Emergence of cosmological scaling behavior in the asymptotic regime

Skugoreva M. A.*, Sami M., Jaman N.

*Kazan (Volga region) Federal University

Introduction

- Scalar fields in General Relativity (GR) are used for the description of inflationary era as well as the phenomenon of late-time acceleration. The models of so-called **quintessential inflation** unite the description of both of these accelerated stages.
- **Scaling solutions** are solutions with the constant ratio of energy densities of scalar field and matter and equal parameters of their equations of state.
- Scaling solutions allow the field energy density to mimic the background being sub-dominant during radiation and matter dominating eras. In dark energy models scaling behavior makes the evolution free of initial conditions. However, one needs late-time exit from scaling regime to acceleration.

Aim of the work

The aim of this work was the investigation of cosmological dynamics in the scalar field model of GR with the Lagrangian of the form

$$L=\frac{1}{2}\sqrt{-g}\left(-\frac{R}{8\pi G}+(\nabla\varphi)^2-2V(\varphi)\right)+L_m \quad ,$$

where $V(\varphi)=V_0\left(\frac{\varphi}{M_{Pl}}\right)^m e^{-\lambda\frac{\varphi^n}{M_{Pl}^n}}$, $V_0>0$, $\lambda>0$, $n\geqslant 1$, $m\geqslant 0$.

$$p=w\rho \quad , \quad w\in[-1;1]$$

The metrics and Planck units are used

$$ds^2=dt^2-a^2(t)dl^2$$
$$c=\hbar=1$$

Important earlier works

The models of GR with the scalar field potential

$$V(\varphi) = V_0 \left(\frac{\varphi}{M_{Pl}} \right)^m \exp(-\lambda \varphi^n / M_{Pl}{}^n)$$

have been studied in the following papers:

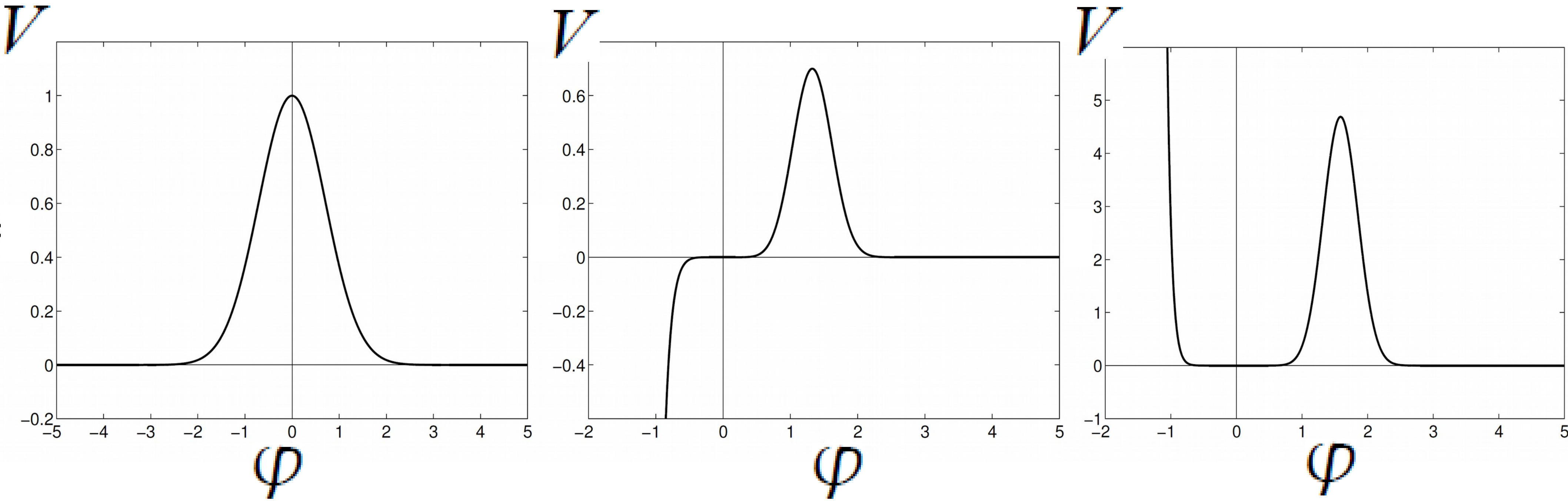
1). *m = 0, n = 1* – the exponential potential. E. J. Copeland, A. R. Liddle and D. Wands. **Phys. Rev. D 57, 4686 (1998)**.

2). *m = 0, n > 1* – the generalized exponential potential. C.-Q. Geng, C.-C. Lee, M. Sami, E. N. Saridakis and A. A. Starobinsky. **J. Cosmol. Astropart. Phys. 06 (2017) 011.**

3). *m > 1, n > 1* – the generalized potential. P. Parsons and J. D. Barrow. **Phys. Rev. D 51, 6757 (1995).** J. Rubio and C. Wetterich. Phys. **Rev. D 96, 063509 (2017)**.

5

Fig. 1. The generalized potential $V(\varphi)=V_0\left(\frac{\varphi}{M_{Pl}}\right)^m \exp(-\lambda\varphi^n/M_{Pl}{}^n)$ for several sets of papramaters *n*, *m* and $V_0=1$, $\lambda=1$, $M_{Pl}{}^2=1$.



$n=2$, $m=0$ | $n=3$, $m=7$ | $n=3$, $m=12$

Methods of the investigation

Methods of **dynamical system theory**,

the numerical integration,

methods of theory of differential equations,

algebraic methods

are applyed in this work.

Main equations

Equations of gravitation and scalar fields are derived by varying the action with the considered Lagrangian

$$3H^2 {M_{Pl}}^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \rho \quad , \qquad (1)$$

$$(2\dot{H} + 3H^2) {M_{Pl}}^2 = -\frac{1}{2}\dot{\varphi}^2 + V(\varphi) + w\rho \quad , \qquad (2)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0 \quad , \qquad (3)$$

here $\quad {M_{Pl}}^2 = \frac{1}{8\pi G}$, $\qquad ' = \frac{d}{d\varphi}$.

Main equations

The scalar field energy density $$\rho_\varphi = \frac{\dot{\varphi}^2}{2} + V(\varphi) \ ,$$

the scalar field pressure $$p_\varphi = \frac{\dot{\varphi}^2}{2} - V(\varphi) \ ,$$

the scalar field equation of state parameter $$w_\varphi = \frac{p_\varphi}{\rho_\varphi} = \frac{\dot{\varphi}^2/2 - V(\varphi)}{\dot{\varphi}^2/2 + V(\varphi)} \ ,$$

the important quantity for the scaling solution $$\Gamma \equiv \frac{V''}{(V')^2 V} \ ,$$

the fractional density of the scalar field $$\Omega_\varphi = \frac{\rho_\varphi}{3 M_{Pl}^2 H^2} \ .$$

In the scaling solution $\Gamma \to 1 \ , \ w_\varphi \to w \ , \ |\varphi| \to \infty \ .$

9 Scheme of the basic method

- Introduction of new variables $(a, \rho, \varphi, \ldots) \rightarrow (x, y, z, \ldots)$

$$\begin{cases} \dfrac{dx}{d(\ln(a))} = f_1(x, y, \ldots) \\ \dfrac{dy}{d(\ln(a))} = f_2(x, y, \ldots) \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

- We find stationary points

$$\begin{cases} f_1 = 0 \\ f_2 = 0 \\ \ldots\ldots \end{cases} \Rightarrow \begin{matrix} (x_{stat1}, y_{stat1}, \ldots) \\ (x_{stat2}, y_{stat2}, \ldots) \\ \ldots\ldots\ldots\ldots \end{matrix}$$

- Type of stability is determined by **Lyapunov (1892**) in linear approach

$$\begin{pmatrix} (\delta x)' \\ (\delta y)' \\ \ldots \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \ldots \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \ldots \end{pmatrix} \Rightarrow \lambda_1, \quad \lambda_2, \quad \lambda_3$$

eigenvalues

10 Asymptotic solutions in the model with $V(\varphi)=V_0\left(\dfrac{\varphi}{M_{Pl}}\right)^m \exp(-\lambda\varphi^n/M_{Pl}{}^n)$.

1. $$\varphi(t)=\varphi_0+\varphi_1\frac{w+1}{w-1}t^{\frac{w-1}{w+1}}\approx\varphi_0\,,\quad a(t)=a_0 t^{\frac{2}{3(w+1)}}\,,$$

$$\rho(t)=\rho_0 t^{-2},\quad t\to+\infty\,,\quad w\neq-1\,.$$

2. $$\varphi(t)=\varphi_0+\frac{\varphi_1}{2}t^2\approx\varphi_0\,,\quad a(t)=a_0 t^{\frac{2}{3(w+1)}}\,,$$

$$\rho(t)=\rho_0 t^{-2},\quad t\to+\infty\,,\quad w\neq-1\,.$$

1a, 2a. $$\varphi(t)=\varphi_0-\frac{\varphi_1}{3H_0}e^{-3H_0 t}\to\varphi_0\,,\quad a(t)=a_0 e^{H_0 t}\,,$$

$$\rho(t)=\rho_0\,,\quad t\to+\infty\,,\quad w=-1\,.$$

11 Asymptotic solutions in the model with $V(\varphi)=V_0\left(\frac{\varphi}{M_{Pl}}\right)^m \exp(-\lambda\varphi^n/M_{Pl}{}^n)$.

3. $\lambda\left(\frac{\varphi}{M_{Pl}}\right)^n = f_0 \ln\left(\frac{t}{t_1}\right) + f_1 \ln\left(\ln\left(\frac{t}{t_1}\right)\right) + \dots$, **scaling solution**

$$f_0=2\ ,\ \ f_1=2+\frac{m-2}{n}\ ,\ \ t_1{}^2=\frac{M_{Pl}{}^2(1-w)}{V_0 n^2(1+w)}\, 2^{\frac{2-n-m}{n}}\,\lambda^{\frac{m-2}{n}}\ ,$$

$$a(t)=a_0 t^{\frac{2}{3(w+1)}},\quad \rho(t)=\rho_0 t^{-2},\quad t\to+\infty,\quad w\neq-1\ .$$

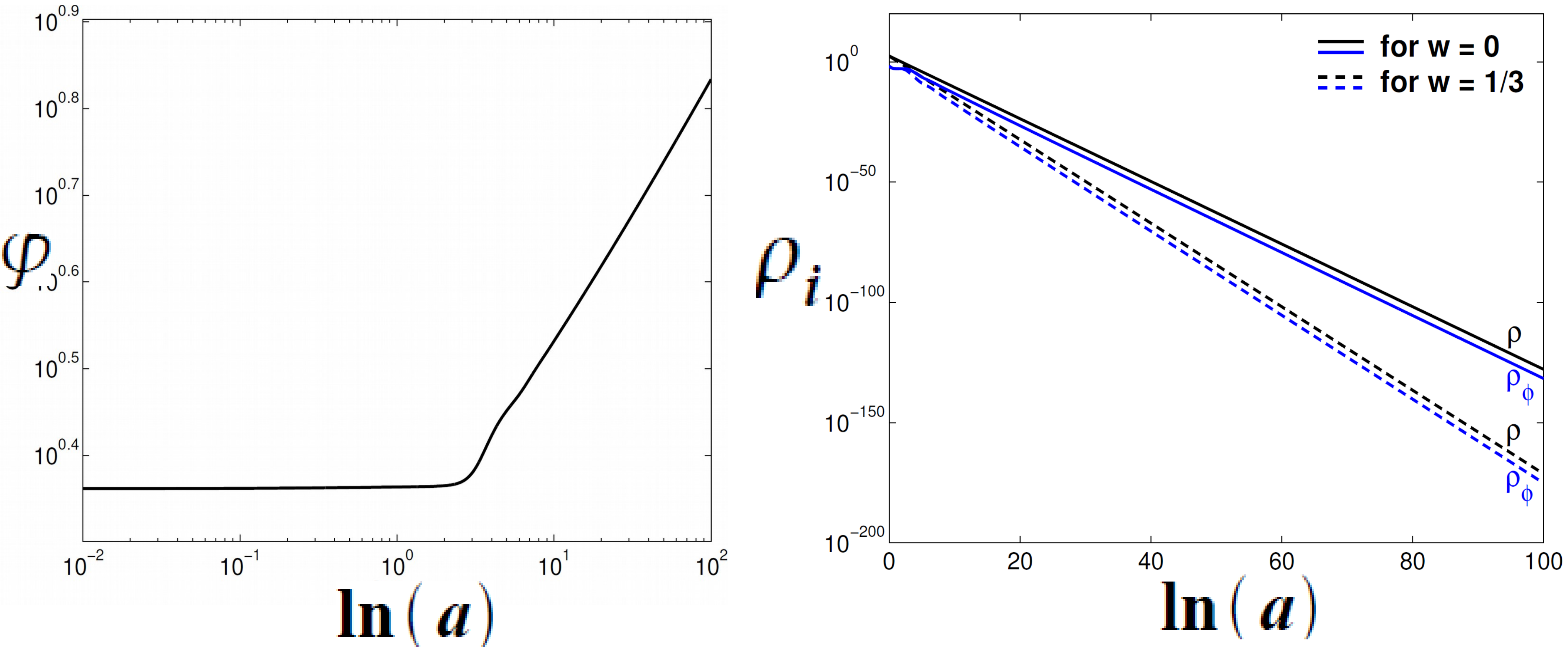
4. $\varphi(t)=\varphi_0(t-t_0)^{\frac{2}{2-m}}$, $a(t)=a_0 t^{\frac{2}{3(w+1)}}$, $\rho(t)=\rho_0 t^{-2}$,

$\varphi_0=\frac{2M_{Pl}{}^m(m(1-w)-4)}{V_0 m(1+w)(2-m)^2}$, exists for ***0 < m < 2*** $t\to t_0$

n > 2 , $t\to+\infty$

It is **stable** for m > 2, $w\in\left(-1;\frac{m-6}{m+2}\right)$. $w\neq-1$.

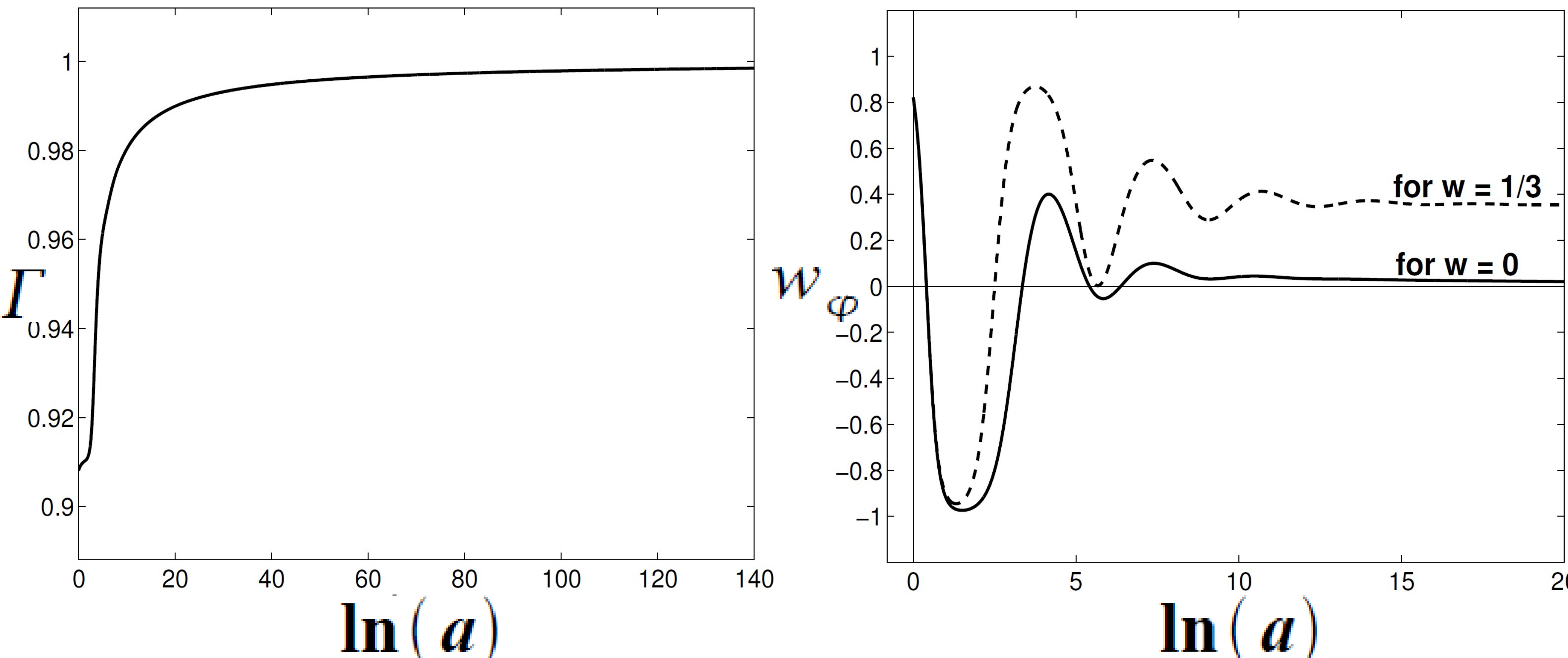
12 **Fig. 2. The evolutions of φ, ρ, ρ_φ in the model with $V(\varphi)=V_0\left(\frac{\varphi}{M_{Pl}}\right)^m \exp(-\lambda\varphi^n/M_{Pl}{}^n)$. The cosmological evolution ends in scaling solution.**



$$n=3 \quad , \quad m=7$$

$$V_0=1 \quad , \quad \lambda=1 \quad , \quad M_{Pl}{}^2=1$$

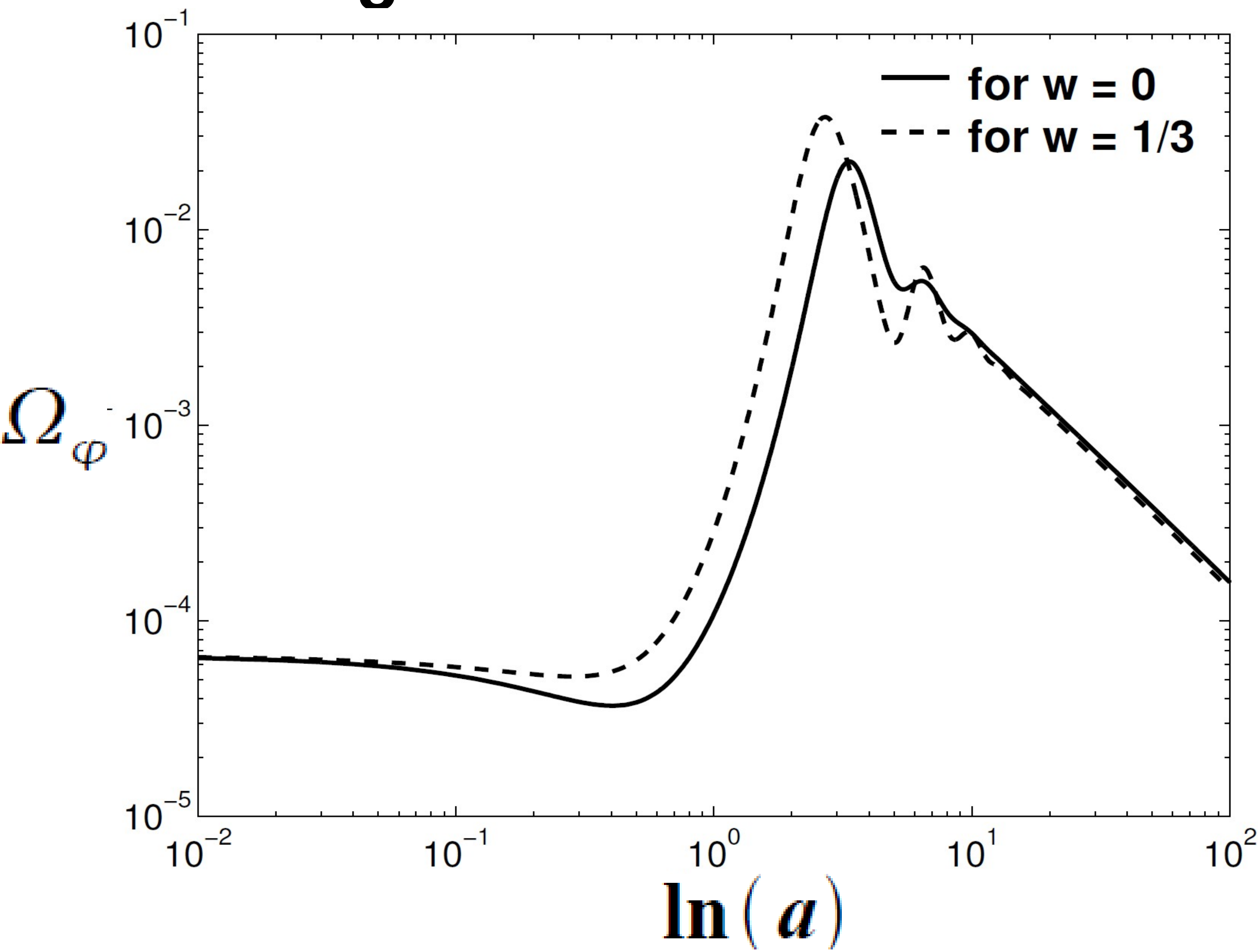
Fig. 3. The evolutions of $\Gamma \equiv \frac{V''}{(V')^2 V}$, w_φ in the model with $V(\varphi)=V_0\left(\frac{\varphi}{M_{Pl}}\right)^m \exp(-\lambda\varphi^n/M_{Pl}{}^n)$. The cosmological evolution ends in scaling solution.



$$n=3 \quad , \quad m=7$$
$$V_0=1 \quad , \quad \lambda=1 \quad , \quad M_{Pl}{}^2=1$$

14

Fig. 4. The evolutions of $\Omega_\varphi = \frac{\rho_\varphi}{3 M_{Pl}^2 H^2}$ in the model with $V(\varphi) = V_0 \left(\frac{\varphi}{M_{Pl}}\right)^m \exp(-\lambda \varphi^n / M_{Pl}^n)$. The cosmological evolution ends in scaling solution.



$n = 3$

$m = 7$

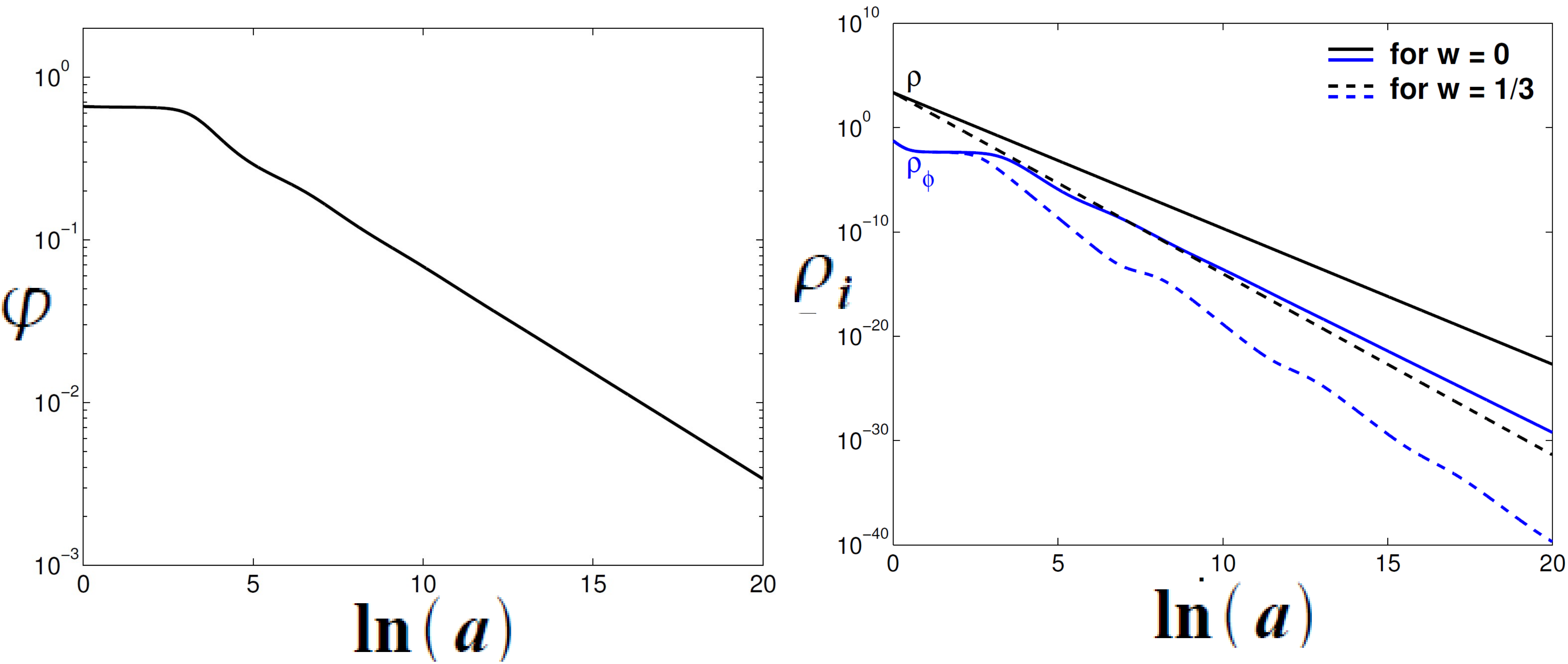
$V_0 = 1$

$\lambda = 1$

$M_{Pl}^2 = 1$

Ω_φ decreases as $(\ln(a))^{\frac{2(1-n)}{n}}$.

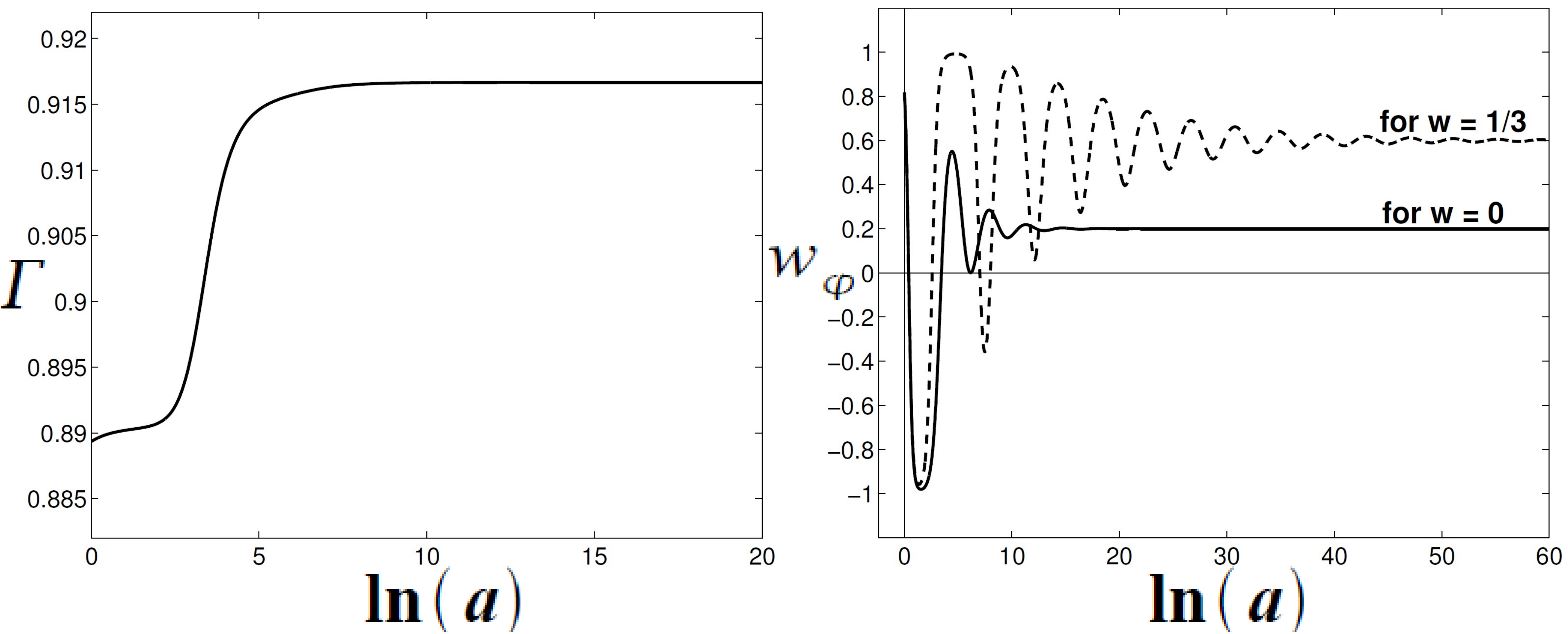
Fig. 5. The evolutions of φ, ρ, ρ_φ in the model with $V(\varphi)=V_0\left(\frac{\varphi}{M_{Pl}}\right)^m \exp(-\lambda\varphi^n/M_{Pl}{}^n)$. The cosmological evolution ends in power-law solution.



$$n=3\quad,\quad m=12$$

$$V_0=1\quad,\quad\lambda=1\quad,\quad M_{Pl}{}^2=1$$

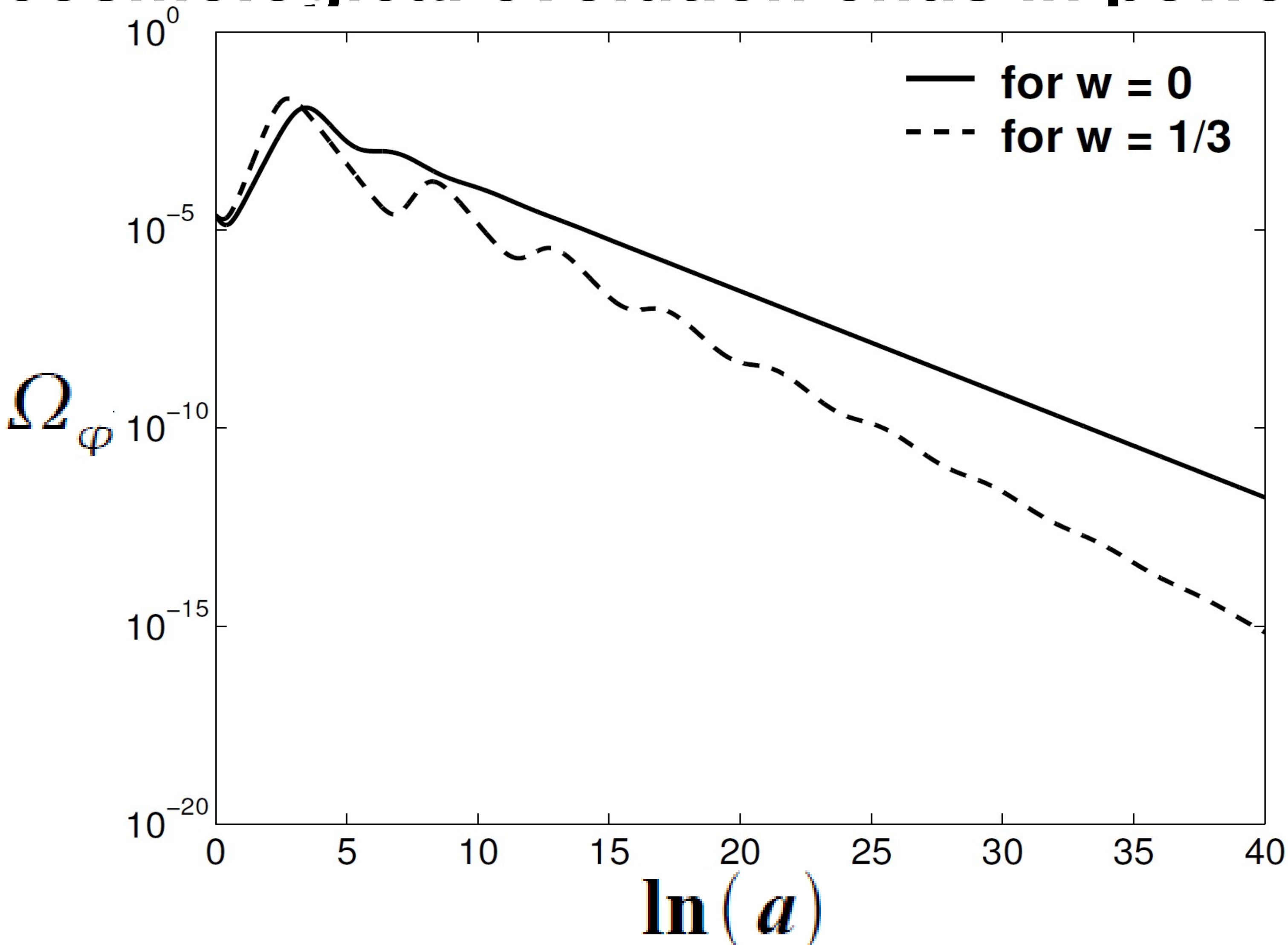
Fig. 6. The evolutions of $\Gamma \equiv \frac{V''}{(V')^2 V}$ **,** w_φ **in the model with** $V(\varphi) = V_0 \left(\frac{\varphi}{M_{Pl}} \right)^m \exp(-\lambda \varphi^n / M_{Pl}{}^n)$ **.The cosmological evolution ends in power-law solution.**



$$n = 3 \quad , \quad m = 12$$

$$V_0 = 1 \quad , \quad \lambda = 1 \quad , \quad M_{Pl}{}^2 = 1$$

Fig. 7. The evolutions of $\Omega_{\varphi}=\frac{\rho_{\varphi}}{3M_{Pl}^{2}H^{2}}$ in the model with $V(\varphi)=V_0\left(\frac{\varphi}{M_{Pl}}\right)^{m}\exp(-\lambda\varphi^{n}/M_{Pl}^{n})$. The cosmological evolution ends in power-law solution.



$n=3$

$m=12$

$V_0=1$

$\lambda=1$

$M_{Pl}^{2}=1$

Conclusion

In the model of GR with the potential

$$V(\varphi)=V_0\left(\frac{\varphi}{M_{Pl}}\right)^m \exp\left(-\lambda\varphi^n/M_{Pl}{}^n\right)$$

1. we have shown that the asymptotic power-law solution exists and its **stability conditions** are m > 2, $w\in\left(-1;\frac{m-6}{m+2}\right)$. In this asymptotic $\varphi\to\mathbf{0}$,

$$w_\varphi\to\frac{wm+2}{m-2}\ ,\quad \Gamma=\frac{V''}{(V')^2 V}\to\frac{m-1}{m}\ .$$

2. The **scaling solution** exists and **stable** for some region of initial data. In this asymptotic regime $|\varphi|\to\infty$, $w_\varphi\to w$, $\Gamma\to 1$. The scalar field energy density ρ_φ decreases slightly faster than ρ and Ω_φ tends to zero. The said behavior is suited to models of quintessential inflation. One needs mechanism responsible for late-time exit from scaling regime to accelerated expansion.

Thanks for attention!

Proposals of the work send to

Maria A. Skugoreva

masha-sk@mail.ru