Massive gravity theories

Mikhail S. Volkov

Institut Denis Poisson, University of Tours, FRANCE

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Motivations for massive gravity

Cosmic acceleration $\Rightarrow$ dark energy problem.

- either Λ-term, very natural phenomenologically,

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad \rightarrow \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

  but unnatural from the QFT viewpoint

- or modification of gravity (many options). Massive gravity:

$$\text{Newton} \ \frac{1}{r} \quad \rightarrow \quad \text{Yukawa} \ \frac{1}{r} e^{-mr}$$

  $m \sim 1/(\text{Hubble radius}) \sim 10^{-33}$ eV. If $r <$ Hubble, then Yukawa=Newton, usual physics. Screening for $r \geq$ Hubble $\Rightarrow$ gravity is weaker at large distance = cosmic acceleration.

- From QFT viewpoint small m is more natural (multiplicative renormalization) than small Λ (additive renormalization).

Contents

References

Reviews:

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Rubakov and Tyniakov, UFN (2008)

Key names classical: Fierz-Pauli, van-Dam-Veltman-Zakharov, Vainshtein, Boulwre-Deser, Ogievetsky-Polybarinov

Key names recent: de Rham-Gabadadze-Tolley, Hassan-Rosen

Other names: Comelli-Pilo, Mukohyama et al., Visser et al., Deser-Waldron, Akrami-Kovisto et at., Deffayet et al., etc.

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Fierz-Pauli massive gravity

Linear massless gravitons – linearized GR

$\mathcal{L} = \frac{1}{2\kappa} R\sqrt{-g} + \mathcal{L}_{\mathrm{matter}}$ /$\kappa = 8\pi G$, signature $(-+++)$ /

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}}$$

If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ then /check this/

$$\begin{aligned} - & \frac{1}{2}\left\{\Box h_{\mu\nu} - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \eta_{\mu\nu}(\partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h) + \partial_{\mu\nu} h\right\} \\ \equiv & -\frac{1}{2}(\Box h_{\mu\nu} + \ldots) = \kappa T_{\mu\nu} \end{aligned}$$

so that

$$\boxed{\Box h_{\mu\nu} + \ldots = -2\kappa T_{\mu\nu}}$$

Gauge invariance $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ does not change the l.h.s. $\Rightarrow$ Bianchi identities

$$0 \equiv \partial^\mu(\Box h_{\mu\nu} + \ldots) \quad \Rightarrow \quad \partial^\mu T_{\mu\nu} = 0$$

DoF counting

Gauge invariance $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ implies that one can impose gauge conditions. With $\mathbf{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}$ one requires

$$\partial^\mu \mathbf{h}_{\mu\nu} = 0 \qquad\qquad \text{4 gauge conditions}$$

and the equations reduce to

$$\Box \mathbf{h}_{\mu\nu} = -2\kappa T_{\mu\nu}$$

Residual gauge freedom with $\Box \xi_\mu = 0 \Rightarrow$ one can impose 4 more conditions $\Rightarrow$ $2 = 10 - 4 - 4$ DoF. If $T_{\mu\nu} = 0$

$$\mathbf{h} = 0, \quad \mathbf{h}_{0k} = 0 \qquad \Rightarrow \qquad \mathbf{h}_{00} = 0, \quad \partial_i \mathbf{h}_{ik} = 0$$

the solution is

$$\mathbf{h}_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_+ & D_\times & 0 \\ 0 & D_\times & -D_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik(t-z)}$$

Quadratic action

The equations can be obtained from $S = \int \mathcal{L}\, d^4x$ with

$$\mathcal{L} = \frac{1}{\kappa}\,\mathcal{L}_0 + \frac{1}{2}\,h_{\mu\nu}T^{\mu\nu}$$

with

$$\begin{aligned} \mathcal{L}_0 &= \text{quadratic part of } \left\{\frac{1}{2}\,R\sqrt{-g}\right\} \\ &= \frac{1}{4}\left(-\frac{1}{2}\,\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \partial_\mu h_{\nu\alpha}\partial^\nu h^{\mu\alpha} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\,\partial_\mu h \partial^\mu h\right) \end{aligned}$$

which is invariant under diffeomorphisms

$$\mathcal{L}_0(h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu) = \mathcal{L}_0(h_{\mu\nu})$$

The matter term is also invariant since $\partial_\mu T^{\mu\nu} = 0$.

Linear massive gravitons – Fierz and Pauli /1939/

$\Box\phi = 0 \Rightarrow \Box\phi = m^2\phi$. Similarly for gravitons $/h = \eta^{\mu\nu}h_{\mu\nu}/$

$$\Box h_{\mu\nu} + \ldots = m^2(h_{\mu\nu} - \alpha\, h\, \eta_{\mu\nu}) - 2\kappa T_{\mu\nu}$$

$\Rightarrow$ no gauge invariance anymore. Taking the divergence gives 4 constraints

$$m^2(\partial^\mu h_{\mu\nu} - \alpha\partial_\nu h) = 0$$

Taking the trace and using the 4 constraints gives

$$2(\alpha - 1)\,\Box h = m^2(1 - 4\alpha)\, h - 2\kappa T$$

$\Rightarrow$ for $\alpha = 1$ one gets the fifth constraint

$$h = -\frac{2\kappa}{3m^2}\, T$$

$\Rightarrow$ $10 - 5 = 5$ DoF=graviton polarizations.

FP action

The FP equations can be obtained from $S = \int \mathcal{L}_{\rm FP} d^4x$ with

$$\mathcal{L}_{\rm FP} = \frac{1}{\kappa}\left(\mathcal{L}_0 - m^2 U\right) + \frac{1}{2} h_{\mu\nu} T^{\mu\nu}$$

where the kinetic term is the same as in GR,

$$\mathcal{L}_0 \;=\; \frac{1}{4}\left(-\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \partial_\mu h_{\nu\alpha}\partial^\nu h^{\mu\alpha} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h \partial^\mu h\right)$$

while the mass term

$$U = \frac{1}{8}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)$$

breaks the diff. invariance.

FP equations

$$\begin{aligned}\Box h_{\mu\nu} &- \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\alpha\mu} + \eta_{\mu\nu}(\partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h) \\ &+ \partial_{\mu\nu} h = m^2(h_{\mu\nu} - h\,\eta_{\mu\nu}) - 2\kappa T_{\mu\nu}\end{aligned}$$

are equivalent to

$$\begin{aligned}\Box h_{\mu\nu} - \partial_{\mu\nu} h &= m^2(h_{\mu\nu} - h\,\eta_{\mu\nu}) - 2\kappa T_{\mu\nu} \\ \partial^\mu h_{\mu\nu} &= \partial_\nu h \\ h &= -\frac{2\kappa}{3m^2}\,T\end{aligned}$$

They describe free massive gravitons in flat space. Each graviton has 5 degrees of freedom = 5 spin polarizations.

Theory is NOT invariant under $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$

Free massive gravitons

If $T_{\mu\nu} = 0$ then

$$\begin{aligned} \Box h_{\mu\nu} &= m^2 (h_{\mu\nu} - h\,\eta_{\mu\nu}) \\ \partial^\mu h_{\mu\nu} &= h = 0 \end{aligned}$$

the solution is, with $\omega = \sqrt{k^2 + m^2}$,

$$\mathbf{h}_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_+ & D_\times & 0 \\ 0 & D_\times & -D_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\omega t - kz)}$$

$$+ \begin{pmatrix} 0 & V_1 & V_2 & 0 \\ V_1 & 0 & 0 & 0 \\ V_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\omega t - kz)} + \begin{pmatrix} 2S & 0 & 0 & 0 \\ 0 & -S & 0 & 0 \\ 0 & 0 & -S & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\omega t - kz)}$$

Contribution of vectors and scalar to the GW170817 signal is less than 0.1%. Taking $m \to 0$, tensor modes become massless gravitons. Vectors and scalars can probably be set to zero.

Veltman-van Dam-Zakharov (VdVZ) discontinuity

VdVZ discontinuity /1970/

If $T_{\mu\nu} \neq 0$ then the FP equations are

$$\begin{aligned} \Box h_{\mu\nu} + \ldots &= m^2 (h_{\mu\nu} - h\,\eta_{\mu\nu}) - 2\kappa T_{\mu\nu} \\ \partial^{\mu} h_{\mu\nu} &= \partial_{\nu} h \\ h &= -\frac{2\kappa}{3m^2}\,T \end{aligned}$$

The $m \to 0$ limit is apparently singular. How to take it ?

Introducing the Stueckelberg fields $\chi_{\mu\nu}$, A_{μ}, and ϕ one decomposes $h_{\mu\nu}$ into tensor, vector, and the scalar parts as

VdVZ, limit $m \to 0$

$$h_{\mu\nu} = \chi_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{1}{m^2}\partial_\mu \partial_\nu \phi$$

This is invariant under the local

$$\chi_{\mu\nu} \to \chi_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \qquad A_\mu \to A_\mu - m\,\xi_\mu,$$
$$A_\mu \to A_\mu + \partial_\mu \Psi \qquad \phi \to \phi - m\,\Psi$$

Setting $\chi_{\mu\nu} = \mathbf{h}_{\mu\nu} + (\phi/2)\,\eta_{\mu\nu}$ and taking the $m \to 0$ limit gives

$$\begin{aligned} \Box\,\mathbf{h}_{\mu\nu} + \ldots &= -2\kappa T_{\mu\nu} && \text{tensor modes} \\ \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) &= 0 && \text{vector modes} \\ \Box\phi &= -\frac{2\kappa}{3} T && \text{scalar mode} \end{aligned}$$

Vector modes decouple. Scalar rests coupled the matter $\Rightarrow$ additional attractive field (5th force) $\Rightarrow$ wrong Newton law but correct light bending. One can rescale κ $\Rightarrow$ correct Newton law but wrong light bending.

VdVZ – two source interaction

The tree amplitude of interaction of two matter sources is

$$\mathcal{A}_{12} = \kappa T_1^{\mu\nu} P_{\mu\nu\alpha\beta} T_2^{\alpha\beta}.$$

One has in the FP theory

$$P_{\mu\nu\alpha\beta} = P^{\mathrm{FP}}_{\mu\nu\alpha\beta} = \sum_{i=1}^{5} e^i_{\mu\nu} e^i_{\alpha\beta} \frac{1}{p^2 - m^2},$$

while in GR

$$P_{\mu\nu\alpha\beta} = P^{\mathrm{GR}}_{\mu\nu\alpha\beta} = \sum_{i=1}^{2} e^i_{\mu\nu} e^i_{\alpha\beta} \frac{1}{p^2}.$$

If $m \to$ then

$$P^{\mathrm{FP}}_{\mu\nu\alpha\beta} = P^{\mathrm{GR}}_{\mu\nu\alpha\beta} + \frac{\eta_{\mu\nu}\eta_{\alpha\beta}}{p^2} + \dots$$

extra term gives an extra attraction due to the scalar graviton coupled to T.

FP does not agree with GR, however small m is.

VdVZ solution

Scalar graviton mode can propagate in the spherically-symmetric sector

$$ds^2 = -e^{\nu(R)}dt^2 + e^{\lambda(R)}dR^2 + R^2 d\Omega^2 \qquad (\star)$$

Let $R \to R(r) = re^{\mu(r)/2}$ then

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}R'^2(r)dr^2 + r^2 e^{\mu(r)}d\Omega^2 \qquad (\star\star)$$

- In GR metrics $(\star)$ and $(\star\star)$ are equivalent and μ is a pure gauge parameter, one can set $\mu = 0$ by changing back $r \to r(R)$.
- In FP there is no invariance, $(\star)$ and $(\star\star)$ are NOT equivalent, $\mu(r)$ is not a pure gauge but describes the scalar graviton.
- Linearizing $(\star\star)$ gives

$$h_{00} = \nu, \quad h_{rr} = -\lambda - (r\mu)', \quad h_{\vartheta\vartheta} = -r^2\mu, \quad h_{\varphi\varphi} = -r^2\mu\sin^2\vartheta$$

The FP equations

FP equations

$$
\begin{aligned}
\frac{1}{r}\,\lambda' + \frac{1}{r^2}\,\lambda &= -\frac{m^2}{2}(\lambda + 3\mu + r\mu') \\
-\frac{1}{r}\,\nu' + \frac{1}{r^2}\,\lambda &= -m^2(\mu + \frac{\nu}{2}) \\
m^2(\frac{\nu'}{2} - \frac{\lambda}{r}) &= 0 \qquad (\dagger)
\end{aligned}
$$

For $m = 0$ one gets the GR solution (μ is arbitrary = pure guge)

$$\lambda = -\nu = \frac{r_g}{r} \equiv \frac{2\kappa M}{r} \qquad \boxed{\nu + \lambda = 0}$$

For $m \neq 0$ this does not pass through ($\dagger$), one finds instead

VdVZ potential

$$\nu = -\frac{2C}{r}\, e^{-mr}, \qquad \lambda = \frac{C}{r}\,(1 + mr)\, e^{-mr}$$
$$\mu = C\, \frac{1 + mr + (mr)^2}{m^2 r^3}\; e^{-mr}$$

In the near zone, for $r \ll 1/m$, this reduces to the VdVZ solution

$$\boxed{\nu = -\frac{2C}{r}, \qquad \lambda = \frac{C}{r}, \qquad \mu = \frac{C}{r(mr)^2} \sim \frac{1}{r^3}}$$

therefore

$$\boxed{\nu + \lambda \neq 0}$$

$\Rightarrow$ depending on choice of C either the Newton law is wrong or the light bending is wrong.

Does this rule out the massive gravity ?

No, there is a remedy at the non-linear level.

Non-linear Fierz-Pauli – the bimetric theory

Non-linear FP

$$S = \frac{1}{\kappa} \int \sqrt{-g} \left(\frac{1}{2} R(g) - m^2 \, U(g, f) \right) d^4 x + S_{\rm mat}$$

where U is a scalar function of $g_{\mu\nu}$. One cannot construct a scalar using only $g_{\mu\nu}$. However, if there is a second fixed non-dynamical reference metric $f_{\mu\nu} = \eta_{\mu\nu}$ then one defines

$$\hat{\mathcal{S}} = \hat{1} - \hat{g}^{-1} \hat{f} \qquad \Rightarrow \qquad \mathcal{S}^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} - g^{\mu\sigma} f_{\sigma\nu}$$

and then one can choose any function (infinitely many options)

$$U = U([\hat{\mathcal{S}}], [\hat{\mathcal{S}}^2], [\hat{\mathcal{S}}^3], \det \hat{\mathcal{S}}).$$

In the weak field limit $g_{\mu\nu} = f_{\mu\nu} + h_{\mu\nu}$ and $\mathcal{S}_{\mu\nu} = h_{\mu\nu} + \ldots$ The correct FP limit for small $\hat{\mathcal{S}}$ is achieved if

$$U = \frac{1}{8} \left([\hat{\mathcal{S}}^2] - [\hat{\mathcal{S}}]^2 \right) + \mathcal{O}(\mathcal{S}^3)$$

One can allow for diffeomorphisms by setting

$$f_{\mu\nu} = \eta_{AB} \partial_\mu \Phi^A \partial_\nu \Phi^B$$

where Φ^A are Stueckelberg scalars.

Equations

One can define two energy-momentum tensors

$$T_{\mu\nu} = 2\frac{\partial U}{\partial g_{\mu\nu}} - U g_{\mu\nu}, \qquad \mathcal{T}_{\mu\nu} = 2\frac{\partial U}{\partial f_{\mu\nu}} - U f_{\mu\nu},$$

the equations are

$$G_{\mu\nu} \quad = \quad m^2\, T_{\mu\nu} \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0$$

The diff. invariance of U implies the identity

$$\sqrt{-g}\,\nabla^\mu T_{\mu\nu} - \sqrt{-\eta}\,\partial^\mu \mathcal{T}_{\mu\nu} \equiv 0$$

and therefore one has on-shell

$$\partial^\mu \mathcal{T}_{\mu\nu} = 0$$

Theory of Ogievetsky-Polubarinov /1965/

$$S = \frac{1}{\kappa}\int \sqrt{-g}\left(\frac{1}{2}R(g) - m^2\, U(g,\eta)\right) d^4x$$

the primary object is the graviton field $h^{\mu\nu}$ defining the metric

$$\left(\frac{\sqrt{-g}}{\sqrt{-\eta}}\right)^{s+1}((\hat{g}^{-1})^n)^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}.$$

the equations

$$\begin{aligned} G_{\mu\nu} = m^2 T_{\mu\nu} \quad &\Rightarrow \quad \Box h_{\mu\nu} = m^2 h_{\mu\nu} + \text{non-linear terms} \\ \partial^\mu \mathcal{T}_{\mu\nu} = 0 \quad &\Rightarrow \quad \partial^\mu h_{\mu\nu} = \lambda \partial_\nu h \end{aligned}$$

the OP potential, with $S^\mu_\nu = g^{\mu\sigma}\eta_{\sigma\nu}$,

$$U = \frac{1}{4n^2}\left(\det(\hat{S})\right)^{-s/2}[\hat{S}^n]$$

which gives $\lambda = -s/(2n)$.

VdVZ and Vainshtein mechanism

Vainshtein /1972/

Let us consider a non-linear FP

$$S = \frac{1}{\kappa}\int\left(\frac{1}{2}R - \frac{m^2}{8}(\mathcal{S}^{\alpha}_{\ \beta}\mathcal{S}^{\beta}_{\ \alpha} - (\mathcal{S}^{\alpha}_{\ \alpha})^2)\right)\sqrt{-g}\,d^4x + S_{\rm mat}$$

with $\mathcal{K}^{\mu}_{\nu} = \delta^{\mu}_{\nu} - g^{\mu\alpha}\eta_{\alpha\nu}$ and consider a spherically symmetric metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}R'^2dr^2 - R^2d\Omega^2$$

with $R = r\,e^{\mu/2}$ and compute non-linear corrections to the VdVZ. At large r, one looks for solutions of $G_{\mu\nu} = m^2\,T_{\mu\nu}$ in the form

$$\nu(r) = \sum_{n\geq 1}\kappa^n\nu_n(r), \qquad \lambda(r) = \sum_{n\geq 1}\kappa^n\lambda_n(r), \qquad \mu(r) = \sum_{n\geq 1}\kappa^n\mu_n(r).$$

the $n = 1$ terms being the VdVZ solution

Large r solution

$$\nu = -\frac{2r_g}{r}\left(1 + c_1\,\frac{r_g}{m^4 r^5} + \ldots\right)$$
$$\lambda = \frac{r_g}{r}\left(1 + c_2\,\frac{r_g}{m^4 r^5} + \ldots\right)$$
$$\mu = \frac{r_g}{m^2 r^3}\left(1 + c_3\,\frac{r_g}{m^4 r^5} + \ldots\right)$$

Leading terms are the VdVZ solution. For $m \sim (10^{25} cm)^{-1}$ the next-to-leading terms are $\sim r_g/(m^4 r^5) \sim 10^{32}$ at the edge of solar system. They become small only for

$$r \gg r_V = \left(r_g/m^4\right)^{1/5} \sim 100\ Kps$$

The VdVZ problem therefore arises only for $r \gg r_V$.

Small r solution

$$\nu(r)=\sum_{n\geq 0} m^{2n}\nu_n(r),\quad \lambda(r)=\sum_{n\geq 0} m^{2n}\lambda_n(r),\quad \mu(r)=\sum_{n\geq 0} m^{2n}\mu_n(r),$$

it is assumed that ν_0, λ_0 are small, their equations are linearized, while μ_0 is not small and its equation is fully non-linear. For $r \gg r_g$ one finds

$$\begin{aligned}\nu &= -\frac{r_g}{r}\left(1+a_1\,(mr)^2\sqrt{r/r_g}+\ldots\right)\\ \lambda &= \frac{r_g}{r}\left(1+a_2\,(mr)^2\sqrt{r/r_g}+\ldots\right)\\ \mu &= \sqrt{\frac{ar_g}{r}}\left(1+a_3\,(mr)^2\sqrt{r/r_g}+\ldots\right)\end{aligned}$$

so ν,λ show the GR behavior. Corrections are small for $r \ll r_V$ $\Rightarrow$ one recovers GR in the non-linear regime.

Vainshtein scenario

- The VdVZ discontinuity is only visible in the linear regime, for

$$r \gg r_V = \left(\frac{r_g}{m^4}\right)^{1/5} \sim 100Kps$$

.

- For $r \ll r_V$ the scalar graviton is frozen by non-linear effects and does not propagate $\Rightarrow$ GR is recovered.
- For $r \sim r_V$ there is a transition between the two regimes.

The VdVZ problem is cured by the non-linear effects.
This restores GR.

A model for Vainshtein

$$S = \frac{1}{\kappa}\int\left(\frac{1}{2}R - \frac{m^2}{8}(\mathcal{K}^{\alpha}{}_{\beta}\mathcal{K}^{\beta}{}_{\alpha} - (\mathcal{K}^{\alpha}{}_{\alpha})^2)\right)\sqrt{-g}\,d^4x + S_{\rm mat}$$

$$\mathcal{K}^{\mu}_{\nu} = \delta^{\mu}_{\nu} - g^{\mu\alpha}f_{\alpha\nu} \qquad f_{\mu\nu} = \eta_{AB}\partial_{\mu}\Phi^{A}\partial_{\nu}\Phi^{B}$$

In static, spherically symmetric case

$$\begin{aligned} g_{\mu\nu}dx^{\mu}dx^{\nu} &= -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2 \\ f_{\mu\nu}dx^{\mu}dx^{\nu} &= -dt^2 + dR^2 + R^2 d\Omega^2 \\ R(r) &= re^{\mu(r)/2} \qquad \text{Stuckelberg field} \end{aligned}$$

One looks for an asymptotically flat solution describing a localized object (star). Field equations

$$G_{\mu\nu} = m^2 T_{\mu\nu} + \kappa T^{\rm mat}_{\mu\nu}$$

$$T_{\mu\nu} = 2\frac{\partial\mathcal{U}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{U}, \qquad T^{{\rm mat}\ \mu}{}_{\nu} = {\rm diag}[-\rho, P, P, P]$$

Field equations

$$H^\mu_\nu = \text{diag}\left[1-e^{-\nu}, 1-e^{-\lambda}R'^2, 1-e^{\mu}, 1-e^{\mu}\right]$$

$$T^\mu_\nu = \delta^\mu_\nu \frac{1}{8}((1-H^\mu_\nu)(H^\mu_\nu - [H]) + [H^2] \qquad /\text{no sum over } \mu,\nu/$$

4 independent field equations determine ν, λ, μ, P

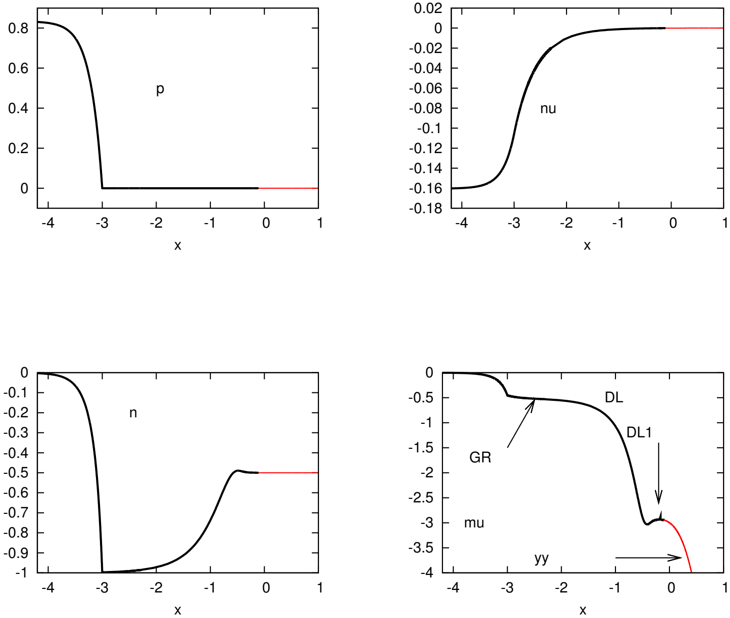
$$G^0_0 = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2} = m^2 T^0_0 - \kappa\rho$$

$$G^r_r = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) - \frac{1}{r^2} = m^2 T^r_r + \kappa P$$

$$(T^r_r)' = -\frac{\nu'}{2}(T^r_r - T^0_0) + \frac{2}{r}(T^\vartheta_\vartheta - T^r_r) \qquad /\text{conservation of } T_{\mu\nu}/$$

$$P' = -\frac{\nu'}{2}(P+\rho) \qquad /\text{conservation of } T^{\rm mat}_{\mu\nu}/$$

while $\rho(r) = \rho_\star \Theta(r_\star - r) \Rightarrow$ star of radius $r_\star$ and density $\rho_\star$.


p
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Summary

- Free massive gravitons are described by the linear Fierz-Pauli theory.
- This theory gives different from GR predictions in the $m \to 0$ limit due to the additional attraction mediated by the scalar graviton (VdVZ problem).
- In non-linear generalizations of the FP theory the scalar graviton is strongly bound by non-linear effects within the Vainshtein radius
$$r_V = \left(\frac{r_g}{m^4}\right)^{1/5}$$
This pushes the VdVZ effect to the region $r \gg r_V$ and restores GR for $r \ll r_V$.
- As a result, theories with massive gravitons can agree with observations.

Boulware-Deser problem:
non-linear effects bring back the
ghost $=$ sixth DoF.

Fierz and Pauli with 6 DoF

$$\Box h_{\mu\nu} + \ldots = m^2(h_{\mu\nu} - \alpha\, h\,\eta_{\mu\nu}) - 2\kappa T_{\mu\nu}$$

Taking the divergence gives 4 constraints

$$m^2(\partial^\mu h_{\mu\nu} - \alpha\partial_\nu h) = 0$$

Taking the trace gives

$$2(\alpha - 1)\,\Box h = m^2(1 - 4\alpha)\,h - 2\kappa T$$

$\Rightarrow$ for $\alpha = 1$ one gets the fifth constraint

$$h = -\frac{2\kappa}{3m^2}\,T$$

$\Rightarrow$ $10 - 5 = 5$ DoF=graviton polarizations.
If $\alpha \neq 1 \Rightarrow$ there are 6 DoF. The additional mode is a ghost: its kinetic energy is negative.

Sixth DoF

Let $\alpha \neq 1$. One can always decompose $h_{\mu\nu}$ as

$$h_{\mu\nu} = \psi_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \partial_{\mu\nu}\phi \quad \Rightarrow \quad h = \psi + \Box\phi$$

where $\partial^\mu \psi_{\mu\nu} = \partial^\mu \xi_\mu = 0$. The last FP equation

$$2(\alpha - 1)\,\Box h = m^2(1 - 4\alpha)\,h - 2\kappa T$$

then gives $\boxed{\Box^2\phi + \ldots = 0}$, the corresponding term in the action

$$\begin{aligned}
(\Box\phi)^2 &= \chi\Box\phi - \frac{1}{4}\chi^2 & \qquad /\chi = 2\Box\phi/ \\
&= (\phi_1 - \phi_2)\Box(\phi_1 + \phi_2) - \frac{1}{4}(\phi_1 - \phi_2)^2 \\
&= \phi_1\Box\phi_1 - \phi_2\Box\phi_2 - \frac{1}{4}(\phi_1 - \phi_2)^2
\end{aligned}$$

The minus sign = negative kinetic energy = Ostrogradsky ghost.

Boulware-Deser problem /1972/

The ghost can be removed in the linear FP theory by choosing $\alpha = 1$. However, it comes back in the non-linear FP. Therefore the latter make no sense.

This stopped all developments of massive gravity for almost 40 years.

Hamiltonian formulation

The Lagrangian

$$\mathcal{L} = \left(\frac{1}{2}R - m^2\,\mathcal{U}\right)\sqrt{-g}$$

after the ADM decomposition

$$\begin{aligned} ds_g^2 &= -N^2 dt^2 + \gamma_{ik}(dx^i + N^i dt)(dx^k + N^k dt) \\ ds_f^2 &= -dt^2 + \delta_{ik} dx^i dx^k \end{aligned}$$

becomes

$$\mathcal{L} = \frac{1}{2}\sqrt{\gamma}\,N\left(K_{ik}K^{ik} - K^2 + R^{(3)}\right) - m^2\mathcal{V}(N^\mu, \gamma_{ik}) + \text{total derivative}$$

where $\mathcal{V} = \sqrt{\gamma}\,N\,\mathcal{U}$ and the second fundamental form

$$K_{ik} = \frac{1}{2N}\left(\dot{\gamma}_{ik} - \nabla_i^{(3)}N_k - \nabla_k^{(3)}N_i\right)$$

Variables are γ_{ik} and $N^\mu = (N, N^k)$.

Hamiltonian

Conjugate momenta

$$\pi^{ik} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ik}} = \frac{1}{2}\sqrt{\gamma}(K^{ik} - K\gamma^{ik}), \qquad \boxed{p_{N_\mu} = \frac{\partial \mathcal{L}}{\partial \dot{N}^\mu} = 0} \quad \text{constraints}$$

$\Rightarrow$ N^ν are non-dynamical $\Rightarrow$ phase space is spanned by 12 variables (π^{ik}, γ_{ik}) = 6 DoF. Hamiltonian

$$\boxed{H = \pi^{ik}\dot{\gamma}_{ik} - \mathcal{L} = N^\mu \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \mathcal{V}(N^\mu, \gamma_{ik})}$$

with

$$\mathcal{H}_0 = \frac{1}{\sqrt{\gamma}}(2\pi_{ik}\pi^{ik} - (\pi^k_k)^2) - \frac{1}{2}\sqrt{\gamma}R^{(3)}, \quad \mathcal{H}_k = -2\nabla^{(3)}_i \pi^i_k$$

Secondary constrints

$$-\dot{p}_{N_\mu} = \frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0$$

Degrees of freedom, $m = 0$

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0 \qquad (\star)$$

- If $m = 0$ this gives 4 constraints

$$\mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) = 0$$

They are first class

$$\{\mathcal{H}_\mu, \mathcal{H}_\nu\} \sim \mathcal{H}_\alpha$$

and generate gauge symmetries, one can impose 4 gauge conditions, there remain 4 independent phase space variables

$$12 - 4 - 4 = 4 = 2 \times (2 \text{ DoF}) \quad \Rightarrow \quad 2 \text{ graviton polarizations}$$

Energy vanishes on the constraint surface (up to a surface term)

$$H = N^\mu \mathcal{H}_\mu = 0$$

Degrees of freedom, $m \neq 0$

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0 \qquad (\star)$$

- If $m \neq 0$ this gives 4 equations for laps and shifts whose solution is $N^\mu(\pi^{ik}, \gamma_{ik})$. No constraints arise $\Rightarrow$ there are

$$12 = 2 \times (6 \text{ degrees of freedom})$$

Inserting $N^\mu = N^\mu(\pi^{ik}, \gamma_{ik})$ back to the Hamiltonian

$$H = N^\mu \mathcal{H}_\mu + m^2 \mathcal{V}(N^\mu, \gamma_{ik})$$

yields $H(\pi^{ik}, \gamma_{ik})$ whose kinetic energy part is not positive-definite $\Rightarrow$ the energy is unbounded from below. This is related to the sixth DoF=ghost. The ghost is removed on flat background by choosing $\alpha = 1$, but it comes back on arbitrary background.

In non-linear Fierz-Pauli theory the VdVZ is cured but the ghost comes back /Boulware-Deser 1972/

Ghost-free massive gravity

Ghost-free massive gravity /2010/

One has

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0 \qquad (\star)$$

with

$$\mathcal{V}(N^\mu, \gamma_{ik}) = \frac{1}{8}\sqrt{-g}\left([H^2] - [H]^2\right) + \text{higher order terms}$$

One can choose the higher order terms such that

$$\text{rank}\left(\frac{\partial^2 \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\nu \partial N^\mu}\right) = 3$$

$\Rightarrow$ the 4 equations ($\star$) determine only 3 shifts $N^k = N^k(\pi^{ik}, \gamma_{ik})$, the lapse N remains undetermined, the 4-th equation reduces to a constraint

$$\mathcal{C}(\pi^{ik}, \gamma_{ik}) = 0 \qquad \Rightarrow \qquad \dot{\mathcal{C}} = \{\mathcal{C}, H\} \equiv S = 0.$$

The two constraints $\mathcal{C}, \mathcal{S}$ remove one DoF, there remain 5.

dRGT theory

Explicitely

$$S = M_{\rm Pl}^2 \int \left(\frac{1}{2} R - m^2 \mathcal{U}\right) \sqrt{-g} d^4x$$

$$\mathcal{U} = b_0 + b_1 \sum_a \lambda_a + b_2 \sum_{a<b} \lambda_a \lambda_b + b_3 \sum_{a<b<c} \lambda_a \lambda_b \lambda_c + b_4 \lambda_0 \lambda_1 \lambda_2 \lambda_3$$

where b_k are parameters and λ_a are eigenvalues of the matrix

$$\boxed{\gamma^\mu_{\ \nu} = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}}$$

/de Rham, Gabadadze, Tolley 2010/

A different parameter choice

$$S = M_{\rm Pl}^2 \int \left(\frac{1}{2} R - m^2 \mathcal{U} \right) \sqrt{-g} d^4 x$$

$$\mathcal{U} = c_0 + c_1 \sum_a \lambda_a + c_2 \sum_{a<b} \lambda_a \lambda_b + c_3 \sum_{a<b<c} \lambda_a \lambda_b \lambda_c + c_4 \lambda_0 \lambda_1 \lambda_2 \lambda_3$$

where λ_a are eigenvalues of $\boxed{\mathcal{K}^\mu_{\ \nu} = \delta^\mu_\nu - \sqrt{g^{\mu\alpha} f_{\alpha\nu}}}$

Flat space is a solution if $c_0 = c_1 = 0$, $c_1 = -1/2$ then

$$\begin{aligned} \mathcal{U} &= \frac{1}{2}([\mathcal{K}^2] - [\mathcal{K}]^2) \\ &+ \frac{c_3}{3!}([\mathcal{K}]^3 - 3[\gamma][\mathcal{K}]^2) + 2[\mathcal{K}^3]) \\ &+ \frac{c_4}{4!}([\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2) + 8[\mathcal{K}][\mathcal{K}^3] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]). \end{aligned}$$

In the simplest case $c_3 = c_4 = 0 \Rightarrow$

$$\mathcal{U} = \frac{1}{2}([\mathcal{K}^2] - [\mathcal{K}]^2)$$

Theory cutoff

$$S = M_{\rm Pl}^2 \int \left(\frac{1}{2} R - m^2 \mathcal{U} \right) \sqrt{-g}\, d^4 x$$

$$\mathcal{U} = \frac{1}{8} \left(H^\mu_\nu H^\nu_\mu - (H^\alpha_\alpha)^2 \right) + \dots$$

$$H^\mu_\nu = \delta^\mu_\nu - g^{\mu\alpha} f_{\alpha\nu} \qquad f_{\mu\nu} = \eta_{\alpha\beta} \partial_\mu \Phi^\alpha \partial_\nu \Phi^\beta$$

Let

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\rm Pl}} h_{\mu\nu} \qquad \partial_\mu \Phi^\alpha = \delta^\alpha_\mu + \frac{1}{m M_{\rm Pl}} \partial_\mu A^\alpha + \frac{1}{m^2 M_{\rm Pl}^2} \partial_\mu \partial^\alpha \phi$$

then expanding the kinetic term (similarly for $g_{\mu\nu} = g^0_{\mu\nu} + \frac{1}{M_{\rm Pl}} h_{\mu\nu}$)

$$\frac{M_{\rm Pl}^2}{2} R\sqrt{-g} = \underbrace{\frac{1}{8}\left(- \partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} + \dots \right)}_{\text{classical part}} + \underbrace{\frac{1}{M_{\rm Pl}} \mathcal{O}(h^3) + \dots}_{\text{quantum corrections}}$$

Quantum corrections become important only for $E \sim M_{\rm Pl}$.

Raising the cutoff

Expanding the potential gives (if $A_\mu = 0$)

$$m^2 M_{\rm Pl}^2 \mathcal{U}\sqrt{-g} = (\partial\phi)^2 + \underbrace{\frac{1}{(\Lambda_5)^5}\left((\partial^2\phi)^2 + \dots\right) + \frac{1}{(\Lambda_3)^3}\left(h(\partial^2\phi)^2 + \dots\right) + \dots}_{\text{quantum corrections}}$$

The quantum corrections become important when $E \sim \Lambda_5$ where the lowest cutoff scale is

$$\Lambda_5 = (M_{\rm Pl} m^4)^{1/5} \sim 1/(10^{11} km)$$

One can adjust the higher order terms in $\mathcal{U}$ such that all terms suppressed by Λ_5 are total derivatives and vanish upon integration. The rest sums up to $\mathcal{U}_{\rm dRGT}$. This raises the cutoff to

$$\Lambda_3 = (M_{\rm Pl} m^2)^{1/3} \sim 1/(10^3 \text{km})$$

$\Rightarrow$ reliable predictions within Solar System.

Galileons in the decoupling limit: $M_{\rm Pl}\to\infty$, $m\to 0$, $\Lambda_3=(M_{\rm Pl}m^2)^{1/3}$=const.

$$g_{\mu\nu}=\eta_{\mu\nu}+\frac{1}{M_{\rm Pl}}h_{\mu\nu} \qquad \partial_\mu\Phi^\alpha=\delta^\alpha_\mu+\frac{1}{mM_{\rm Pl}}\,\partial_\mu A^\alpha+\frac{1}{m^2M^2_{\rm Pl}}\,\partial_\mu\partial^\alpha\phi$$

with $h_{\mu\nu}=\mathbf{h}_{\mu\nu}+a_1\,\phi\eta_{\mu\nu}+a_2\,\partial_\mu\phi\partial_\nu\phi$ one obtains (if $A_\mu=0$)

$$\mathcal{L}_{\Lambda_3}=\mathcal{L}_0(\mathbf{h}_{\mu\nu})+\sum_{n=2}^{5}\frac{d_n}{\Lambda_3^{3(n-2)}}\,\mathcal{L}^{(n)}_{\rm Gal}[\phi]+\frac{q}{\Lambda_3^6}\,\mathbf{h}^{\mu\nu}X^{(3)}_{\mu\nu}$$

where the Galileon terms (shift inv. $\phi\to\phi+\phi_0$) $/\Pi_{\mu\nu}=\partial_{\mu\nu}\phi/$

$$\begin{aligned}
\mathcal{L}^{(2)} &= (\partial\phi)^2,\\
\mathcal{L}^{(3)} &= (\partial\phi)^2[\Pi],\\
\mathcal{L}^{(4)} &= (\partial\phi)^2([\Pi]^2-[\Pi^2]),\\
\mathcal{L}^{(5)} &= (\partial\phi)^2([\Pi]^3-3[\Pi][\Pi^2]+3[\Pi^3])
\end{aligned}$$

$$X^{(3)}_{\mu\nu}=([\Pi]^3-3[\Pi][\Pi^2]+3[\Pi^3])\eta_{\mu\nu}-3([\Pi]\Pi_{\mu\nu}-2[\Pi]\Pi^2_{\mu\nu}-[\Pi^2]\Pi_{\mu\nu}+2\Pi^3_{\mu\nu})$$

Galileon model of Vainshtein screening

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{\Lambda^3}(\partial\phi)^2\Box\phi + \phi T$$

$$\text{let} \quad T = -4\pi M\delta^{(3)}(\vec{r}) = -M\frac{\delta(r)}{r^2}$$

then

$$\frac{\phi'}{r} + \frac{1}{\Lambda^3}\left(\frac{\phi'}{r}\right)^2 = \frac{M}{r^3}$$

The Vainshtein radius is $\boxed{r_{\rm V} = M^{1/3}/\Lambda}$

$$\frac{M}{r_{\rm V}^{3/2}\sqrt{r}} \quad \underbrace{\leftarrow}_{r\ll r_{\rm V}} \quad \phi' \quad \underbrace{\rightarrow}_{r\gg r_{\rm V}} \quad \frac{M}{r^2} = \text{Newton force}$$

for $r \ll r_{\rm V}$ the force ratio

$$\frac{\phi'}{\text{Newton force}} = \left(\frac{r}{r_{\rm V}}\right)^{3/2} \ll 1$$

$\Rightarrow$ scalar graviton is screened at small distances.

Other massive gravities with 5 DoF

To have 5 DoF one needs constraints which arise if in

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\mu, \gamma_{ik})}{\partial N^\mu} = 0 \qquad (\star)$$

one has

$$\det\left(\frac{\partial^2 \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu \partial N^\nu}\right) = 0$$

This is the Monge-Ampere equation, all its solutions have been studied. Only the dRGT choice is Lorentz-invariant. Other solutions define theories which reduce in the weak field not to Fierz-Pauli

$$\mathcal{U} = (1/8)(h_{\mu\nu}h^{\mu\nu} - (h^\mu_\mu)^2)$$

but to a non-Lorentz-invariant potential

$$\mathcal{U} = (1/8)(a\, h_{00}^2 + b\, h_{0k}^2 + c\, h_{ik}^2 + d\, h_{kk}^2 + e\, h_{00}h_{0k} + \ldots)$$

which could be relevant in cosmology. They have a higher cutoff

$$\Lambda_2 = \sqrt{m M_{\rm Pl}} \sim 1/(1\text{mm})$$

Summary

- Non-linear Fierz-Pauli models generically propagate 5+1 DoF, the extra DoF being the BD ghost rendering the theory unstable. For almost 40 years this was considered to be an inevitable obstacle.
- However, a careful analysis by dRGT has shown that there is a unique (up to 5 free parameters) way to choose the potential $\mathcal{U}$ such that a constraints arise in the Hamiltonian formulation. The constraints remove one DoF. The resulting theory propagates 5 DoF and is called ghost-free.
- The dRGT theory is valid up to the energies of the order $\Lambda_3 = (M_{\rm Pl} m^2)^{1/3} \sim 1/(10^3 \text{km})$, so that it can be used to make predictions within Solar System.
- In the decoupling limit, $M_{\rm Pl} \to \infty$, $m \to 0$, fixed Λ_3, the theory describes linear gravitons interacting with non-linear vector and scalar. The scalar part describes the scalar graviton polarization and has the Galileon structure.
- The theory cutoff can be raised even higher, up to $\Lambda_2 = \sqrt{M_{\rm Pl} m} \sim 1/\text{mm})$, via braking the Lorentz invariance.

Properties of the dRGT potential

Properties of the dRGT potential

$$S = M_{\rm Pl}^2 \int \left(\frac{1}{2} R - m^2 \mathcal{U} \right) \sqrt{-g} d^4 x$$

with

$$\mathcal{U} = \sum_{k=0}^{4} b_k \mathcal{U}_k(\gamma)$$

where $\mathcal{U}_k(\gamma)$ are symmetric polynomials of eigenvalues λ_a of

$$\gamma^\mu{}_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$$

which means that

$$\gamma^\mu{}_\alpha \gamma^\alpha{}_\beta = g^{\mu\alpha} f_{\alpha\nu}$$

or

$$\hat{\gamma}^2 = \hat{g}^{-1} \hat{f}$$

$\mathcal{U}_k$

$$
\begin{aligned}
\mathcal{U}_0(\gamma) &= 1 \\
\mathcal{U}_1(\gamma) &= \sum_a \lambda_a = [\gamma] \\
\mathcal{U}_2(\gamma) &= \sum_{a<b} \lambda_a \lambda_b = \frac{1}{2}([\gamma]^2 - [\gamma^2]) \\
\mathcal{U}_3(\gamma) &= \sum_{a<b<c} \lambda_a \lambda_b \lambda_c = \frac{1}{3!}([\gamma]^3 - 3[\gamma][\gamma]^2) + 2[\gamma^3]) \\
\mathcal{U}_4(\gamma) &= \lambda_0 \lambda_1 \lambda_2 \lambda_3 \\
&= \frac{1}{4!}([\gamma]^4 - 6[\gamma^2][\gamma]^2) + 8[\gamma][\gamma^3] + 3[\gamma^2]^2 - 6[\gamma^4]).
\end{aligned}
$$

Varying the action

$$S = M_{\rm Pl}^2 \int \left(\frac{1}{2} R - m^2 \sum_k b_k \mathcal{U}_k(\gamma) \right) \sqrt{-g}\, d^4x$$

To vary this with respect to $g_{\mu\nu}$ one uses $\boxed{\hat{\gamma}^2 = \hat{g}^{-1}\hat{f}}$ hence

$$\delta\hat{\gamma}\hat{\gamma} + \hat{\gamma}\delta\hat{\gamma} = \delta\hat{g}^{-1}\hat{f}$$

This is the matrix Sylvestre equation for $\delta\hat{\gamma}$ whose solution is extremely complex. Fortunately, $\mathcal{U}_k$ depend only on $[\gamma^n] \equiv [\hat{\gamma}^n]$. One has

$$\delta\hat{\gamma} + \hat{\gamma}\delta\hat{\gamma}\hat{\gamma}^{-1} = \delta\hat{g}^{-1}\hat{f}\hat{\gamma}^{-1} = \delta\hat{g}^{-1}\hat{g}\hat{\gamma}$$

and taking the trace

$$\delta[\hat{\gamma}] = \frac{1}{2}[\delta\hat{g}^{-1}\hat{g}\hat{\gamma}] \quad \text{or} \quad \delta\gamma^{\alpha}_{\ \alpha} = \frac{1}{2}\delta g^{\mu\alpha} g_{\alpha\beta}\gamma^{\beta}_{\ \mu} \equiv \frac{1}{2}\delta g^{\mu\alpha}\gamma_{\alpha\mu}$$

Similarly,

$$\delta(\gamma^n)^{\alpha}_{\ \alpha} = \frac{n}{2}\delta g^{\mu\alpha} g_{\alpha\beta}(\gamma^n)^{\beta}_{\ \mu} \equiv \frac{1}{2}\delta g^{\mu\alpha}(\gamma^n)_{\alpha\mu}$$

One has $\boxed{(\gamma^n)_{\mu\nu} = (\gamma^n)_{\nu\mu}}$ /check this !/

Field equations

$$G_{\mu\nu} = m^2 T_{\mu\nu}$$

with

$$T^{\mu}{}_{\nu} = g^{\mu\alpha} T_{\alpha\nu} = \tau^{\mu}{}_{\nu} - \mathcal{U}\,\delta^{\mu}_{\nu}$$

where

$$\begin{aligned}
\tau^{\mu}{}_{\nu} &= \{b_1\,\mathcal{U}_0 + b_2\,\mathcal{U}_1 + b_3\,\mathcal{U}_2 + b_4\,\mathcal{U}_3\}\,\gamma^{\mu}{}_{\nu} \\
&- \{b_2\,\mathcal{U}_0 + b_3\,\mathcal{U}_1 + b_4\,\mathcal{U}_2\}\,(\gamma^2)^{\mu}{}_{\nu} \\
&+ \{b_3\,\mathcal{U}_0 + b_4\,\mathcal{U}_1\}\,(\gamma^3)^{\mu}{}_{\nu} \\
&- \{b_4\,\mathcal{U}_0\}\,(\gamma^4)^{\mu}{}_{\nu}
\end{aligned}$$

Equivalent form

Consider the characteristic polynomial

$$
\begin{aligned}
f_\gamma(\lambda) &\equiv \det(\hat{\gamma} - \lambda \hat{I}) = (\lambda_0 - \lambda)(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \\
&= \lambda^4 - \lambda^3 \sum_a \lambda_a + \lambda^2 \sum_{a<b} \lambda_a \lambda_b - \lambda \sum_{a<b<c} \lambda_a \lambda_b \lambda_c + \lambda_0 \lambda_1 \lambda_2 \lambda_3 \\
&= \mathcal{U}_0\, \lambda^4 - \mathcal{U}_1\, \lambda^3 + \mathcal{U}_2\, \lambda^2 - \mathcal{U}_3\, \lambda + \mathcal{U}_4
\end{aligned}
$$

The Hamilton-Caly theorem tells that

$$
f_\gamma(\hat{\gamma}) = \mathcal{U}_0\, \hat{\gamma}^4 - \mathcal{U}_1\, \hat{\gamma}^3 + \mathcal{U}_2\, \hat{\gamma}^2 - \mathcal{U}_3\, \hat{\gamma} + \mathcal{U}_4 = 0
$$

therefore

$$
\begin{aligned}
\tau^\mu{}_\nu &= \{b_1\, \mathcal{U}_0 + b_2\, \mathcal{U}_1 + b_3\, \mathcal{U}_2\}\, \gamma^\mu{}_\nu \\
&- \{b_2\, \mathcal{U}_0 + b_3\, \mathcal{U}_1\}\, (\gamma^2)^\mu{}_\nu \\
&+ \{b_3\, \mathcal{U}_0\}\, (\gamma^3)^\mu{}_\nu \\
&+ \{b_4\, \mathcal{U}_4\}\, \delta^\mu{}_\nu \equiv \sigma^\mu{}_\nu + b_4\, \mathcal{U}_4 \delta^\mu{}_\nu
\end{aligned}
$$

Field equations – simplified form

$$G^{\mu}_{\ \nu} = m^2 \, T^{\mu}_{\ \nu}$$

with

$$T^{\mu}_{\ \nu} = \sigma^{\mu}_{\ \nu} - \left(\sum_{k=0}^{3} b_k \mathcal{U}_k \right) \delta^{\mu}_{\nu}$$

where

$$\begin{aligned} \sigma^{\mu}_{\ \nu} &= \{ b_1 \mathcal{U}_0 + b_2 \mathcal{U}_1 + b_3 \mathcal{U}_2 \} \, \gamma^{\mu}_{\ \nu} \\ &- \{ b_2 \mathcal{U}_0 + b_3 \mathcal{U}_1 \} \, (\gamma^2)^{\mu}_{\ \nu} \\ &+ \{ b_3 \mathcal{U}_0 \} \, (\gamma^3)^{\mu}_{\ \nu} \end{aligned}$$

One more representation of $\mathcal{U}_k$

$$
\begin{aligned}
\mathcal{U}_0(\gamma) &= \frac{1}{4!}\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \\
\mathcal{U}_1(\gamma) &= \frac{1}{3!}\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\nu\rho\sigma}\gamma^{\mu}{}_{\alpha} \\
\mathcal{U}_2(\gamma) &= \frac{1}{2!2!}\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho\sigma}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta} \\
\mathcal{U}_3(\gamma) &= \frac{1}{3!}\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\sigma}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\gamma^{\rho}{}_{\gamma} \\
\mathcal{U}_4(\gamma) &= \frac{1}{4!}\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\gamma^{\rho}{}_{\gamma}\gamma^{\sigma}{}_{\delta}
\end{aligned}
$$

assuming that $\epsilon_{0123} = \epsilon^{0123} = 1$.

Tetrad formulation

Let us introduce two tetrads e^a_μ and f^a_μ such that

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \qquad\qquad f_{\mu\nu} = \eta_{ab} f^a{}_\mu f^b{}_\nu$$

Let e^μ_a abd f^μ_a be the inverse tetrads, so that

$$g^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu \qquad\qquad f^{\mu\nu} = \eta^{ab} f_a{}^\mu f_b{}^\nu$$

and define $\boxed{\Gamma^\mu{}_\nu = e_a{}^\mu f^a{}_\nu}$ and also

$$\Gamma_{\mu\nu} = g_{\mu\alpha}\Gamma^\alpha{}_\nu = \eta_{ab} e^a{}_\mu \underbrace{e^b{}_\alpha e_c{}^\alpha}_{\delta^b_c} f^c{}_\nu = \eta_{ac} e^a{}_\mu f^c{}_\nu \equiv e^a{}_\mu f_{a\nu}$$

Let us assume that

$$\boxed{\Gamma_{\mu\nu} = \Gamma_{\nu\mu}} \quad \Rightarrow \quad e^a{}_\mu f_{a\nu} = e^a{}_\nu f_{a\mu} \quad \Rightarrow \quad e_a{}^\mu f_{b\mu} = e_b{}^\mu f_{a\mu} \qquad (!)$$

then

$$\Gamma^\mu{}_\alpha \Gamma^\alpha{}_\nu = e_a{}^\mu f^a{}_\alpha e_b{}^\alpha f^b{}_\nu = e^{a\mu} f_{a\alpha} e_b{}^\alpha f^b{}_\nu = e^{a\mu} f_{b\alpha} e_a{}^\alpha f^b{}_\nu = g^{\mu\alpha} f_{\alpha\nu}$$

$$\Rightarrow \qquad \boxed{\Gamma^\mu{}_\nu = \gamma^\mu{}_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}}$$

Useful identities

$$
\begin{aligned}
\frac{1}{4!}\,\epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta}e^{a}{}_{\mu}e^{b}{}_{\nu}e^{c}{}_{\alpha}e^{d}{}_{\beta} &= |e^{a}{}_{\mu}| \equiv e = \sqrt{-g} \\
\frac{1}{3!}\,\epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta}e^{b}{}_{\nu}e^{c}{}_{\alpha}e^{d}{}_{\beta} &= e\,e_{a}{}^{\mu} \\
\frac{1}{2!}\,\epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta}e^{c}{}_{\alpha}e^{d}{}_{\beta} &= e\left(e_{a}{}^{\mu}e_{b}{}^{\nu} - e_{a}{}^{\nu}e_{b}{}^{\mu}\right)
\end{aligned}
$$

Yet one more representation of $\mathcal{U}_k$

$$\mathcal{U}_0(\gamma)\sqrt{-g} = \frac{1}{4!}\epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta}e^a{}_\mu e^b{}_\nu e^c{}_\alpha e^d{}_\beta = e = \sqrt{-g}$$

$$\mathcal{U}_1(\gamma)\sqrt{-g} = \frac{1}{3!}\epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta}e^a{}_\mu e^b{}_\nu e^c{}_\alpha f^d_\beta = e\, e_d{}^\beta f^d_\beta = e\,\Gamma^\beta{}_\beta = e\,[\Gamma]$$

$$\mathcal{U}_2(\gamma)\sqrt{-g} = \frac{1}{2!2!}\epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta}e^a{}_\mu e^b{}_\nu f^c_\alpha f^d_\beta$$

$$= \frac{1}{2}e\,(e_c{}^\alpha e_d{}^\beta - e_c{}^\beta e_d{}^\alpha)f^c_\alpha f^d_\beta = e\,\frac{1}{2}\left([\Gamma]^2 - [\Gamma^2]\right)$$

$$\mathcal{U}_3(\gamma)\sqrt{-g} = \frac{1}{3!}\epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta}e^a{}_\mu f^b_\nu f^c_\alpha f^d_\beta = |f_a{}^\mu| e^a{}_\mu f_a{}^\mu \equiv f\,[\Gamma^{-1}]$$

$$\mathcal{U}_4(\gamma)\sqrt{-g} = \frac{1}{4!}\epsilon_{abcd}\epsilon^{\mu\nu\alpha\beta}f^a_\mu f^b_\nu f^c_\alpha f^d_\beta = f$$

Here $(\Gamma^{-1})^\mu{}_\nu = f_a{}^\mu e^a{}_\mu$ where $f_a{}^\mu$ is the inverse of f^a_μ.

These expressions are equivalent to the previous ones provided that $\Gamma^\mu{}_\nu = \gamma^\mu{}_\nu$ which is the case if $\boxed{\Gamma_{\mu\nu} = \Gamma_{\nu\mu}}$.

Field equations – tetrad form

Varying with respect to $e^a{}_\mu$ gives

$$G_{\mu\nu} = m^2 T_{\mu\nu}$$

with

$$\begin{aligned} T_{\mu\nu} &= -b_0\, g_{\mu\nu} + b_1 \{\Gamma_{\mu\nu} - [\Gamma]\, g_{\mu\nu}\} \\ &+ b_2 \frac{f}{e} \{(\Gamma^{-2})_{\mu\nu} - [\Gamma^{-1}](\Gamma^{-1})_{\mu\nu}\} \\ &- b_3 \frac{f}{e} (\Gamma^{-1})_{\mu\nu} \end{aligned}$$

where

$$\Gamma_{\mu\nu} = g_{\mu\alpha}\Gamma^\alpha{}_\nu, \quad (\Gamma^{-1})_{\mu\nu} = g_{\mu\alpha}(\Gamma^{-1})^\alpha{}_\nu, \quad (\Gamma^{-2})_{\mu\nu} = g_{\mu\alpha}(\Gamma^{-1})^\alpha{}_\beta(\Gamma^{-1})^\beta{}_\nu$$

Since $T_{\mu\nu} = T_{\nu\mu}$ this generically implies that $\Gamma_{\mu\nu} = \Gamma_{\nu\mu} = \gamma_{\mu\nu}$.
Therefore the tetrad formulation is generically equivalent to the square root formulation.
For special values of b_k one can have $T_{\mu\nu} = T_{\nu\mu}$ but $\Gamma_{\mu\nu} \neq \Gamma_{\nu\mu}$

Form formalism

$$
\begin{aligned}
S &= M_{\rm Pl}^2 \int \left(\frac{1}{2}R - m^2\mathcal{U}\right)\sqrt{-g}\,d^4x \\
&= \int \Bigg\{ \frac{1}{4}\epsilon_{abcd}R^{ab}\wedge e^c \wedge e^d \\
&- m^2\epsilon_{abcd}\left(\frac{b_0}{4!}\,e^a\wedge e^b\wedge e^c\wedge e^d\right. \\
&+ \frac{b_1}{3!}\,e^a\wedge e^b\wedge e^c\wedge f^d + \frac{b_2}{2!2!}\,e^a\wedge e^b\wedge f^c\wedge f^d \\
&+ \left.\frac{b_3}{3!}\,e^a\wedge f^b\wedge f^c\wedge f^d + \frac{b_4}{4!}\,f^a\wedge f^b\wedge f^c\wedge f^d\right)\Bigg\}
\end{aligned}
$$

with $e^a = e^a_{\ \mu}dx^\mu$ and $f^a = f^a_{\ \mu}dx^\mu$. In the ADM formulation

$$
ds_g^2 = -N^2dt^2+\gamma_{ik}(dx^i+N^i dt)(dx^k+N^k dt) = -e^0\otimes e^0+\delta_{ik}e^i\otimes e^k
$$

$\Rightarrow$ N enters only $e^0 = Ndt$. The potential is linear in e^0 $\Rightarrow$ it is liner in N so that $\mathcal{V} = \mathcal{U}\sqrt{-g} = AN + B$ $\Rightarrow$ constraints=5 DoF.

Dimensional reconstruction

$$S_{\rm dRGT} = \frac{1}{2} M_{\rm Pl}^2 \int \left(R + m^2([\mathcal{K}]^2 - [\mathcal{K}^2])\right) \sqrt{-g}\, d^4x, \quad \mathcal{K} = 1 - \sqrt{g^{-1}f}$$

$$S_5 = \frac{1}{2} M_5^3 \int R_5 \sqrt{-g_5}\, d^5x \quad \Rightarrow \quad \text{5 DoF}$$

$$ds_5^2 = dy^2 + \mathbf{g}_{\mu\nu} dx^\mu dx^\nu = dt^2 + \eta_{ab} \mathbf{e}^a_\mu \mathbf{e}^b_\nu$$

$$S_5 = \frac{1}{2} M_5^2 \int_{y_1}^{y_2} dy \int (R(\mathbf{g}) + [K]^2 - [K^2]) \sqrt{-\mathbf{g}}\, d^4x$$

$$K_{\mu\nu} = \frac{1}{2} \partial_y \mathbf{g}_{\mu\nu} = \frac{1}{2} \eta_{ab} (\partial_y \mathbf{e}^a_\mu \mathbf{e}^b_\nu + \mathbf{e}^a_\mu \partial_y \mathbf{e}^b_\nu)$$

$$e^a_\mu \equiv \mathbf{e}^a_\mu(y_1), \quad f^a_\mu \equiv \mathbf{e}^a_\mu(y_2), \quad \boxed{\partial_y \mathbf{e}^a_\mu \to \frac{\mathbf{e}^a_\mu(y_2) - \mathbf{e}^a_\mu(y_1)}{y_2 - y_1} \equiv m(e^a_\mu - f^a_\mu)}$$

$$K_{\mu\nu} \to -m\,(g_{\mu\nu} - \eta_{ab} e^a_\mu f^a_\nu), \quad K^\mu_\nu \to -m\,(\delta^\mu_\nu - \underbrace{e^\mu_a f^a_\nu}_{\sqrt{g^{\mu\alpha} f_{\alpha\nu}}}) = -m\,\mathcal{K}^\mu{}_\nu$$

Bigravity

Bigravity

$$
\begin{aligned}
S &= \frac{1}{2\kappa_1}\int R(g)\sqrt{-g}\,d^4x + \frac{1}{2\kappa_2}\int R(f)\sqrt{-f}\,d^4x \\
&- \frac{m^2}{\kappa_1+\kappa_2}\int \mathcal{U}\sqrt{-g}\,d^4x + S_{\mathrm{mat}}[g,\Psi_g] + S_{\mathrm{mat}}[f,\Psi_f]
\end{aligned}
$$

with the same potential as before

$$
\mathcal{U} = b_0 + b_1\sum_a \lambda_a + b_2\sum_{a<b}\lambda_a\lambda_b + b_3\sum_{a<b<c}\lambda_a\lambda_b\lambda_c + b_4\lambda_0\lambda_1\lambda_2\lambda_3
$$

There is interchange symmetry

$$
g_{\mu\nu} \leftrightarrow f_{\mu\nu} \qquad \kappa_1 \leftrightarrow \kappa_2 \qquad b_k \leftrightarrow b_{4-k} \qquad T^{\mathrm{mat}}_{\mu\nu}(g) \leftrightarrow T^{\mathrm{mat}}_{\mu\nu}(f)
$$

7 DoF = one massive + one massless graviton

/Hassan and Rosen 2012/

Field equations

$$G_{\mu\nu}(g) = m^2 \cos^2\eta\, T_{\mu\nu}(g,f) + \kappa_1 T^{\rm mat}_{\mu\nu}(g)$$

$$G_{\mu\nu}(f) = m^2 \sin^2\eta\, \mathcal{T}_{\mu\nu}(g,f) + \kappa_2 T^{\rm mat}_{\mu\nu}(f)$$

with $\tan^2\eta = \kappa_2/\kappa_1$ and

$$T^{\mu}_{\ \nu} = g^{\mu\alpha} T_{\alpha\nu} = \tau^{\mu}_{\ \nu} - \mathcal{U}\,\delta^{\mu}_{\nu}$$

$$\mathcal{T}^{\mu}_{\ \nu} = f^{\mu\alpha}\mathcal{T}_{\alpha\nu} = -\frac{\sqrt{-g}}{\sqrt{-f}}\,\tau^{\mu}_{\ \nu}$$

with

$$\begin{aligned}
\tau^{\mu}_{\ \nu} &= \{b_1\mathcal{U}_0 + b_2\mathcal{U}_1 + b_3\mathcal{U}_2 + b_4\mathcal{U}_3\}\,\gamma^{\mu}_{\ \nu} \\
&- \{b_2\mathcal{U}_0 + b_3\mathcal{U}_1 + b_4\mathcal{U}_2\}\,(\gamma^2)^{\mu}_{\ \nu} \\
&+ \{b_3\mathcal{U}_0 + b_4\mathcal{U}_1\}\,(\gamma^3)^{\mu}_{\ \nu} \\
&- \{b_4\mathcal{U}_0\}\,(\gamma^4)^{\mu}_{\ \nu}
\end{aligned}$$

In the limit where $\kappa_2 \to 0$ and $f_{\mu\nu} \to \eta_{\mu\nu}$ the theory reduces to the dRGT massive gravity $\Rightarrow$ dRGT is contained in the bigravity.

Flat space

Let us require the flat space $g_{\mu\nu} = f_{\mu\nu} = \eta_{\mu\nu}$ to be a solution. This imposes two conditions $T_{\mu\nu} = 0$, $\mathcal{T}_{\mu\nu} = 0$. Requiring in addition m to be the FP mass of gravitons in flat space gives a third condition. These three conditions are fulfilled by adjusting the 5 b_k's as $b_k(c_3, c_4)$

$$\begin{aligned} b_0 &= 4c_3 + c_4 - 6, \quad b_1 = 3 - 3c_3 - c_4, \quad b_2 = 2c_3 + c_4 - 1 \\ b_3 &= -(c_3 + c_4), \quad b_4 = c_4 \end{aligned}$$

Small fluctuations $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$ and $f_{\mu\nu} = \eta_{\mu\nu} + \delta f_{\mu\nu}$

$$h^{\mathrm{m}}_{\mu\nu} = \cos\eta\,\delta g_{\mu\nu} + \sin\eta\,\delta f_{\mu\nu} \qquad h^{0}_{\mu\nu} = \cos\eta\,\delta f_{\mu\nu} - \sin\eta\,\delta g_{\mu\nu}$$

fulfill

$$\begin{aligned} (\Box + \ldots)h^{\mathrm{m}}_{\mu\nu} &= m^2(h^{\mathrm{m}}_{\mu\nu} - h^{\mathrm{m}}\eta_{\mu\nu}) \\ (\Box + \ldots)h^{0}_{\mu\nu} &= 0 \end{aligned}$$

Cosmologies and black holes

- Proportional solutions
- Non-bidiagonal solutions
- Hairy solutions

I. Proportional solutions

Proportional solutions

$$\boxed{f_{\mu\nu} = C^2 g_{\mu\nu}} \;\Rightarrow\; G^{\mu}_{\nu}(g)+\Lambda_g(C)\delta^{\mu}_{\nu} = 0, \quad G^{\mu}_{\nu}(f)+\Lambda_f(C)\delta^{\mu}_{\nu} = 0$$

where, with $P_m = b_m + 2b_{m+1}C + b_{m+2}C^2$,

$$\Lambda_g = m^2 \cos^2\eta\,(P_0 + CP_1), \quad \Lambda_f = m^2 \frac{\sin^2\eta}{C^3}\,(P_1 + CP_2)$$

/show this/ Since $G^{\mu}_{\nu}(f) = G^{\mu}_{\nu}(g)/C^2 \Rightarrow \boxed{\Lambda_g = C^2\Lambda_f} \Rightarrow$ quartic algebraic equation for C.

- Fours roots $C = \{C_k\}$
- $\Lambda_g(C_k)$ can be positive, negative or zero, depending on C_k. If $b_k = b_k(c_3, c_4)$ then $C = 1$ is a root and $\Lambda_g(1) = 0$.
- If $\Lambda_g > 0$ then there is de Sitter solution $\Rightarrow$ late time acceleration. Since one has to have $\Lambda_g \sim 1/H^2 \Rightarrow$ either $m \sim 1/H$ or $\cos^2\eta\,(P_0 + CP_1) \sim 1/H^2$.

Proportional solutions

- If there is matter then proportional solutions are possible if only the matter is fine-tuned such that $T^{\mu}_{\nu} = \mathcal{T}^{\mu}_{\nu}/C^2$. However, matter becomes negligible at late times $\Rightarrow$ proportional de Sitter is the late time attractor for generic cosmologies = inhomogeneous, anisotropic, with any matter.
- Proportional black holes are the same as in GR = Schwarzschild (Kerr)-(anti)-de Sitter. However, when perturbed, solutions show a mild ($\sim m$) instability due to the scalar graviton polarization mode.
- Proportional solutions exist only in bigravity, not in massive gravity with a fixed f-metric.

II. Decoupled solutions

$$T^\mu_\nu = \tau^\mu_\nu - \mathcal{U}\,\delta^\mu_\nu \qquad \mathcal{T}^\mu_\nu = -\frac{\sqrt{-g}}{\sqrt{-f}}\,\tau^\mu_\nu$$

with $\tau^\mu{}_\nu = \sigma^\mu{}_\nu + b_4 \mathcal{U}_4 \delta^\mu{}_\nu$

$$\begin{aligned} \sigma^\mu{}_\nu &= \{b_1\,\mathcal{U}_0 + b_2\,\mathcal{U}_1 + b_3\,\mathcal{U}_2\}\,\gamma^\mu{}_\nu \\ &- \{b_2\,\mathcal{U}_0 + b_3\,\mathcal{U}_1\}\,(\gamma^2)^\mu{}_\nu + \{b_3\,\mathcal{U}_0\}\,(\gamma^3)^\mu{}_\nu \end{aligned}$$

Let us require that $\boxed{\sigma^\mu{}_\nu = 0}$ then

$$T^\mu_\nu = -\Lambda_g \delta^\mu_\nu \qquad \mathcal{T}^\mu_\nu = -\Lambda_f \delta^\mu_\nu$$

with

$$\Lambda_g = \sum_{k=0}^{3} b_k \mathcal{U}_k \qquad \Lambda_f = b_4\,\frac{\sqrt{-g}}{\sqrt{-f}}\,\mathcal{U}_4 = b_4$$

and the field equations require these to be constants.

II. Non-bidiagonal solutions

Common SO(3)

$$ds_g^2 = -A\,dt^2 + \frac{dr^2}{B} + r^2 d\Omega^2$$
$$ds_f^2 = -C\,dT^2 + \frac{dU^2}{D} + U^2 d\Omega^2$$

A, B depend on t, r while C, D depend on $T(t, r)$, $U(t, r)$. Field equations reduce to

- $$U = Cr \quad \text{where} \quad b_1 + 2b_2 C + b_3 C^2 = 0$$
- $$G_{\mu\nu}(g) + \Lambda_g\, g_{\mu\nu} = 0 \qquad G_{\mu\nu}(f) + \Lambda_f\, f_{\mu\nu} = 0$$

  $$\Lambda_g = m^2 \cos^2\eta(b_0 + 2b_1 C + b_2 C^2)$$
  $$\Lambda_f = m^2 \frac{\sin^2}{C^2}\eta(b_2 + 2b_3 C + b_4 C^2)$$
- A differential condition for $T(t, r)$.

Explicit non-bidiagonal solutions

Schwarzschild-(anti)-de Sitter

$$ds_g^2 = -\Sigma dt^2 + \frac{dr^2}{\Sigma} + r^2 d\Omega^2, \qquad \Sigma = 1 - \frac{2M_g}{r} - \frac{\Lambda_g}{3} r^2$$

$$ds_f^2 = C^2 \left(-\Delta dT^2 + \frac{dr^2}{\Delta} + r^2 d\Omega^2 \right), \quad \Delta = 1 - \frac{2M_f}{r} - \frac{C^2 \Lambda_f}{3} r^2$$

$$\boxed{\frac{\Delta}{\Sigma} (\partial_t T)^2 + \frac{\Delta\Sigma}{\Delta - \Sigma} (\partial_r T)^2 = 1}$$

infinitely many inequivalent solutions, the simplest one

$$T = t + \int \left(\frac{1}{\Sigma} + \frac{1}{\Delta} \right) dr$$

- If $M_g = M_f = 0 \Rightarrow$ de Sitter cosmology, one can add matter.
- If $M_g \neq 0,\ M_f \neq 0 \Rightarrow$ black holes.
- If $M_f = 0$ and $\eta \to 0$ then $\Lambda_f \sim \sin^2 \eta \to 0 \Rightarrow$ f-metric is flat $\Rightarrow$ all known cosmologies and black holes in massive gravity
- Same linear perturbations as in GR $\Rightarrow$ scalar graviton is strongly bound

Massive gravity cosmologies

Cosmological constant $\Lambda = m^2(b_0 + 2b_1 C + b_2 C^2)$ where $b_1 + 2b_2 C + b_3 C^2 = 0$. The g-metric is de Sitter, f is flat

$$\begin{aligned} ds_g^2 &= \frac{3}{\Lambda}\left\{-dt^2 + dr^2 + dx^2 + dy^2 + dz^2\right\} \\ 1 &= -t^2 + r^2 + x^2 + y^2 + z^2 \equiv -t^2 + r^2 + R^2 \\ ds_f^2 &= \frac{3C^2}{\Lambda}\left\{-dT^2 + dx^2 + dy^2 + dz^2\right\} \end{aligned}$$

whear the Stuckelberg field $T(t, r)$ fulfills

$$\boxed{(\partial_t T)^2 - (\partial_r T)^2 = 1}$$

Infinitely many solution. Only one solution $\boxed{T = t}$ has been studied. When expressed in different slicings reads

Different slicings

- flat slicing $t = \sinh\tau + \frac{\rho^2}{2}\, e^\tau,\ r = \cosh\tau - \frac{\rho^2}{2}\, e^\tau,\ R = \rho e^\tau$

$$ds_g^2 = \frac{3}{\Lambda}\left(-d\tau^2 + e^{2\tau}(d\rho^2 + \rho^2 d\Omega^2)\right)$$

- close slicing $t = \sinh\tau,\ r = \cosh\tau\cos\rho,\ R = \cosh\tau\sin\rho$

$$ds_g^2 = \frac{3}{\Lambda}\left(-d\tau^2 + \cos^2\tau(d\rho^2 + \sinh^2\rho d\Omega^2)\right)$$

In both cases f-metric is not diagonal and depends on ρ

- close slicing $t = \sinh\tau\cosh\rho,\ r = \cosh\tau,\ R = \sinh\tau\sinh\rho$

$$\begin{aligned} ds_g^2 &= \frac{3}{\Lambda}\left(-d\tau^2 + \sinh^2\tau(d\rho^2 + \sinh^2\rho d\Omega^2)\right) \\ ds_f^2 &= \frac{3C^2}{\Lambda}\left(-\cosh^2\tau d\tau^2 + \sinh^2\tau(d\rho^2 + \sinh^2\rho d\Omega^2)\right) \end{aligned}$$

The two metrics share the same symmetries – "the only genuinely homogeneous and isotropic dRGT cosmology". However, it is unstable /Mukohyama et al/

Summary of non-bidiagonal solutions

- The only solutions which exist both in bigravity and massive gravity with fixed f. Exhaust all massive gravity solutions.
- Comprise an infinite family. The g-metric is the same s in GR – dS(AdS) or Schwarzschild-(A)dS – but the Stuckelberg scalars are different.
- For all of them the scalar graviton is strongly bound – the linear perturbations are the same as in GR. The difference arises only in higher orders.
- Poorly understood. Only one solution (open FRLW cosmology of Mukohyama) was thoroughly studied and a ghost was detected at the third perturbation order. It is unclear if this result extends higher orders.

III. “Hairy” cosmologies

FLRW cosmologies with bidiagonal metrics

FLRW ansatz

$$ds_g^2 = -dt^2 + e^{2\Omega}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \qquad /k = 0, \pm 1/$$

$$ds_f^2 = -\mathcal{A}^2 dt^2 + e^{2\mathcal{W}}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

Friedmann equations $/\xi = e^{\mathcal{W}-\Omega}/$

$$\dot{\Omega}^2 = \frac{\Lambda_g + \rho_g}{3} - \frac{k}{4}\, e^{-2\Omega} \qquad \frac{\dot{\mathcal{W}}^2}{\mathcal{A}^2} = \frac{\Lambda_f + \rho_f}{3} - \frac{k}{4}\, e^{-2\mathcal{W}}$$

$$\Lambda_g = m^2 \cos^2\eta (b_0 + 3b_1\xi + 3b_2\xi^2 + b_3\xi^3)$$

$$\Lambda_f = m^2 \frac{\sin^2\eta}{\xi^3}(b_1 + 3b_2\xi + 3b_3\xi^2 + b_4\xi^3)$$

Conservation condition $\left[\left(e^{\mathcal{W}}\right)^{\cdot} - \mathcal{A}\left(e^{\Omega}\right)^{\cdot}\right](b_1 + 2b_2\xi + b_3\xi^2) = 0$

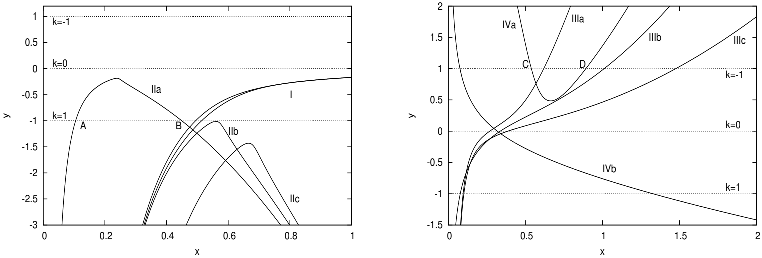
$$\Rightarrow \boxed{\xi^2(\Lambda_f + \rho_f) = \Lambda_g + \rho_g} \quad (\star) \quad \Rightarrow \quad \xi = \xi(\rho_g, \rho_f) = \xi(\Omega)$$

Solutions

With $\mathbf{a} = e^{\Omega}$ equations reduce to

$$\dot{\mathbf{a}}^2 + \mathbf{U}(\mathbf{a}) = -k$$

where $\mathbf{U}(\mathbf{a})$ is defined by roots of the algebraic relation ($\star$)



Various solutions, at late times generically approaching the proportional de Sitter.

Anisotropic cosmologies

/Kei-ichi Maeda, M.S.V/

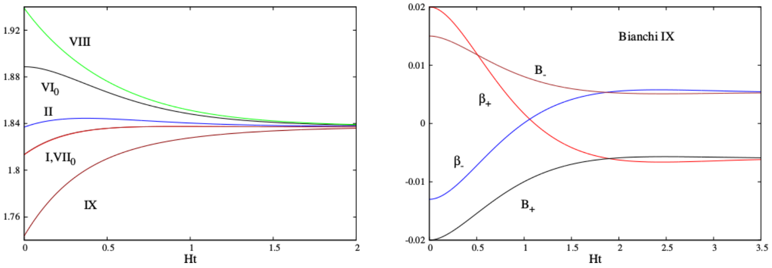
Bianchi class A types

$$ds_g^2 = -dt^2 + dl_g^2 \qquad ds_f^2 = -\mathcal{A}^2(t)dt^2 + dl_f^2$$

$$\begin{aligned} dl_g^2 &= e^{2\Omega}\left(e^{2\beta_+ + 2\sqrt{2}\beta_-}(\omega^1)^2 + e^{2\beta_+ - 2\sqrt{2}\beta_-}(\omega^2)^2 + e^{-4\beta_+}(\omega^3)^2\right) \\ dl_f^2 &= e^{2\mathcal{W}}\left(e^{2\mathcal{B}_+ + 2\sqrt{2}\mathcal{B}_-}(\omega^1)^2 + e^{2\mathcal{B}_+ - 2\sqrt{2}\mathcal{B}_-}(\omega^2)^2 + e^{-4\mathcal{B}_+}(\omega^3)^2\right) \end{aligned}$$

$\langle \omega^a, e_b \rangle = \delta^a_b$ $[e_a, e_b] = C^c_{ab} e_c \Rightarrow$ Bianchi I,II,VI,VII,VIII,IX
Initial data at $t = t_0$: an anisotropic deformation of a finite size FLRW. f-sector is empty, g-sector contains radiation + dust. All solutions rapidly approach proportional backgrounds with constant $H = \dot{\Omega}$ and constant non-zero anisotropies= late time attractor.

Solutions



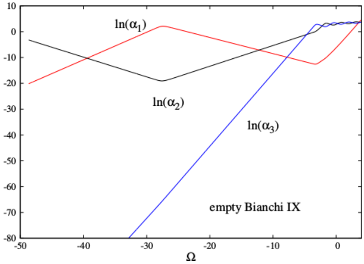
$\dot{\Omega}$ for all Bianchi types (left) and anisotropy parameters for Bianchi IX (right). At late time anisotropies oscillate around constant values $\beta_{\pm} = \beta_{\pm}(\infty) + const. \times e^{-3Ht} \cos(\omega t)$. The shear energy

$$\dot{\beta}_+^2 + \dot{\beta}_-^2 \sim e^{-3H} \sim 1/\mathbf{a}^3$$

behaves as a non-relativistic (dark ?) matter, while in GR it is $\sim 1/\mathbf{a}^6$.

Chaos

In the past solutions show singularity where e^{Ω} and $e^{\mathcal{W}}$ vanish, anisotropies oscillate near singularity.



Sequence of Kasner-type periods during which eigenvalues of the three-metric

$$\alpha_a \sim t^{p_a} \quad \text{with} \quad p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^1$$

$$1/\mathbf{a}^6 \quad \leftarrow \quad \text{shear energy} \;\; \dot{\beta}_+^2 + \dot{\beta}_-^2 \quad \rightarrow \quad 1/\mathbf{a}^3$$

Summary of “hairy” cosmologies

- Exist only in bigravity, comprise a large family. At late times approach the proportional de-Sitter with constant anisotropies – late time acceleration.
- Early time behaviours depends crucially on values of b_k, m and η.
- For certain parameter values can be matched to the primary inflationary stage $\Rightarrow$ candidates for describing physical cosmology.

Akrami, Kovisto, Amendola, Solomon, Flanders, Mortshel, ...

Hairy black holes

M.S.V., Phys.Rev. D85 (2012) 124043
Brito, Cardoso, Pani, Phys.Rev. D88 (2013) 064006

Static bidiagonal metrics

$$
\begin{aligned}
ds_g^2 &= -Q^2 dt^2 + \frac{R'^2}{N^2}\, dr^2 + R^2 d\Omega^2 \\
ds_f^2 &= -q^2 dt^2 + \frac{U'^2}{Y^2}\, dr^2 + U^2 d\Omega^2
\end{aligned}
$$

6 functions Q, N, R, q, Y, U depend on r, one can impose 1 gauge condition ($R = r$) $\Rightarrow$ 5 independent equations

$$
\begin{aligned}
G^0_0(g) &= \kappa_1\, T^0_0, \\
G^r_r(g) &= \kappa_1\, T^r_r, \\
G^0_0(f) &= \kappa_2\, \mathcal{T}^0_0, \\
G^r_r(f) &= \kappa_2\, \mathcal{T}^r_r, \\
T^{r\prime}_r &+ \frac{Q'}{Q}\,(T^r_r - T^0_0) + \frac{2}{r}(T^\vartheta_\vartheta - T^r_r) = 0.
\end{aligned}
$$

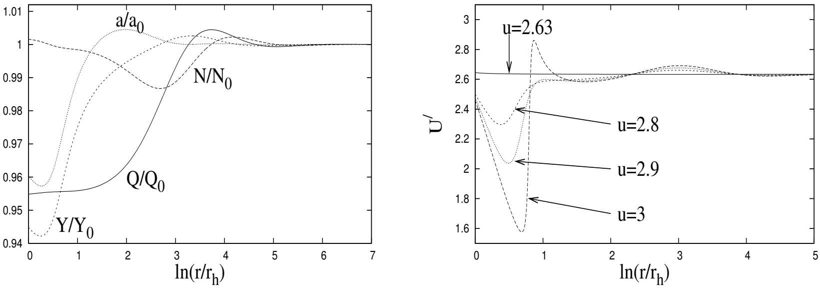
Event horizon at $r = r_h$

Equations reduce to a dynamical system for N, Y, U, one has

$$N^2 = \sum_{n\geq 1} a_n (r - r_h)^n, \quad Y^2 = \sum_{n\geq 1} b_n (r - r_h)^n, \quad U = u_h + \sum_{n\geq 1} c_n (r - r_h)^n$$

- Regular horizon is common for both metrics
- Black hole solutions comprise a two-parameter set labeled by r_h and u_h $\Rightarrow$ horizon radii measured by the two metric.
- Horizon surface gravities and temperatures are the same for both metrics.

Black holes with massive graviton hair



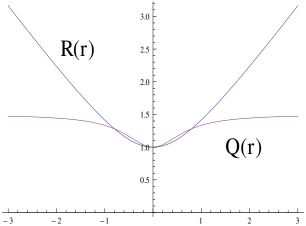
- For generic values of r_h, u_h solutions either show a curvature singularity at a finite distance away from r_h or approach asymptotically the AdS space /M.S.V. 2012/
- For specially fine-tuned r_h, u_h there are asymptotically flat black holes with $r_h \sim 1/m \Rightarrow$ they are cosmologically large /Brito, Cardoso, Pani 2013/

Wormholes

/S.V.Sushkov and M.S.V. 2015/

Wormholes – bridges between universes

$$ds^2 = -Q^2(r)dt^2 + dr^2 + R^2(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$



- $G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow \rho + p < 0,\ p < 0 \Rightarrow$ violation of the null energy conditions $\Rightarrow$ vacuum polarization, or exotic matter (phantoms), or gravity modifications (Gauss-Bonnet, braneworld).
- The structure of $T_{\mu\nu}$ and $\mathcal{T}_{\mu\nu}$ in the bigravity theory generically violates the N.E.C. /Visser et al, 2012/

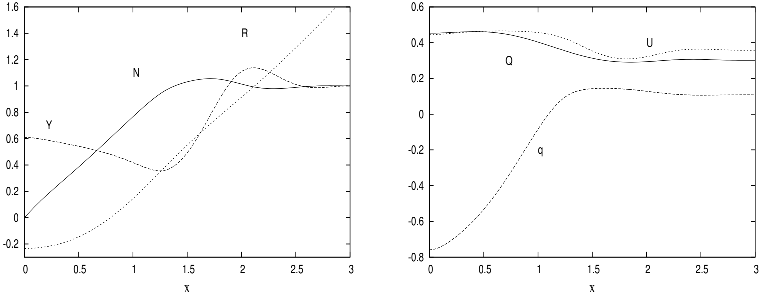
Wormholes – local solution

$$
\begin{aligned}
ds_g^2 &= -Q^2 dt^2 + dr^2 + R^2 d\Omega^2 \\
ds_f^2 &= -q^2 dt^2 + \frac{U'^2}{Y^2} dr^2 + U^2 d\Omega^2
\end{aligned}
$$

$$
\begin{aligned}
Y &= Y_1 r + Y_3 r^3 + \ldots \quad Q = Q_0 + Q_2 r^2 + \ldots \quad R = h + R_2 r^2 + \ldots \\
q &= q_0 + q_2 r^2 + \ldots \qquad U = u h + U_2 r^2 + \ldots
\end{aligned}
$$

Expanding the field equations gives in the leading order algebraic equations for Q_0 and q_0, whose solution exists if only $h \geq 1/\sqrt{3}$ (in units of $1/m$) $\Rightarrow$ wormholes are cosmologically large.

Wormhole solutions



The g-metric is globally regular and asymptotically AdS, has two AdS boundaries. The f-metric shows a Killing horizon at the point where q vanishes. The g-geodesics oscillate around $r = 0 \Rightarrow$ throat is traversable.

Summary of black holes and wormholes

- If the ghost-free bigravity indeed describes the world, then the astrophysical black holes are the same as in GR, up to a tiny ($\sim m$) effect of accretion of massive modes.
- Theory also admits black holes with massive graviton hair. The are generically asymptotically AdS and exceptionally asymptotically flat (but very large).
- Theory admits Lorentzian wormholes. No exotic matter is needed. Wormholes are cosmologically large $\Rightarrow$ in principle we all might live inside a wormhole.

Superluminality

- Characteristic surfaces of the dRGT massive gravity theory can be locally timelike $\Rightarrow$ superluminal signals.
- This has also been detected in the Galileon models.
- It is unclear if this implies aucausality. It is also unclear if timelike characteristics can be global.

Energy

/M.S.V./

Spherical symmetry

$$
\begin{aligned}
ds_g^2 &= -N^2 dt^2 + \frac{1}{\Delta^2}(dr + \beta dt)^2 + R^2 d\Omega^2 \\
ds_f^2 &= -dt^2 + dr^2 + r^2 d\Omega^2
\end{aligned}
$$

N, β, R, Δ depend on t, r. Lapse N and shift β are non-dynamical. Dynamical variables are Δ, R and their momenta

$$
p_\Delta = \frac{\partial \mathcal{L}}{\partial \dot{\Delta}}, \qquad p_R = \frac{\partial \mathcal{L}}{\partial \dot{R}},
$$

Phase space is 4-dimensional, spanned by $(R, \Delta, p_R, p_\Delta)$.

Hamiltonian

$$H = N\mathcal{H}_0 + \beta\mathcal{H}_r + m^2\mathcal{V}$$

where

$$\begin{aligned}
\mathcal{H}_0 &= \frac{\Delta^3}{4R^2}p_\Delta^2 + \frac{\Delta^2}{2R}p_\Delta p_R + \Delta R R'^2 + 2R(\Delta R')' - \frac{1}{\Delta} \\
\mathcal{H}_r &= \Delta'_\Delta + 2\Delta' p_\Delta + R' p_R
\end{aligned}$$

and the potential

$$\mathcal{V} = \frac{NR^2P_0}{\Delta} + \frac{R^2P_1}{\Delta}\sqrt{(\Delta N + 1)^2 - \beta^2} + R^2P_2$$

with

$$P_n = b_n + 2b_{n+1}\frac{r}{R} + b_{n+2}\frac{r^2}{R^2}$$

Number of DoF

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial N} &= \mathcal{H}_0 + m^2 \frac{\partial \mathcal{V}}{\partial N} = 0, \\
\frac{\partial \mathcal{H}}{\partial \beta} &= \mathcal{H}_r + m^2 \frac{\partial \mathcal{V}}{\partial \beta} = 0.
\end{aligned}
$$

- If $m = 0 \Rightarrow$ 2 first class constraints, $\mathcal{H}_0 = 0$ and $\mathcal{H}_r = 0 \Rightarrow$ $4 - 2 - 2 = 0$ DoF $\Rightarrow$ no dynamics = Birkhoff theorem
- If $m \neq 0 \Rightarrow$ the second equations determines β, while the fist one gives the constraint

$$\mathcal{C}(\Delta, R, p_\Delta, p_R) = 0$$

Hamiltonian and constraints

$$H = \mathcal{E} + N\mathcal{C}, \qquad \mathcal{E} = \frac{Y}{\Delta} + m^2 R^2 P_2$$

with

$$\mathcal{C} = \mathcal{H}_0 + Y + m^2 \frac{R^2 P_0}{\Delta} \quad \text{with} \quad Y \equiv \sqrt{(\Delta \mathcal{H}_r)^2 + (m^2 R^2 P_1)^2}$$

Secondary constraint

$$\begin{aligned} \mathcal{S} &= \{\mathcal{C}, H\} = \frac{m^4 R^2 P_1^2}{2Y} (\Delta p_\Delta + R\, p_R) - Y \left(\frac{\Delta \mathcal{H}_r}{Y} \right)' \\ &- \frac{\Delta^2 p_\Delta}{2R} \left\{ \frac{m^4}{2\Delta Y} \partial_R (R^4 P_1^2) + m^2 \partial_R (R^2 P_2) \right\} \\ &- \frac{m^2 \mathcal{H}_r}{Y} \left\{ \Delta (R^2 P_2)' + R^2 \partial_r (P_0 - \Delta P_2) \right\} = 0 \end{aligned}$$

$\Rightarrow 4 - 2 = 2 \times 1$ DoF. Energy $E = \int_0^\infty \mathcal{E} dr$ assuming $\mathcal{C} = \mathcal{S} = 0$.

Conclusions

- The energy is positive in the physical sector of the theory.
- Other sectors shows ghost-like features – negative energies and tachyons, they are unphysical.
- The physical sector is protected from the unphysical ones by a potential barrier.

Remarks

- (A) The energy is claimed to be always positive if the parameters are chosen as $b_k \sim \delta_k^1$ /Comelli and Pilo/
- (B) There is a one-parameter family of theories with $5+1$ DoF which contains (A) as a special case where the energy is claimed to be positive even in the presence of the ghost /Ogievetsky, Polubarinov 1965/.