)$ the general coordinate transformation ~~[illegible]~~ is the consequence of all other transformations

Last step of construction is a formulation of invariant action. Like in Yang-Mills theory and in gravity ~~it is natural~~ to find the action it is natural to use the strengths. The strength $R^a{}_{mn}$ was ~~~~ already used to express the spin connection in terms of $e^a{}_m$, ψ_m, ~~~~ $\omega_m{}^{ab} = \omega_m{}^{ab}(e, \psi)$ (4.29). After that we have $R^{ab}{}_{mn}(\omega(e,\psi))$ and ~~~~ $R_{mnA}(\omega(e,\psi), \psi)$. Take into account that the supergravity action should coincide with pure gravity action at $\psi = 0$ we can write

$$S_{SUGRA} = \frac{1}{2\varkappa^2}\int d^4x\, e\Big\{ e_a{}^m e_b{}^n R^{ab}{}_{mn}(\omega(e,\psi)) + \frac{1}{2}\bar{\psi}_{mA}(\gamma^{[m}\gamma^n\gamma^{k]})_{AB} R_{nkB} \Big\} \quad (4.37)$$

This action must be automatically invariant under general coordinate transformations and local Lorentz transformations. The separate question is to show that it is invariant under local supersymmetry transformations (4.36). It can be proved independently.

$$\delta S = \int d^4x \left(\frac{\delta S}{\delta e^a{}_m}\, \delta e^a{}_m + \frac{\delta S}{\delta \omega_m{}^{ab}}\, \delta \omega_m{}^{ab}(e,\psi) + \bar{\psi}_m \frac{\delta S}{\delta \bar{\psi}_m} \right) \tag{4.38}$$

But, if $\omega_m{}^{ab} = \omega_m{}^{ab}(e,\psi)$, the equation $\frac{\delta S}{\delta \omega_m{}^{ab}} = 0$ is automatically fulfilled. Therefore the second term in right hand side of (4.38) vanishes identically. Then substituting the equations (4.36) one obtains after some tedious calculations $\delta S = 0$. The action (4.37) is invariant under the local supersymmetry transformations. The theory with action (4.37) is called $N=1$, $D=4$ supergravity.