Inhomogeneous states in 2d sigma models in large N limit

Arsenii Pikalov

Institute for Theoretical and Experimental Physics

Moscow Institute of Physics and Technology

Helmholtz International School "Cosmology, Strings, New Physics"
13th August, 2019

Structure of the talks

- 2d sigma models.
- Large N approximation and gap equation
- Solitonic solution
- Periodic solution
- Linear model and finite temperature case

The talk is based on A. Gorsky, A. Pikalov, A. Vainshtein, "On instability of ground states in 2D CP(N-1) and O(N) models at large N" (arXiv:1811.05449).

Motivation

Why to be interested in the sigma model?

- Non-trivial theory with analytical solutions
- Continuum limit of spin chain for $N = 3$
- Similarity to the Yang-Mills theory
- Simple setup to study non-perturbative phenomena

Linear model

Work in $1+1$ Minkowski space-time
There are N real scalar fields
The action of the model

$$S = \int d^2x \left(\frac{1}{2}(\partial n^a)^2 - \frac{g}{2N}(n^2 - r)^2 \right).$$

can be rewritten via an auxiliary field λ

$$S = \int d^2x \left(\frac{1}{2}(\partial n^a)^2 - \frac{\lambda}{2}((n^a)^2 - r) + \frac{N\lambda^2}{8g} \right).$$

The nonlinear model

$g \to \infty$ limit correspond to the nonlinear case

$$S = \int d^2x \left(\frac{1}{2}(\partial n^a)^2 - \frac{\lambda}{2}((n^a)^2 - r) + \cancel{\frac{N\lambda^2}{8g}} \right).$$

Action for $O(N)$ model:

$$S = \frac{1}{2}\int d^2x \left((\partial_\mu n^a)^2 - \lambda\left((n^a)^2 - r\right) \right).$$

λ is a Lagrange multiplier
Constraint $n^a n^a = 1$ defines a $N-1$ dimensional sphere
r can be considered as coupling constant

Large N approximation

Consider non-linear model case.
The action is quadratic in n, so they can be integrated

$$\int Dn\, D\lambda e^{iS[n^a,\lambda]} = \exp(iS_{eff})$$

$$S_{eff} = \frac{iN}{2} tr\log(-\partial^2 - \lambda) + \frac{1}{2}\int d^2x \left((\partial_\mu n)^2 - \lambda\left((n)^2 - r\right)\right)$$

Here we separated "classic" and "quantum" parts $n^a = n^a_q + nl^a$, $(l^a)^2 = 1$. (Background field technique)
If $N \gg 1$ saddle point approximation is valid.

Homogeneous solution

Look for time-independent solutions with $\lambda = \lambda(x);\ n = n(x)$. The equations are

$$n^2 = r - N \sum_n \frac{|f_n(x)|^2}{2E_n};\ (-\partial_x^2 + \lambda(x)) f_n(x) = E_n^2 f_n(x);$$

$$(-\partial_x^2 + \lambda(x)) n(x) = 0.$$

Standard solution:

$$\lambda = const = m^2;\ n = 0.$$

Mass is generated via dimensional transmutation.

Solitonic solution

Another solution (Nitta and Yoshii, 2017):

$$\lambda = m^2\left(1-\frac{2}{\cosh^2 mx}\right);\ n^2 = \frac{N}{2\pi}\frac{1}{\cosh^2 mx}.$$

$$f_k(x) = \frac{ik - m\tanh mx}{\sqrt{m^2+k^2}}e^{ikx};\ E_k^2 = k^2+m^2$$

$$|f_k|^2 = 1-\frac{m^2}{m^2+k^2}\frac{1}{\cosh^2 mx}.$$

Properties of the solution

What is the energy of the soliton?

$$E = -\frac{Nm}{\pi}.$$

Calculation was performed by two independent ways

- calculation of effective action value
- calculation of the average of energy-momentum tensor in inhomogeneous background

The energy is lower than the energy of homogeneous state.
How does the ground state of the model look like?
There are zero modes corresponding to the rotations of the n field in the internal space. Integration over them yields the volume of S_{N-1}

Zero modes problem

Consider again the gap equation

$$n^2 = r - N \sum_n \frac{|f_n(x)|^2}{2E_n}; \; (-\partial_x^2 + \lambda(x)) f_n(x) = E_n^2 f_n(x);$$

$$(-\partial_x^2 + \lambda(x)) n(x) = 0.$$

If the second equation has non-trivial solution, there is an eigenvalue $E_0 = 0$.
The sum in the first equation is not well-defined.
Explanation (no rigorous proof yet!) - zero modes correspond to the rotational moduli of the soliton

$$n^a = n l^a.$$

Zero modes problem

$$n^2 = r - N\sum_n \frac{|f_n(x)|^2}{2E_n}; \; (-\partial_x^2 + \lambda(x))f_n(x) = E_n^2 f_n(x);$$

$$(-\partial_x^2 + \lambda(x))n(x) = 0.$$

We should consider zero modes separately
Integration over the zero modes yields moduli space volume contribution to the partition functions.
Quantized zero modes describe low-energy dynamics of the solitons

Gross-Neveu model

$$S = \frac{1}{2} \int d^2x \left(\bar{\psi}^a (i\hat{\partial} - \sigma)\psi^a - r\sigma^2 \right)$$

It is fermionic part of SUSY $O(N)$ model.
Homogeneous solutions are

$$\sigma = \pm m \sim \langle \bar{\psi}\psi \rangle$$

There are also kink solutions

$$\sigma = \pm m \tanh mx .$$

They correspond to the $O(N)$ soliton:

$$\lambda = \sigma^2 - \partial_x \sigma .$$

Periodic solution

Similarly to Gross-Neveu case look for solution

$$\lambda(x) = m_1^2 \nu (2 sn^2(m_1 x;\, \nu) - 1);\, n \sim dn(m_1 x;\, \nu).$$

Properties

- Corresponds to the ground state of Gross-Neveu model at finite density
- Ill-defined in $O(N)$ case on $\mathbb{R}^2$ due to IR divergences
- Formally, energy is lower than in homogeneous case

About ground state

- Homogeneous solution is (probably) not the ground state at large N
- Periodic solution is not well-behaved on the whole plane
- Euclidean correlators in $O(N)$ model decay exponentially (Kopper, 1998)
- We should consider contributions from many soliton configurations

Other comments and questions

- Topological nature of the soliton
- Difference between $O(N)$ and $\mathbb{CP}^{N-1}$ and role of the gauge field
- SUSY generalizations
- Connection with classical solutions
- Phase transitions in the model on circle (Flachi et al., arXiv:1907.00120; Fujimori et al., arXiv:1907.06925 — lattice simulation)

Linear model

The action of the model

$$S = \int d^2x \left(\frac{1}{2}(\partial n^a)^2 - \frac{\lambda}{2}((n^a)^2 - r) + \frac{N\lambda^2}{8g} \right).$$

We can consider case $g < 0$ if g is small enough.
The system is stable due to quantum corrections to the potential (it can be seen from the gap equation)

Linear model

There is also a case

$$S = \int d^2x \left(\frac{1}{2}(\partial n_a)^2 - \frac{1}{2} m_0^2 n_a^2 - \frac{g}{4N}(n_a^2)^2 \right).$$

$$S = \int d^2x \left(\frac{1}{2}(\partial n_a)^2 - \frac{1}{2}\lambda n_a^2 + \frac{N}{4g}(\lambda - m_0^2)^2 \right).$$

Linear model

We consider the potential

$$V = an^2 + bn^4$$

in following cases

- $a < 0,\ b > 0$ (Higgs like potential)
- $a > 0,\ b < 0$ (Classically unstable)
- $a > 0,\ b > 0$

Higgs-like case

Higgs-like case $(g > 0)$ or "unstable" $(g < 0)$ case

$$S = \int d^2x \left(\frac{1}{2}(\partial n^a)^2 - \frac{\lambda}{2}((n^a)^2 - r) + \frac{N\lambda^2}{8g} \right).$$

For negative g we must have (Abbott, 1976)

$$\frac{|g|}{2\pi m^2} < 1$$

$\lambda = m^2$ is physical mass of the particles

Inhomogeneous solution

We use the same ansatz

$$\lambda = m^2 \left(1 - \frac{2}{\cosh^2 mx}\right)$$

$$n^2 = \frac{N}{2\pi}\left(1 - \frac{2\pi m^2}{g}\right)\frac{1}{\cosh^2 mx}.$$

If $g < 0$ there is always solution.
If $g > 0$ then g should be large enough.

Energy

$$E = -\frac{Nm}{\pi}\left(1 + \log\frac{\Lambda}{m} - \frac{\pi m^2}{3g}\right)$$

First term is the conformal anomaly contribution,
second – change of point of renormalization
third – value of classical potential energy.
Using homogeneous gap equation we obtain

$$E = -\frac{Nm}{\pi}\left(1 + \frac{2\pi m^2}{3g}\right).$$

Energy

$$E = -\frac{Nm}{\pi}\left(1 + \frac{2\pi m^2}{3g}\right).$$

$$n^2 = \frac{N}{2\pi}\left(1 - \frac{2\pi m^2}{g}\right)\frac{1}{\cosh^2 mx}.$$

For positive coupling $g > 0$ energy is always negative, but the solution exists only for strong enough coupling.

For negative coupling there is always a solution, but the energy can change sign. Are there phase transitions?

$$a > 0 \qquad b > 0$$

The solution is

$$n^2 = \frac{N}{2\pi} \frac{1}{\cosh^2 mx} \left(1 - \frac{2\pi m^2}{g}\right).$$

and exists at strong coupling.
The energy is

$$E = -\frac{Nm}{\pi} - \frac{4m^3}{3g} N.$$

and always negative.
The situation is very similar to Higgs-like potential.

Finite temperature

We again consider non-linear model

- Integration over frequencies – summation due to periodic conditions in (Euclidean) time. Gap equation is modified.
- Energy is also modified – there are contributions from thermal excitations.
- At high temperatures the energy of the soliton is positive.
- Thermal phase transition?
- There is no interesting effect due to chemical potential.

Conclusions

- There are solutions with negative energy
- When we vary the parameters of the model, such solutions can appear and disappear, their energy can change sign.
- If there are negative energy solution, the true ground state structure is probably complicated.
- There might be phase transitions.

Thank you for attention!