Q-balls in the U(1) gauged Friedberg-Lee-Sirlin model

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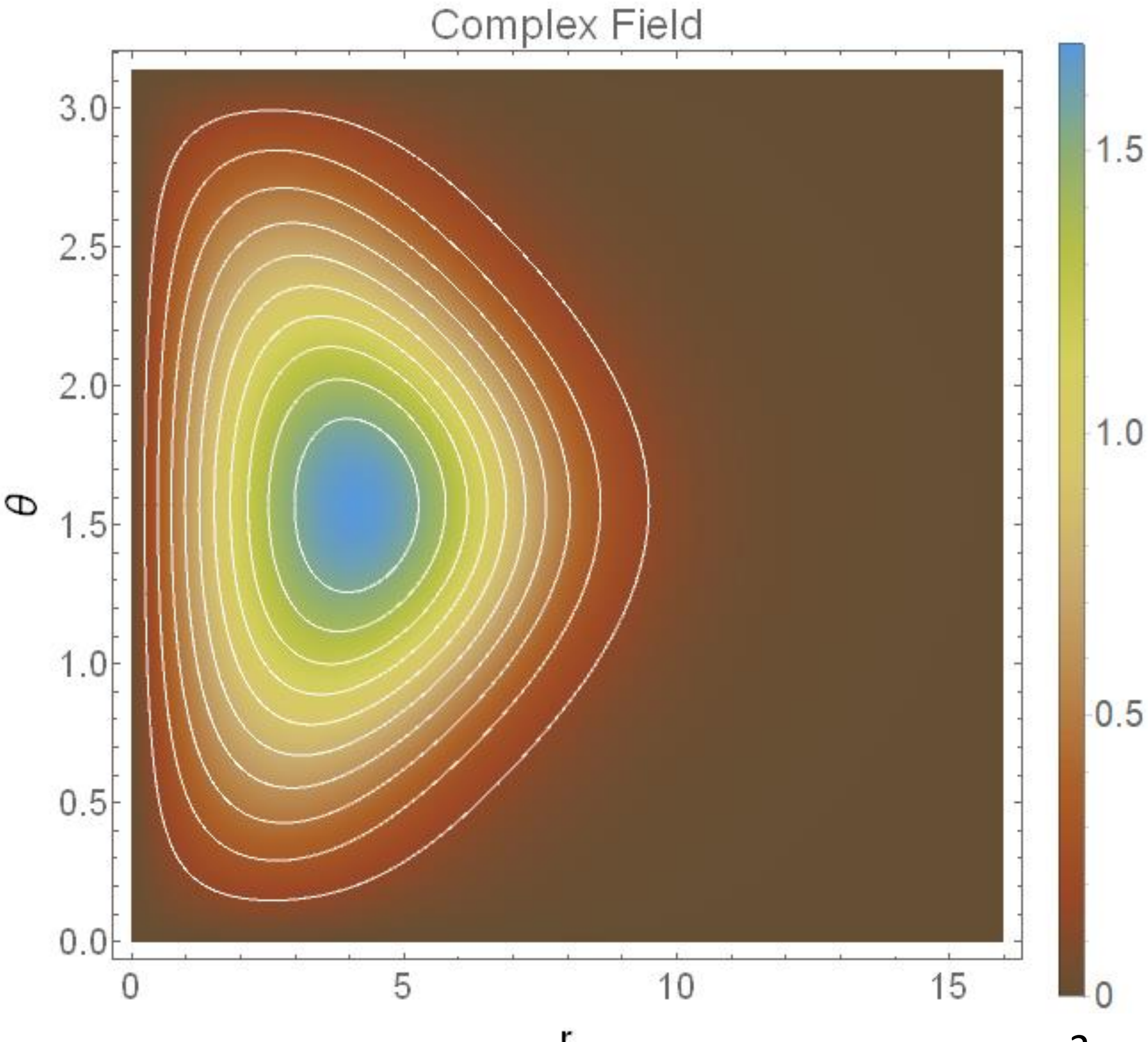
Q-balls

Q-balls – non-topological solitons with time dependent field, carrying conserving Noether charge associated with symmetry.



- Certain types of boson stars with appropriate non-linear self-interaction are linked to the corresponding flat space solutions, which represent Q-balls. (*R. Friedberg, T. D. Lee and Y. Pang, Phys. Rev. D 35 (1987) 3640, B. Kleihaus, J. Kunz and M. List, Phys. Rev. D 72 (2005) 064002, J. Kunz, I. Perapechka and Y. Shnir, arXiv:1904.13379 [gr-qc]*)
- These mini-boson stars contribute to early Universe evolution scenarios.(*R. Friedberg, T. D. Lee and Y. Pang, Phys. Rev. D 35, 3658 (1987), P. Jetzer, Phys. Rept. 220 (1992) 163, T. D. Lee, Phys. Rev. D 35 (1987) 3637.*)
- Q-balls may play an essential role in baryogenesis via the Affleck-Dine mechanism. (*I. Affleck and M. Dine, Nucl. Phys. B 249 (1985) 361.*)
- They were considered as candidates for dark matter. (*A. Kusenko and M. E. Shaposhnikov, Phys. Lett. B 418 (1998) 46.*)

Previous investigations

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 9, NUMBER 7 JULY 1968

Charged Particlelike Solutions to Nonlinear Complex Scalar Field Theories*

GERALD ROSEN
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(Received 15 September 1967)

It is shown that spatially localized singularity-free particlelike solutions exist for Lorentz-covariant complex scalar field theories with minimal gauge-invariant electromagnetic coupling, a positive-definite energy density, and suitably prescribed nonlinear self-interaction. Such a theory provides a perfectly consistent structural model on the classical level for a charged elementary particle of finite positive energy.

G. Rosen, J. Math. Phys. 9 (1968) 996, 999

PHYSICAL REVIEW D VOLUME 13, NUMBER 10 15 MAY 1976

Class of scalar-field soliton solutions in three space dimensions*

R. Friedberg
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T. D. Lee
Columbia University, New York, New York 10027
A. Sirlin
New York University, New York, New York 10003
(Received 19 January 1976)

A class of three-space-dimensional soliton solutions is given; these solitons are made of scalar fields and are of a nontopological nature. The necessary conditions for having such soliton solutions are (i) the conservation of an additive quantum number, say Q, and (ii) the presence of a neutral ($Q = 0$) scalar field. It is shown that there exist two critical values of the additive quantum number, Q_C and Q_S, with Q_C smaller than Q_S. Soliton solutions exist for $Q > Q_C$. When $Q > Q_S$, the lowest soliton mass is $< Qm$, where m is the mass of the free charged meson field; therefore, there are solitons that are stable quantum mechanically as well as classically. When Q is between Q_C and Q_S, the soliton mass is $> Qm$; nevertheless, the lowest-energy soliton solution can be shown to be always classically stable, though quantum-mechanically metastable. The canonical quantization procedures are carried out. General theorems on stability are established, and specific numerical results of the soliton solutions are given.

R. Friedberg, T.D. Lee, A. Sirlin, Phys. Rev. D 13 (1976) 2739.

Q-BALLS*

Sidney COLEMAN
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

Received 4 July 1985

A large family of field theories in 3+1 dimensions contains a new class of extended objects. The existence of these objects depends on (among other conditions) the existence of a conserved charge, Q, associated with an ungauged unbroken continuous internal symmetry. These objects are spherically symmetric, and for large Q their energies and volumes grow linearly with Q; thus they act like homogeneous balls of ordinary matter, with Q playing the role of particle number. This paper proves the fundamental existence theorem for these Q-balls, computes their elementary properties, and finds their low-lying excitations.

S.R. Coleman, Nucl. Phys. B 262 (1985) 263, Nucl. Phys. B 269 (1986) 744, Erratum

E. Radu, M.S. Volkov, Phys. Rep. 468 (2008) 101 p.52.

$[U = \lambda|\Phi|^2(|\Phi|^4 - a|\Phi|^2 + b)]$

The model

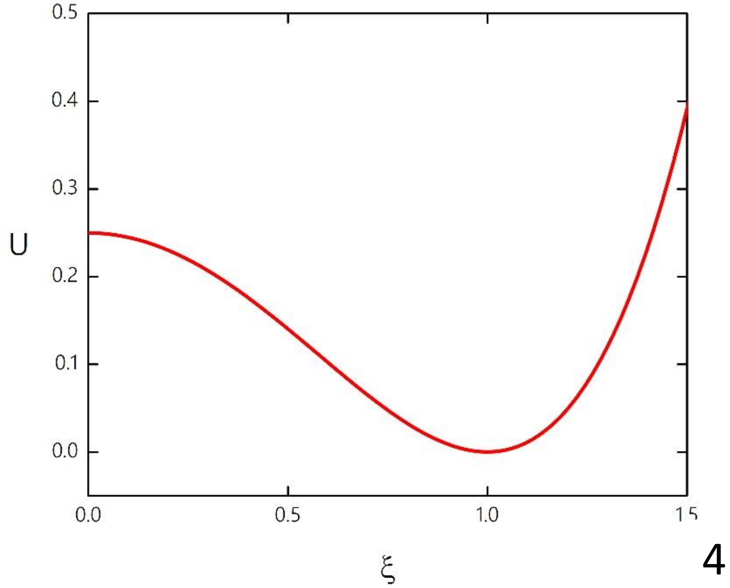
$$L = \left(\partial_\mu \psi\right)^2 + D^\mu \phi^\dagger D_\mu \phi + m^2 \psi^2 \phi^\dagger \phi + U(\psi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D^\mu \phi = \left(\partial_\mu + igA_\mu\right)\phi$$

$$\phi \to e^{i\alpha}\phi$$

$$U(\psi) = \mu(\psi^2 - 1)^2$$



ϕ – complex scalar field
ψ – real scalar field
$U(\psi)$ – scalar field potential
A – gauge field

Noether current

$$j_\mu = i(\phi D_\mu \phi^* - \phi^* D_\mu \phi)$$

$$Q = \int d^3x\, j_0$$

$$\partial^\mu F_{\mu\nu} = g\, j_\nu$$

$$L = \left(\partial_\mu \psi\right)^2 + D^\mu \phi^\dagger D_\mu \phi + m^2 \psi^2 \phi^\dagger \phi + U(\psi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\begin{cases} \partial^\mu \partial_\mu \psi = 2\psi\left(m^2 |\phi|^2 + 2\mu(1 - \psi^2)\right) \\ D^\mu D_\mu \phi = m^2 \psi^2 \phi \end{cases}$$

Different ansatz

Spherically-symmetric model

$$\chi = X(r)$$

$$\phi = Y(r)e^{-i\omega t}$$

$$A_\mu = A_0(r)$$

Axially-symmetric model

$$\psi = X(r,\theta)$$

$$\phi = Y(r,\theta)e^{i(\omega t + n\varphi)}$$

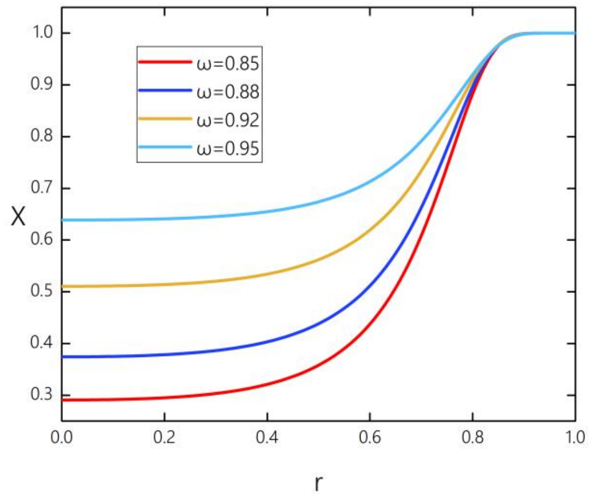
$$A_\mu dx^\mu = A_0(r,\theta)dt + A_\varphi(r,\theta)\sin\theta\, d\varphi$$

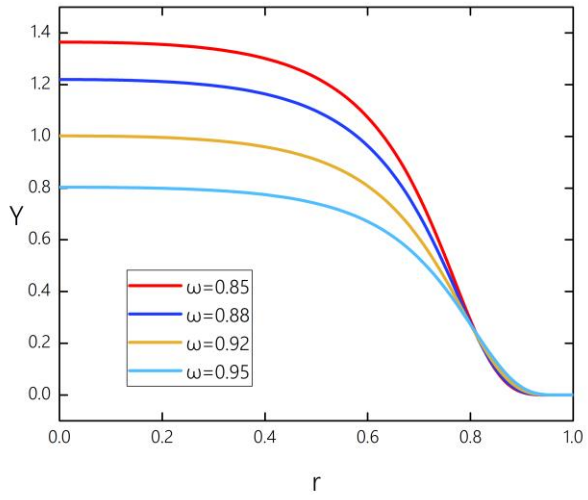
Spherical symmetry with $g \to \mathbf{0}$ (FLS)

$$L = \left(\partial_\mu \xi\right)^2 + \left|\partial_\mu \phi\right|^2 - m^2 \xi^2 |\phi|^2 - U(\xi)$$

$$\xi = X(r)\,; \qquad \phi = Y(r)e^{i\omega t}$$

$$\begin{cases} \dfrac{d^2X}{d} r^2 + \dfrac{2}{r}\dfrac{dX}{dr} + 2\mu^2 X(1 - X^2) - m^2 X Y^2 = 0 \\ \dfrac{d^2Y}{dr^2} + \dfrac{2}{r}\dfrac{dY}{dr} + \omega^2 Y - m^2 X^2 Y = 0 \end{cases}$$





R. Friedberg. T.D. Lee and A. Sirlin. Phys. Rev. D 13 (1976)

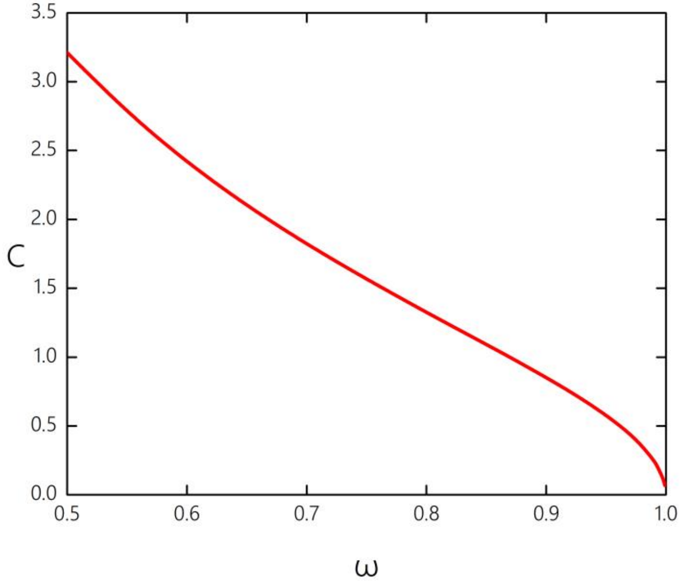
$\mu \to 0$

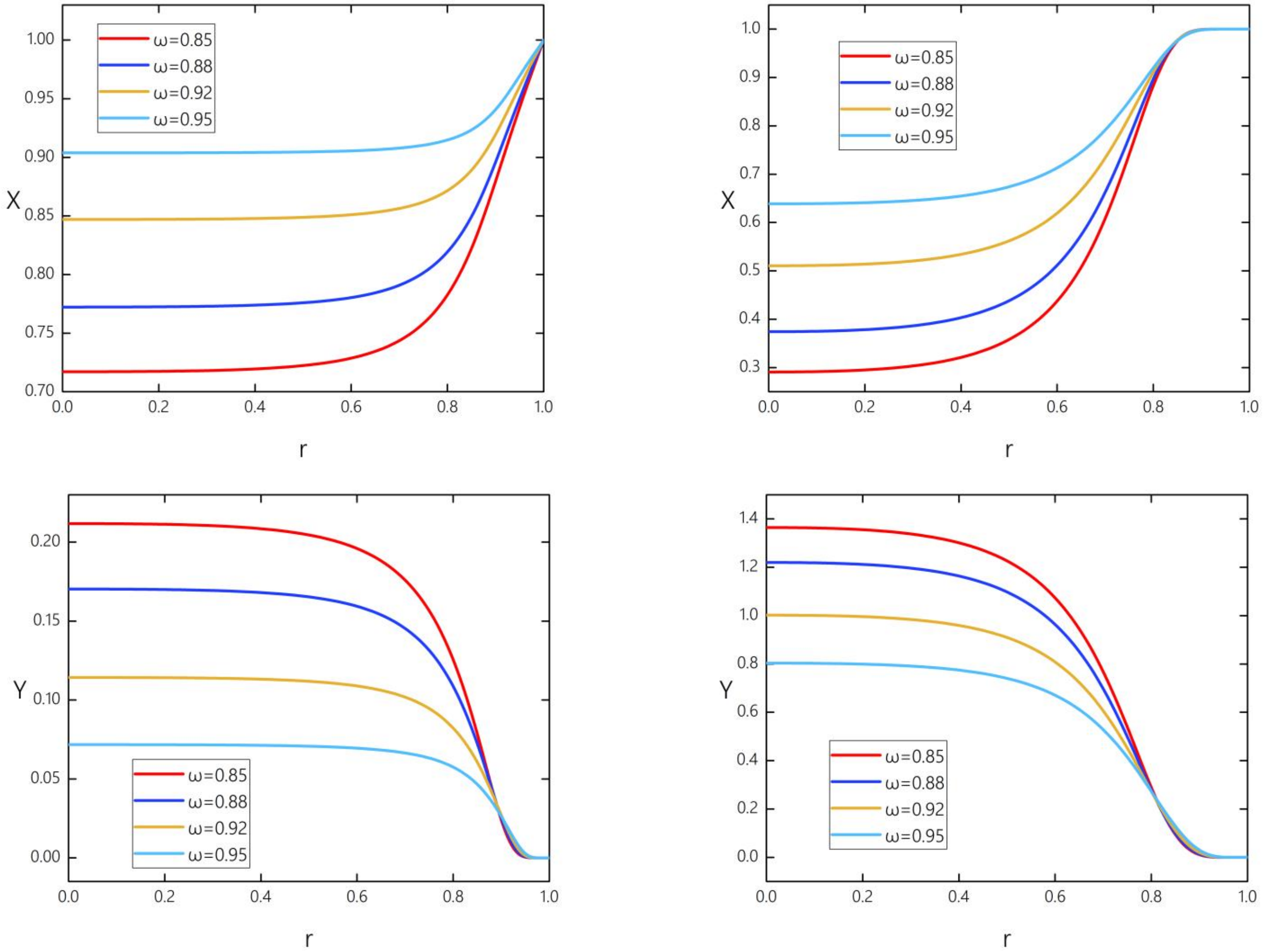
A. Levin, V. Rubakov arXiv:1010.0030v1 (2010)

$$V(X) = \mu(X^2 - 1)^2$$

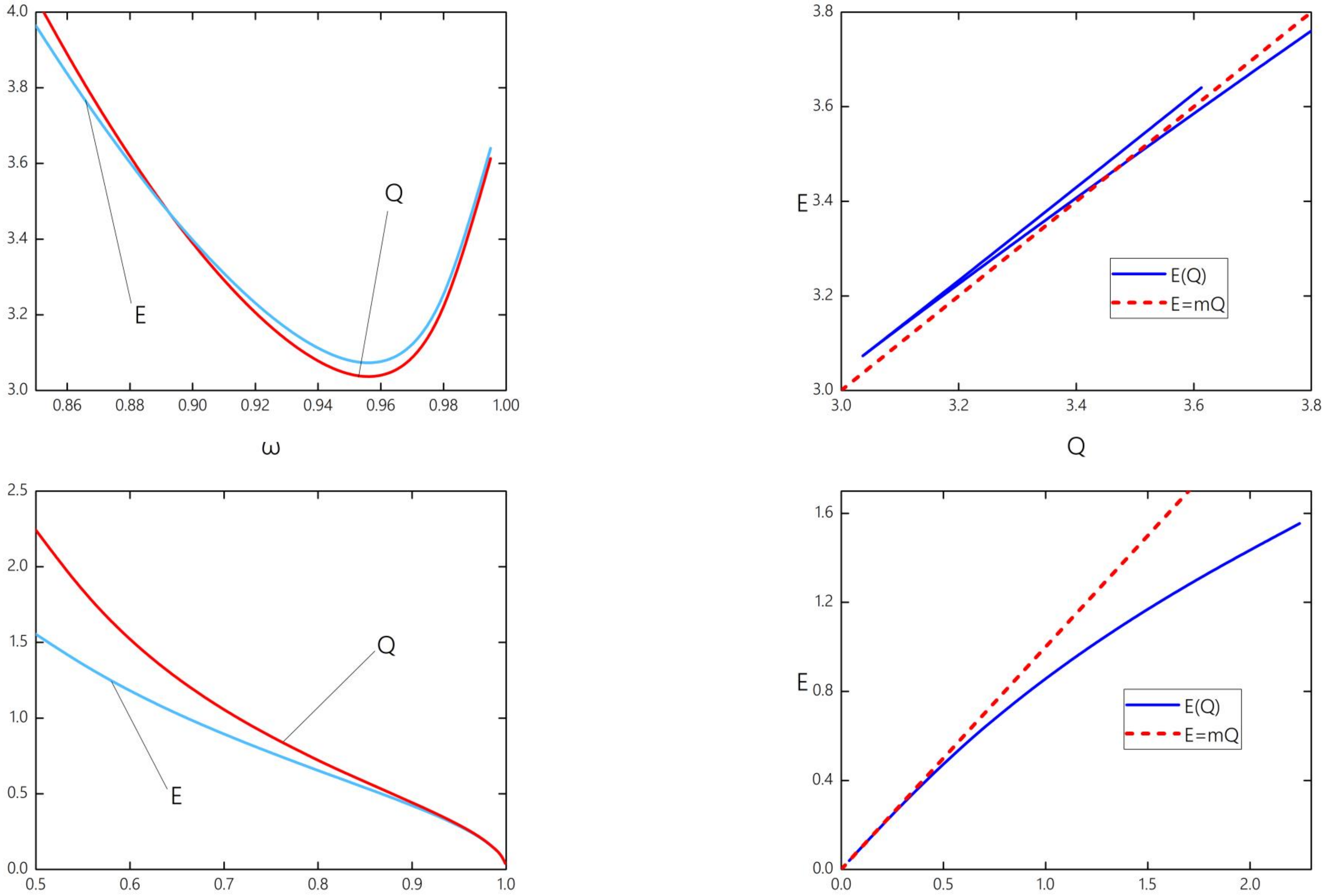
$$\mu \to 0$$

$$X(r) \sim 1 - \frac{C}{r} + \cdots$$





Spherically-symmetric case in FLS model



Spinning FLS Q-balls

$$\chi(r,\theta) = A(r,\theta)$$
$$\phi(r,\theta,\varphi,t) = B(r,\theta)e^{-i\omega t+iN\varphi}$$

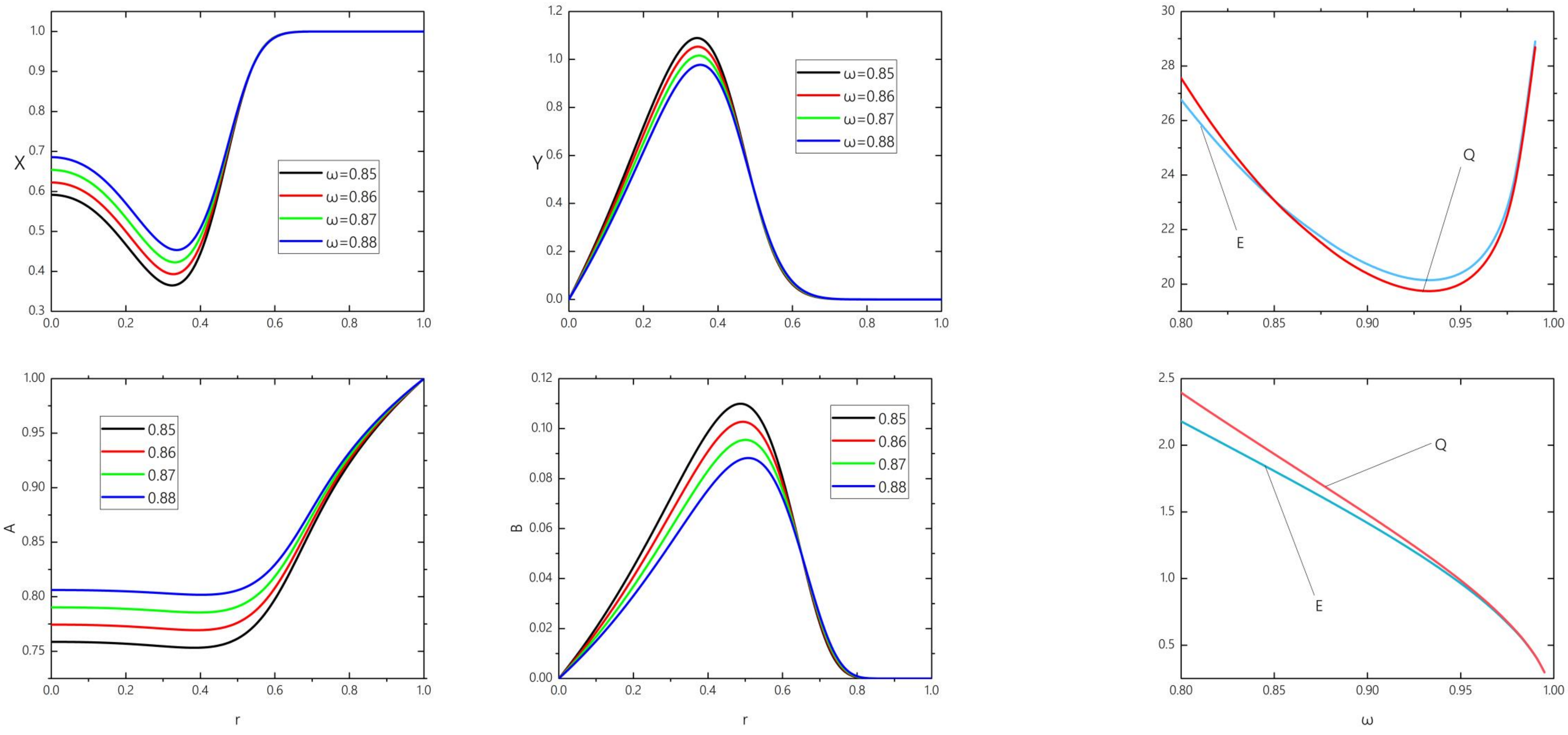
$$\frac{\partial^2 A}{\partial r^2} + \frac{2}{r}\frac{\partial A}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial A}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 A}{\partial\theta^2} - k^2B^2A - 2\lambda(A^2-1)\,A = 0$$

$$\frac{\partial^2 B}{\partial r^2} + \frac{2}{r}\frac{\partial B}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial B}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 B}{\partial\theta^2} - \frac{n^2}{r^2\sin^2(\theta)}B - k^2A^2B + \upsilon^2 B = 0$$

$$\frac{\partial A}{\partial r}(0,\theta) = 0; \quad \frac{\partial A}{\partial\theta}(r,0) = 0$$
$$A(\infty,\theta) = 1; \quad \frac{\partial A}{\partial\theta}(r,\pi/2) = 0$$

$$B(0,\theta) = 0; \quad B(r,0) = 0$$
$$B(\infty,\theta) = 0; \quad \frac{\partial A}{\partial\theta}(r,\pi/2) = 0$$

Profile functions of parity even solutions



FLSM $g \neq 0$

$$\psi = X(r,\theta)$$

$$\phi = Y(r,\theta)e^{i(\omega t + n\varphi)}$$

$$A_\mu dx^\mu = A_0(r,\theta)dt + A_\varphi(r,\theta)\sin\theta\, d\varphi$$

$$Q = \int d^3x(g\,A_0 + \omega)Y^2$$

Charge

$$J = \int d^3x T^0_\varphi = 4\pi \int_0^\pi \int_0^\infty r^2 \sin\theta\, dr d\theta \{(gA_0 + \omega)\big(n + gA_\varphi \sin\theta\big)Y^2 + J_{em}\}$$

Angular momentum

$$J_{em} = \frac{1}{r^2}\partial_\theta A_0\big(A_\varphi \cos\theta + \sin\theta\, \partial_\theta A_\varphi\big) + \sin\theta\, \partial_r A_\varphi \partial_r A_0$$

Contribution of EM field

Energy and field equations

$$E = 2\pi \int_0^{\pi} \int_0^{\infty} r^2 \sin\theta \, dr d\theta \left\{ X_r^2 + Y_r^2 + \frac{X_\theta^2}{r^2} + \frac{Y_\theta^2}{r^2} + \frac{1}{r^2}\left(gA_\varphi + \frac{n}{\sin\theta}\right)^2 Y^2 + (g\,A_0 + \omega)^2 Y^2 + \mu(1 - X^2)^2 + m^2 X^2 Y^2 + E_{em} \right\}$$

$$E_{em} = \frac{1}{2}\left\{ (\partial_r A_0)^2 + \frac{1}{r^2}(\partial_\theta A_0)^2 + \frac{1}{r^2}\left(\partial_r A_\varphi\right)^2 + \frac{1}{r^4 \sin^2\theta}\left[\partial_\theta\left(A_\varphi \sin\theta\right)\right]^2 \right\}$$

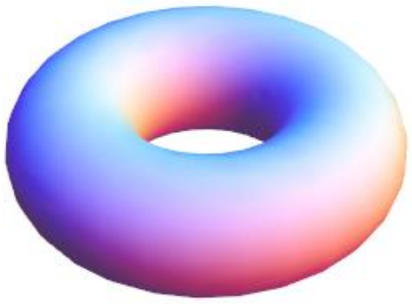
$$\begin{cases} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + 2\mu^2(1 - X^2) - m^2 Y^2\right) X = 0 \\ \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{r^2}\left(gA_\varphi + \frac{n}{\sin\theta}\right)^2 + (gA_0 + \omega^2)^2 - m^2 X^2\right) Y = 0 \\ \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin\theta} - 2g^2 Y^2\right) A_\varphi = \frac{2ng}{\sin\theta} Y^2 \\ \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} - 2g^2 Y^2\right) A_0 = 2g\omega Y^2 \end{cases}$$

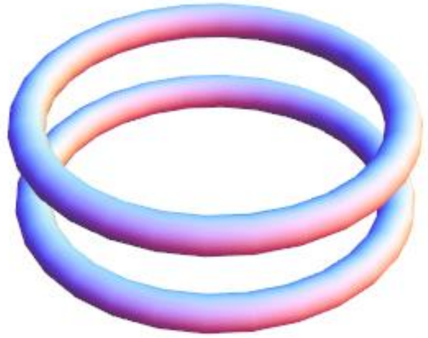
Parity even and odd solutions

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}-\frac{1}{r^2}\left(gA_\varphi+\frac{n}{\sin\theta}\right)^2+(gA_0+\omega^2)^2-m^2X^2\right)Y=0$$

$$Y_l^n(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-n)!}{(l+n)!}}P_l^n(\cos\theta)e^{in\varphi}\,.$$

$n=1$ → Y_1^1, Y_2^1



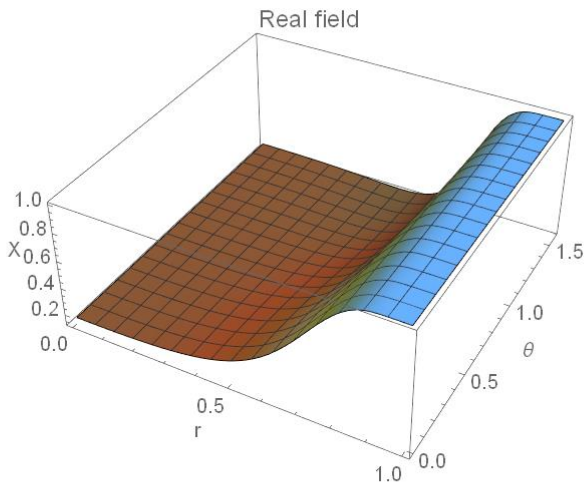


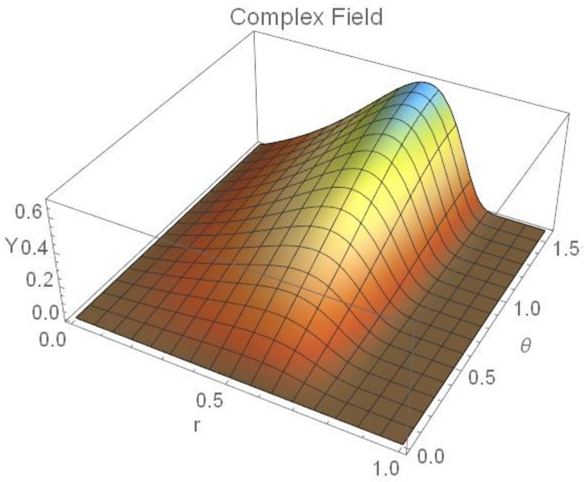
Boundary conditions

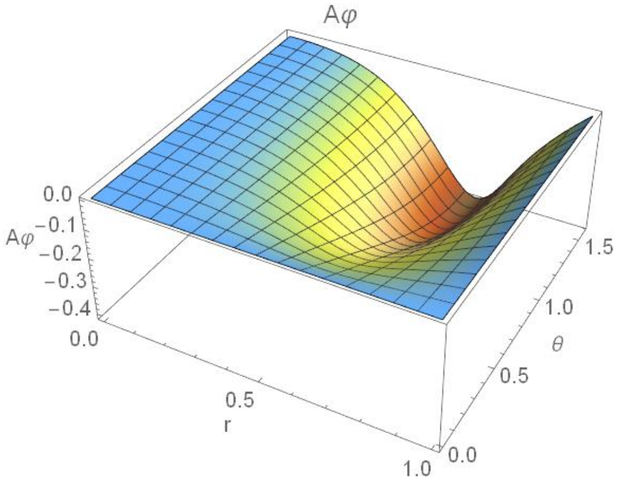
$$x = \frac{r/r_0}{1 + r/r_0} \in [0,1]$$

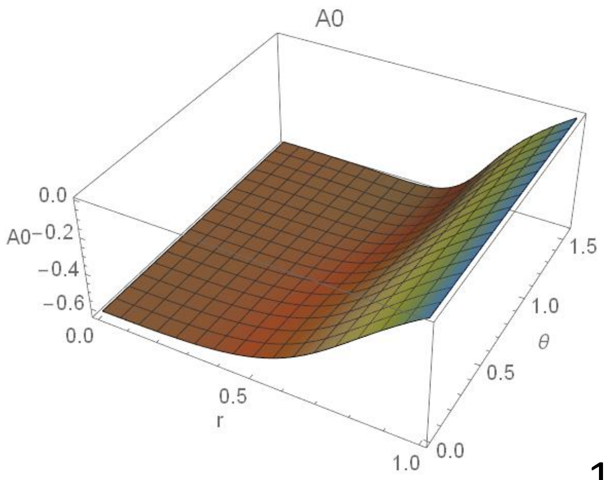
$$\partial_r X \Big|_{r=0,\theta} = 0 \qquad \partial_r Y \Big|_{r=0,\theta} = 0 \qquad X \Big|_{r=\infty,\theta} = 1 \qquad Y \Big|_{r=\infty,\theta} = 0 \qquad \partial_\theta X \Big|_{r,\theta=0} = 0 \qquad Y \Big|_{r,\theta=0} = 0$$

$$\partial_r A_0 \Big|_{r=0,\theta} = 0 \qquad \partial_r A_\varphi \Big|_{r=0,\theta} = 0 \qquad A_0 \Big|_{r=\infty,\theta} = 0 \qquad A_\varphi \Big|_{r=\infty,\theta} = 0 \qquad \partial_\theta A_0 \Big|_{r,\theta=0} = 0 \qquad A_\varphi \Big|_{r,\theta=0} = 0$$

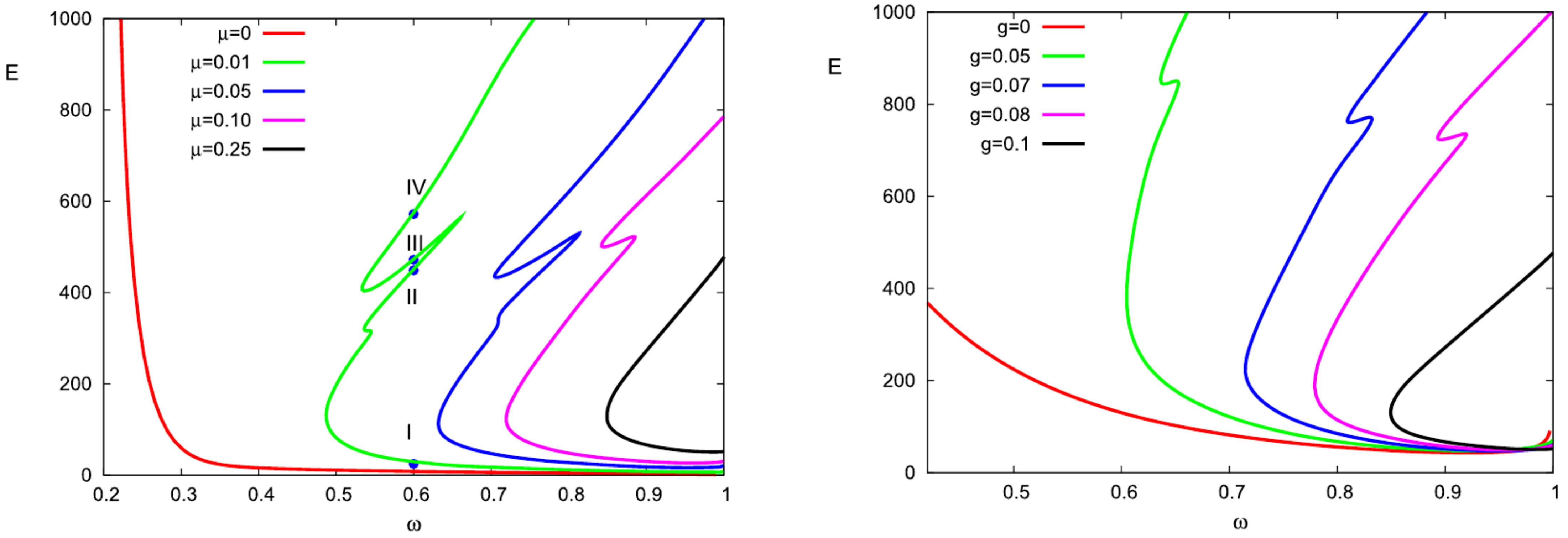






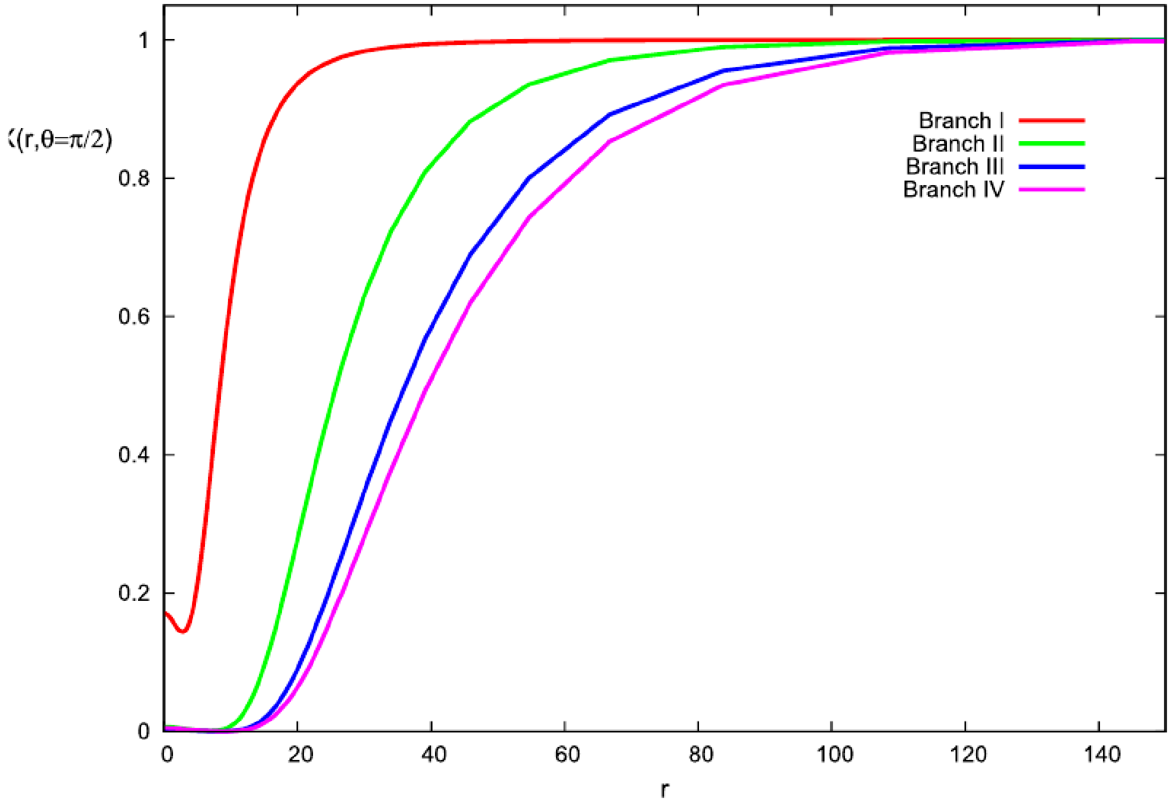


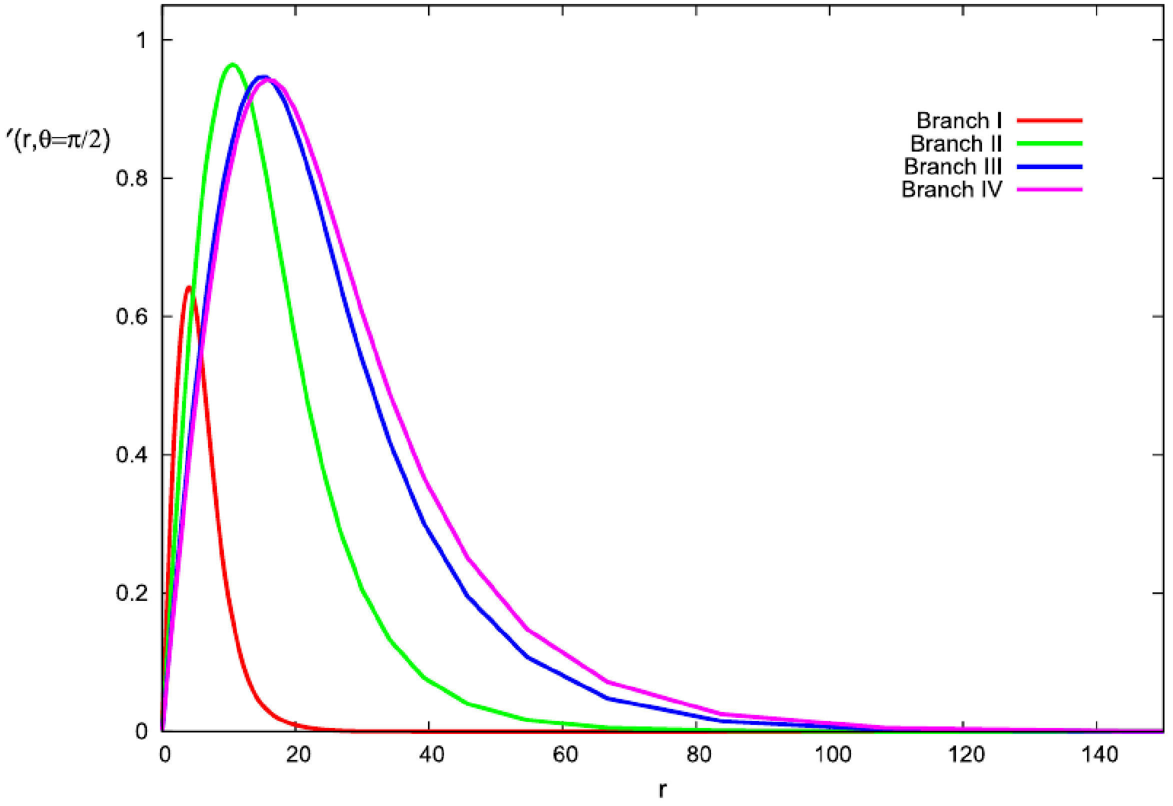
Energy on frequency



The total energy of the parity-even n=1 gauged Q-balls is shown as function of the angular frequency ω for some set of values of mass parameter μ at g = 0.1 (left plot) and for some set of values of the gauge coupling g at μ=0.25 (right plot).

Function profiles

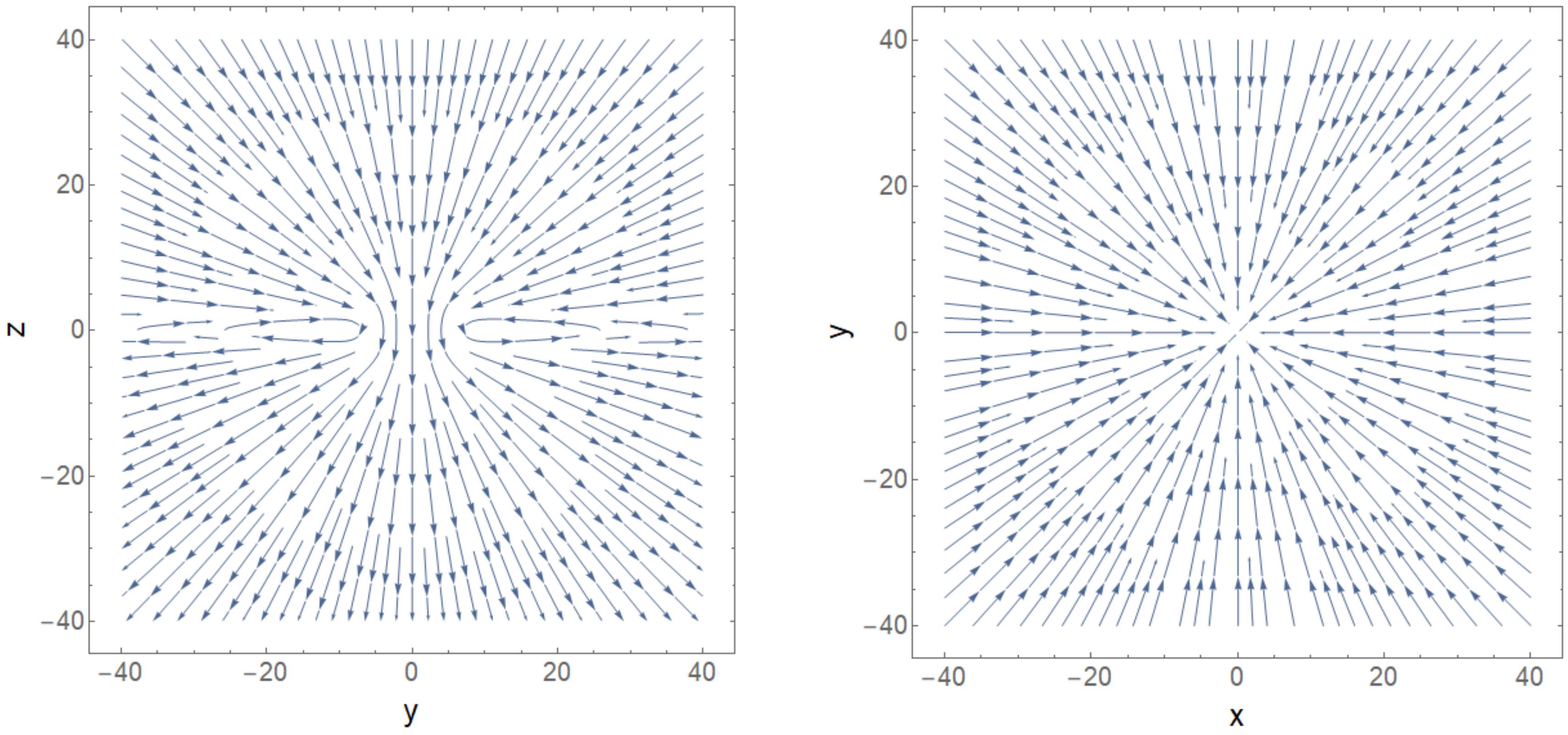




The profiles of the field components of the gauged Friedberg–Lee–Sirlin Q-balls X (left plot) and Y (right plot) at θ = π/2 are plotted on four different branches, at ω=0.60, μ=0.01 and g=0.1.

Magnetic field at I

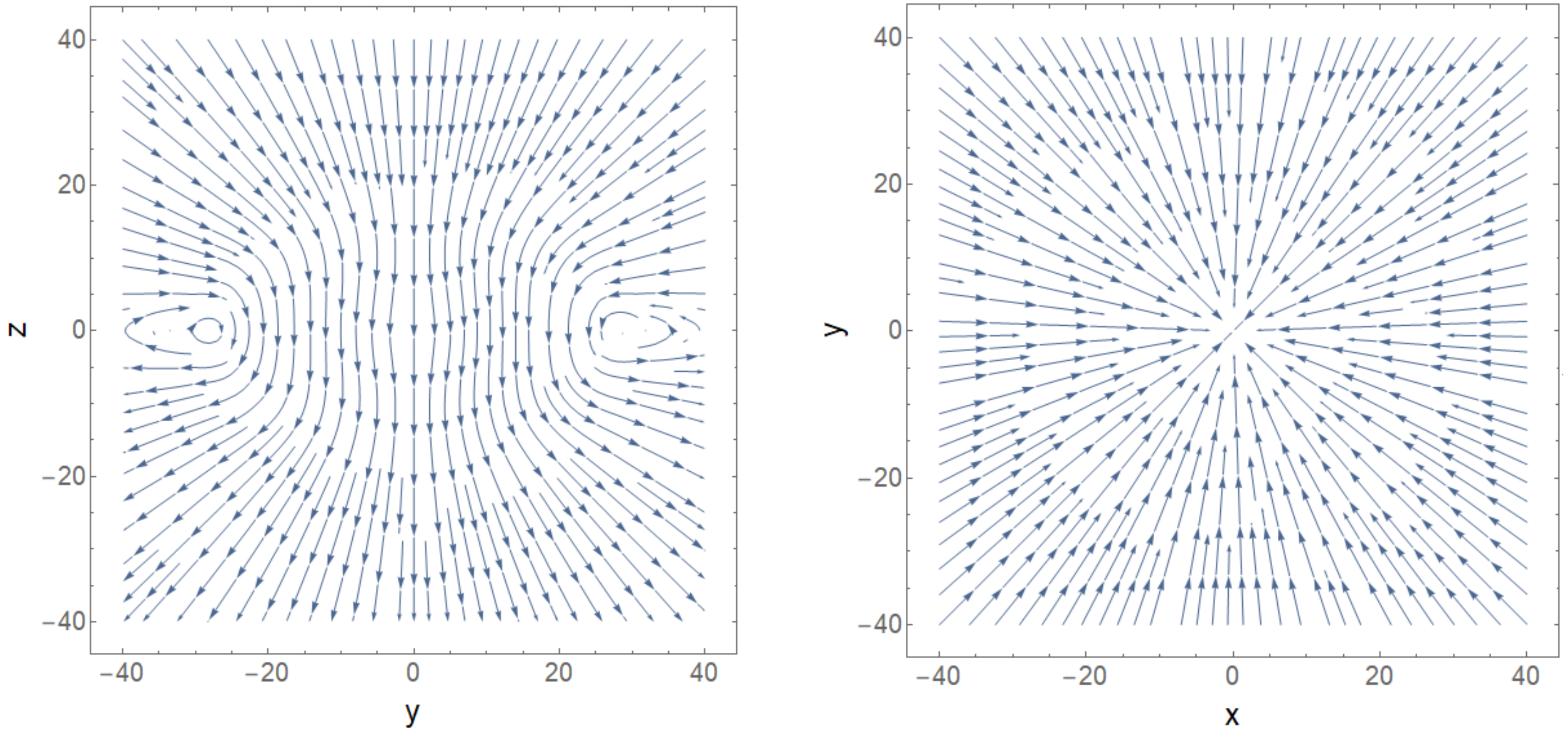
$E_k = F_{k0}$ $\qquad$ $B_k = \varepsilon_{kmn} F^{mn}$



Magnetic field orientation of the gauged n=1 Q-ball at g=0.1, μ=0.01 and ω=0.60 (electric branch); the magnetic flux in the y−z plane (left plot) and in the x−y plane (right plot).

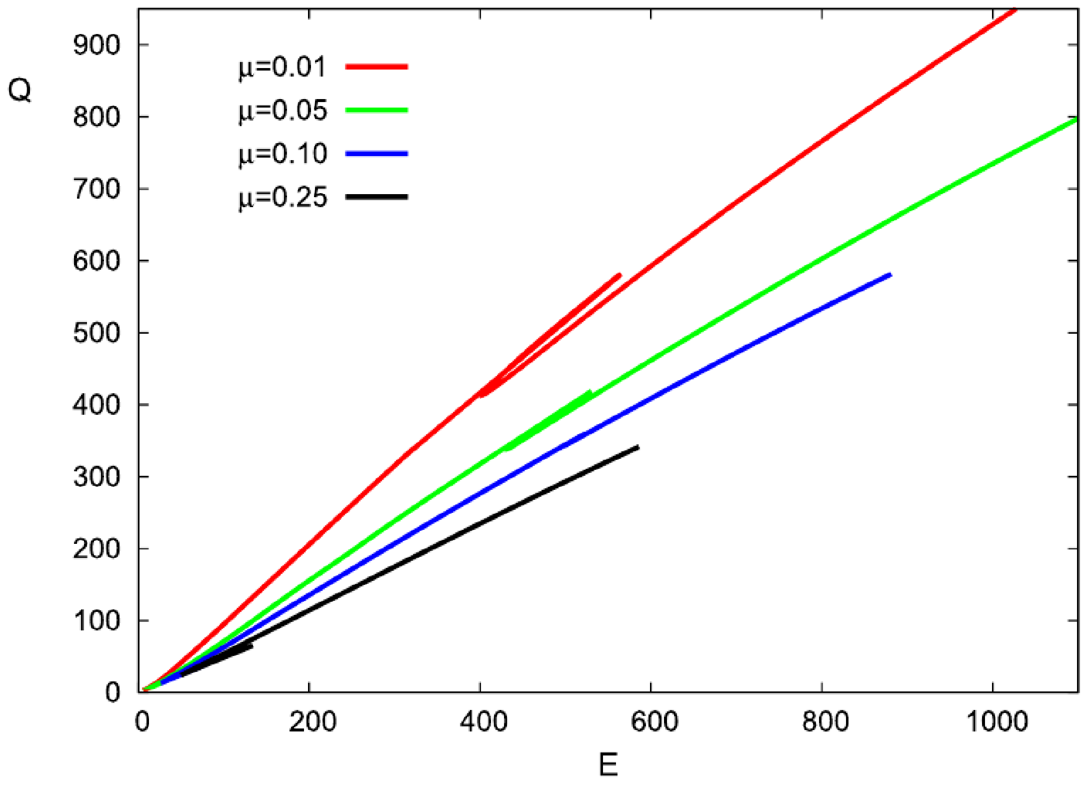
Magnetic field at III

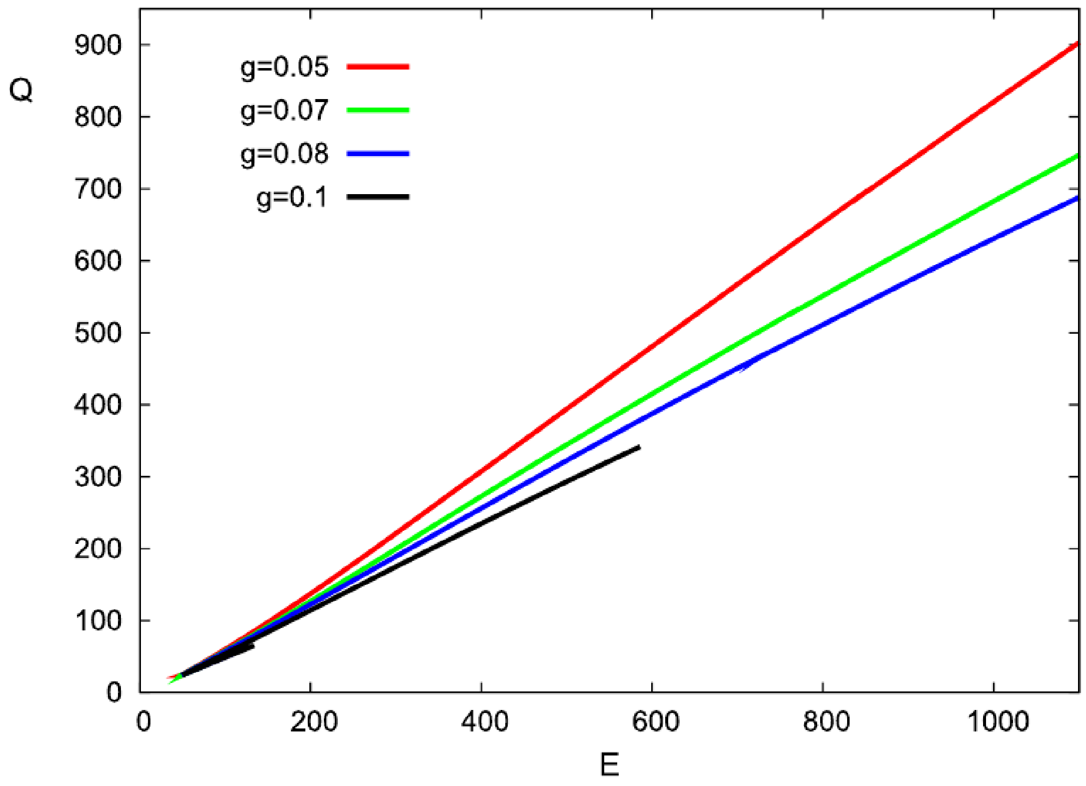
$E_k = F_{k0}$ $\qquad B_k = \varepsilon_{kmn} F^{mn}$



Magnetic field orientation of the gauged n=1 Q-ball at g=0.1, μ=0.01 and ω=0.60 (magnetic branch); the magnetic flux in the y−z plane (left plot) and in the x−y plane (right plot).

Energy curves





The total energy of the parity-even n=1 gauged Q-balls vs the charge Q for some set of values of mass parameter μ at g=0.1 (left plot) and for some set of values of the gauge coupling g at μ=0.25 (right plot)

Vortons in Witten model

$$L_W = -\frac{1}{4}\sum_{a=1,2} F^a_{\mu\nu}F^{a\mu\nu} + D_\mu\phi^* D^\mu\phi + D_\mu\sigma^* D^\mu\sigma - U$$

$$F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu$$

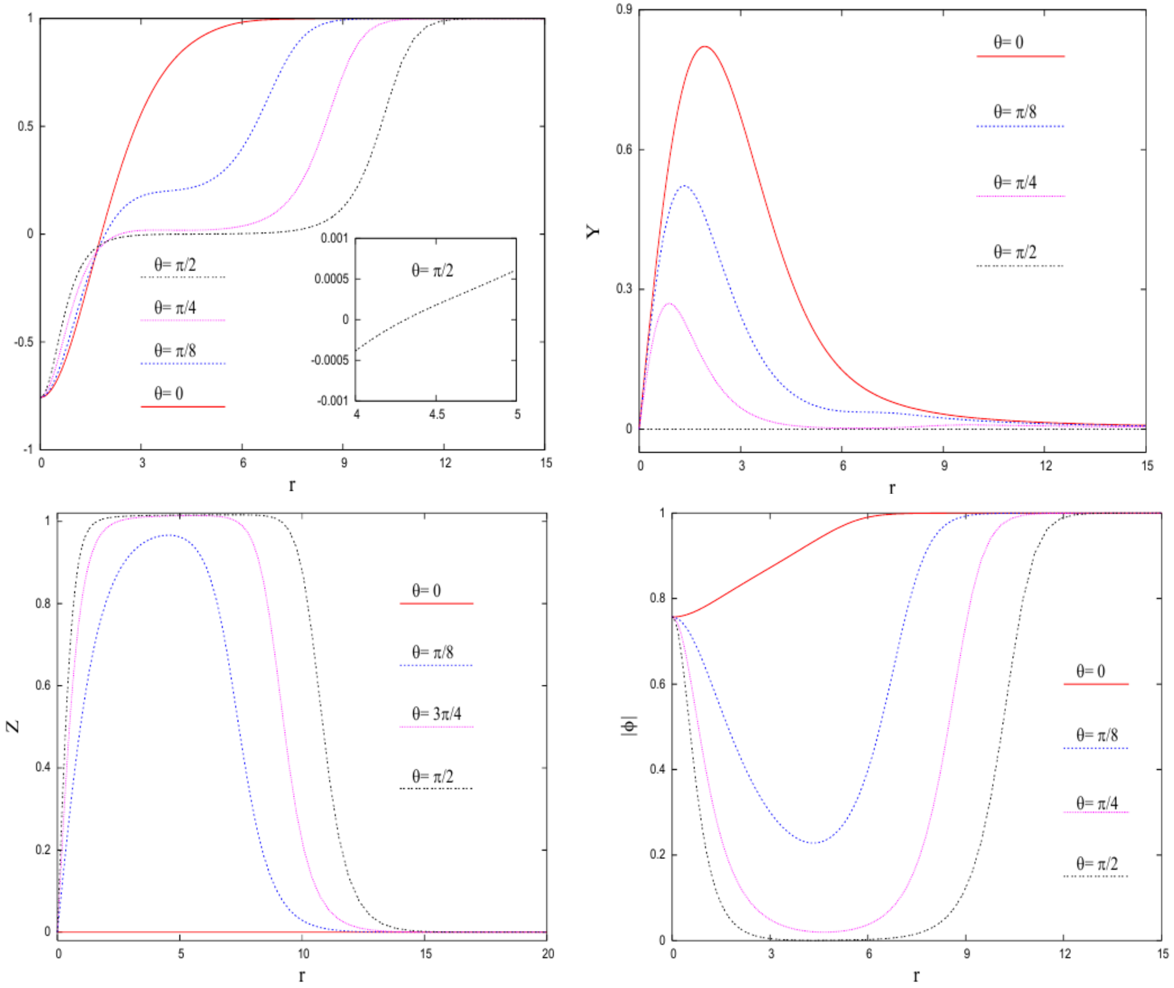
$$D^\mu\phi = \left(\partial_\mu - i g_1 A^{(1)}_\mu\right)\phi \qquad D^\mu\sigma = \left(\partial_\mu - i g_2 A^{(2)}_\mu\right)\sigma$$

$$U = \frac{1}{4}\lambda_\phi\left(|\phi|^2 - \eta_\phi^2\right)^2 + \frac{1}{4}\lambda_\sigma|\sigma|^2\left(|\sigma|^2 - 2\eta_\sigma^2\right) + \gamma|\phi|^2|\sigma|^2$$

$$L = \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\sigma^*\partial^\mu\sigma - U$$

$$\phi = f_1(\rho,z)e^{i\psi_1(\rho,z)}, \sigma = f_2(\rho,z)e^{(im\varphi + i\omega t)}$$

$$X = f_1\cos\psi_1, Y = f_1\sin\psi_1, Z = f_2$$



Conclusions

- Existence of new type of axially-symmetric solutions of the U(1) gauged FLS model was confirmed.
- They exhibit examples of the configurations with both the electric charge and toroidal magnetic field.
- Gauged Q-balls exist for relatively small values of the gauge coupling, increase of the coupling yields stronger electromagnetic repulsion which makes the configuration unstable.
- Minimal allowed value of frequency depends on strength of gauge coupling.
- Presence of toroidal magnetic field may lead to new interesting phenomena in astrophysics and cosmology while investigation of rotating boson stars and corresponding hairy black holes.

Thanks for your attention!

Additional slides

References

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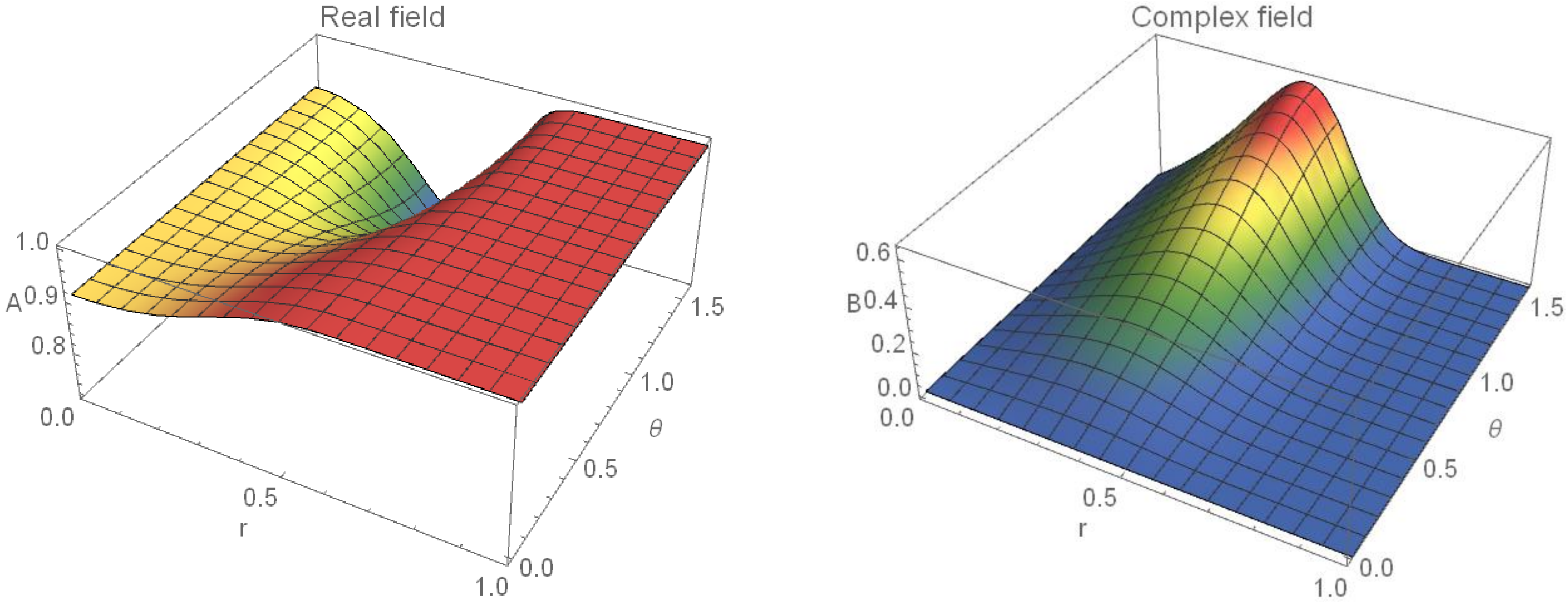
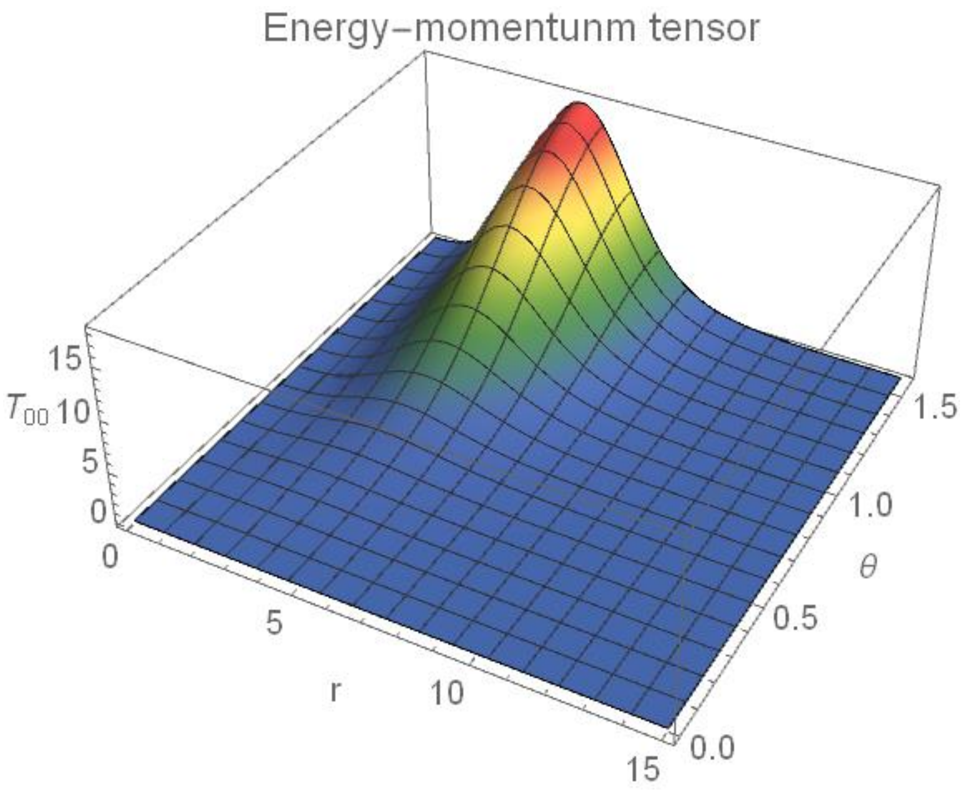
Solutions of axially-symmetric FLS



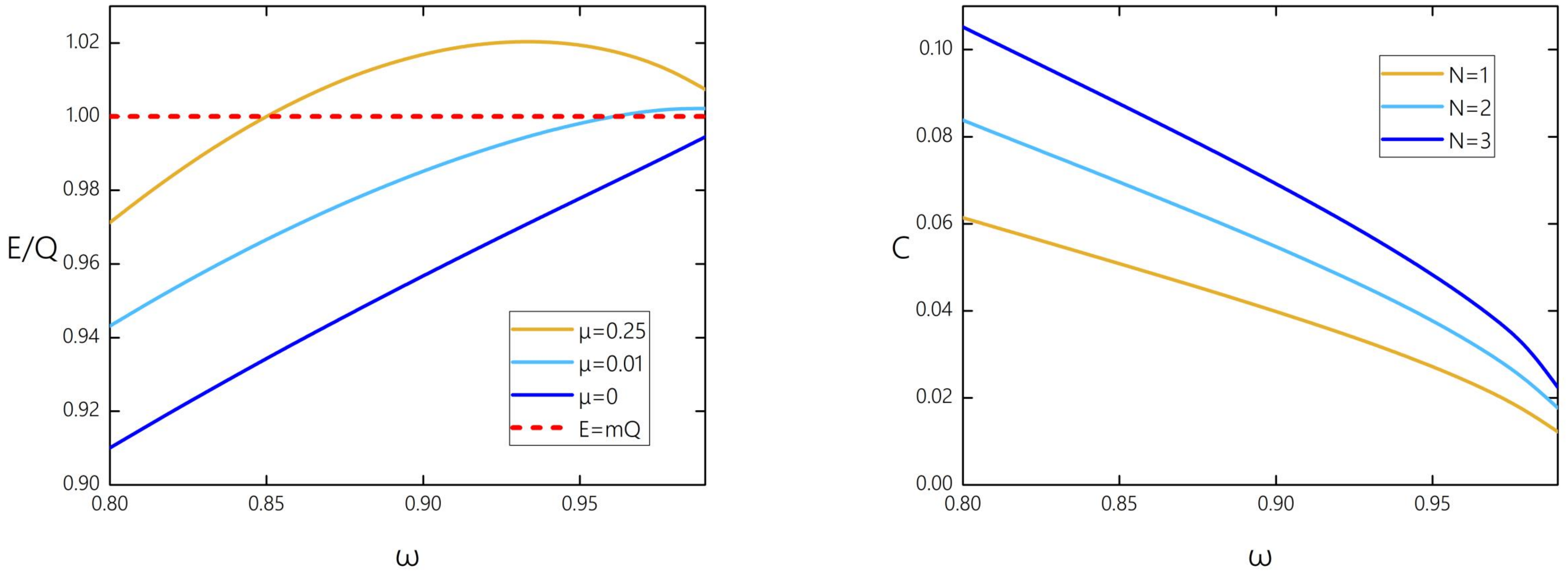
Energy and charge in AS FLS

$$Q = 8\pi\omega \int_0^{\pi/2} d\theta \int_0^{\infty} B^2 r^2 \sin\theta \, dr$$

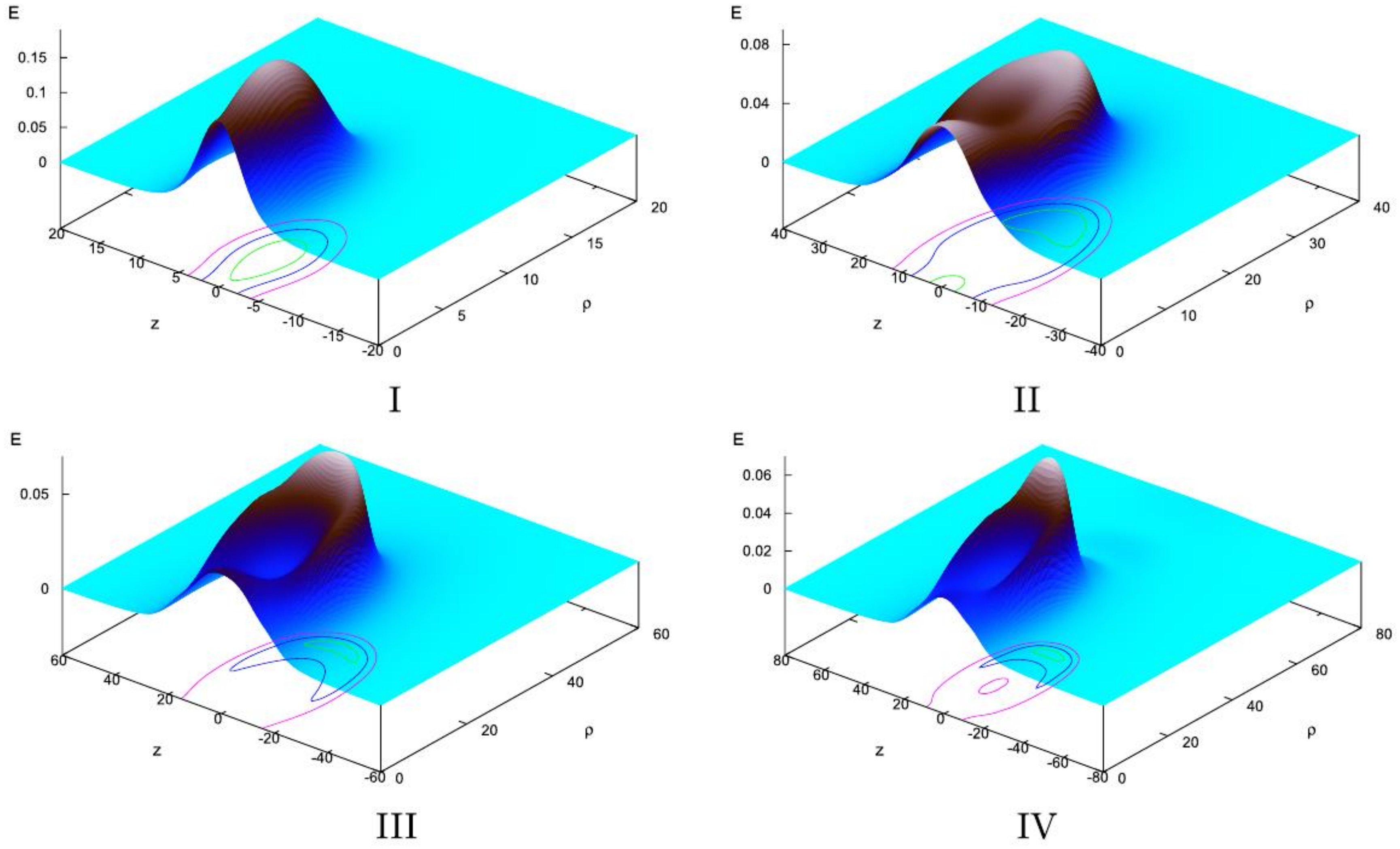
$$E = 4\pi \int_0^{\pi/2} d\theta \int_0^{\infty} \left(\omega^2 f^2 + (\partial_r A)^2 + (\partial_r B)^2 + \frac{1}{r^2}(\partial_\theta A)^2 + \frac{1}{r^2}(\partial_\theta B)^2 + kA^2B^2 + \frac{n^2B^2}{r^2 \sin^2(\theta)} + V \right) r^2 \sin\theta \, dr$$



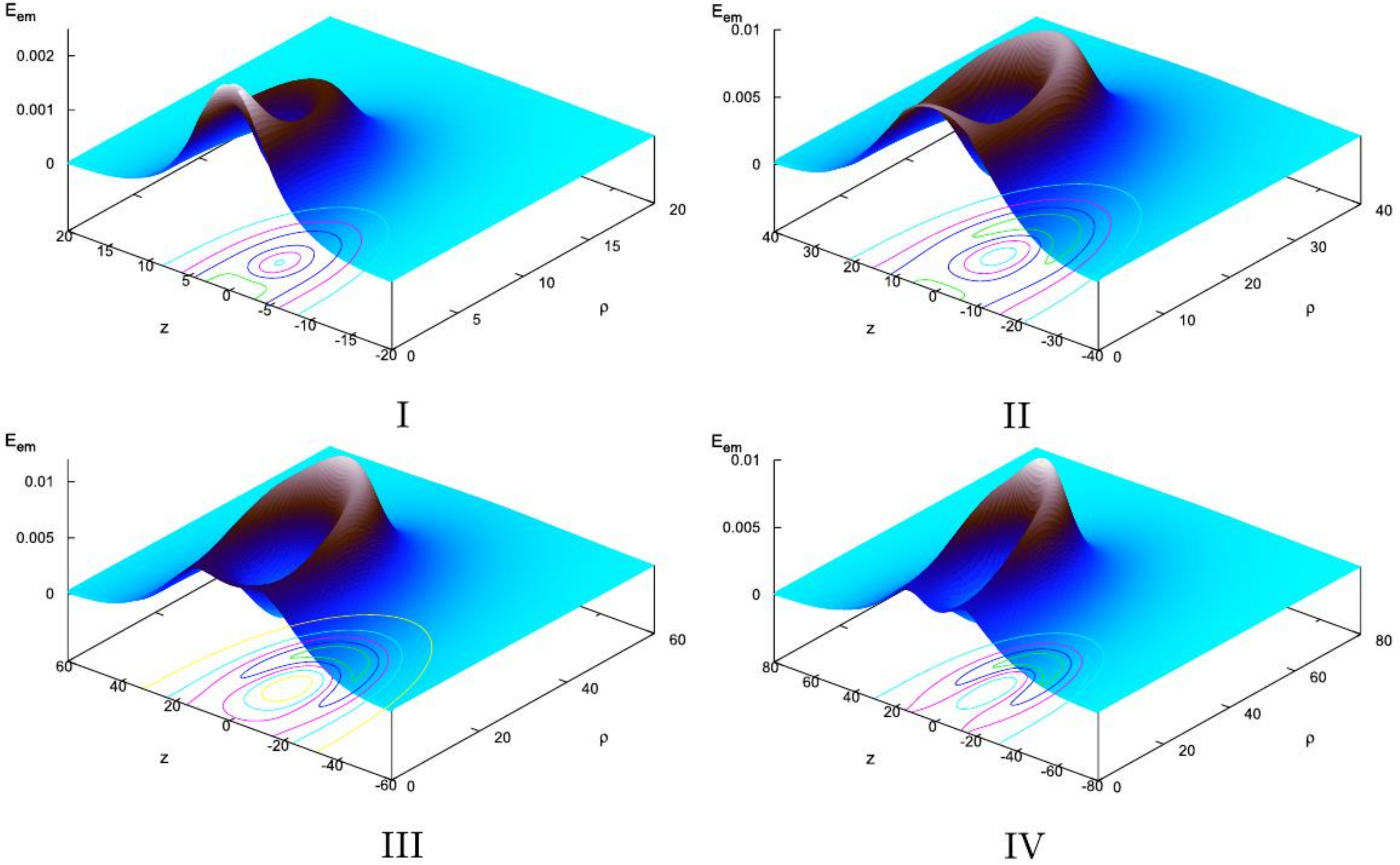
Scalar hair and energy curves



Energy density



Electromagnetic energy density



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Vortons in Witten model

$$L_W = -\frac{1}{4}\sum_{a=1,2} F^{a}_{\mu\nu}F^{a\mu\nu} + D_\mu\phi^* D^\mu\phi + D_\mu\sigma^* D^\mu\sigma - U$$

$$F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu$$

$$D^\mu\phi = \left(\partial_\mu - i g_1 A^{(1)}_\mu\right)\phi \qquad D^\mu\sigma = \left(\partial_\mu - i g_2 A^{(2)}_\mu\right)\sigma$$

$$U = \frac{1}{4}\lambda_\phi\left(|\phi|^2 - \eta_\phi^2\right)^2 + \frac{1}{4}\lambda_\sigma|\sigma|^2\left(|\sigma|^2 - 2\eta_\sigma^2\right) + \gamma|\phi|^2|\sigma|^2$$

$$\Downarrow$$

$$L = \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\sigma^*\partial^\mu\sigma - U$$

$$\phi = f_1(\rho,z)e^{i\psi_1(\rho,z)}, \sigma = f_2(\rho,z)e^{(im\varphi+i\omega t)}$$

$$X = f_1\cos\psi_1, Y = f_1\sin\psi_1, Z = f_2$$