

ASPECTS OF ELLIPTIC HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. General elliptic hypergeometric functions are defined by elliptic hypergeometric integrals. They comprise the elliptic beta integral, elliptic analogues of the Euler-Gauss hypergeometric function and Selberg integral, as well as elliptic extensions of many other plain hypergeometric and q -hypergeometric constructions. In particular, the Bailey chain technique, used for proving Rogers-Ramanujan type identities, has been generalized to integrals. At the elliptic level it yields a solution of the Yang-Baxter equation as an integral operator with an elliptic hypergeometric kernel. We give a brief survey of the developments in this field.

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1. ELLIPTIC HYPERGEOMETRIC INTEGRALS

Hypergeometric functions lie at the center of the world of special functions [3]. Ramanujan obtained many important results in the theory of hypergeometric functions and their q -analogues. It is therefore natural to give at his jubilee conference a survey of the top known special functions of hypergeometric type – the elliptic hypergeometric functions. General representatives of these functions are defined by elliptic hypergeometric integrals introduced by the author in 2000 [39] and a general setup of their theory was formulated in [41, 43]. An overview of the results obtained prior to 2008 is given in [47].

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In the univariate case the contour integrals

$$I = \int_C \Delta(u) du$$

are called elliptic hypergeometric integrals, if the (meromorphic) kernel function $\Delta(u)$ satisfies a first order finite difference equation

$$\Delta(u + \omega_1) = h(u; \omega_2, \omega_3) \Delta(u),$$

where $h(u; \omega_2, \omega_3)$ is an elliptic function,

$$h(u + \omega_2) = h(u + \omega_3) = h(u), \quad \text{Im}(\omega_2/\omega_3) \neq 0,$$

and $\omega_{1,2,3}$ are some (in general incommensurate) complex variables. Define two bases

$$p = e^{2\pi i \omega_3 / \omega_2}, \quad q = e^{2\pi i \omega_1 / \omega_2}$$

and demand that $\Delta(u) := \rho(z)$ is a meromorphic function in the variable $z = e^{2\pi i u / \omega_2} \in \mathbb{C}^*$. Then we can write

$$I = \int \rho(z) \frac{dz}{z}, \quad \rho(qz) = h(z; p) \rho(z), \quad h(pz; p) = h(z; p),$$

where

$$h(z; p) = \prod_{k=1}^m \frac{\theta(t_k z; p)}{\theta(w_k z; p)}, \quad \prod_{k=1}^m t_k = \prod_{k=1}^m w_k,$$

is an arbitrary elliptic function defined as a ratio of products of theta functions

$$\theta(z; p) = (z; p)_\infty (pz^{-1}; p)_\infty, \quad (z; p)_\infty = \prod_{j=0}^{\infty} (1 - zp^j), \quad |p| < 1.$$

The integer parameter $m \geq 2$ is called the order of the elliptic function $h(z; p)$ and t_k, w_k are arbitrary parameters fixing its divisor.

Now, due to the factorization of $h(z; p)$, it is sufficient to solve the following linear first order q -difference equation

$$f(qz) = \theta(z; p) f(z).$$

Its particular solution is given by the (standard) elliptic gamma function

$$\Gamma(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - zp^j q^k}, \quad |p|, |q| < 1. \quad (1)$$

The multiple gamma functions were proposed by Barnes [6]. Known generalizations of Euler's gamma function, including (1), can be built as combinations of Barnes' gamma functions. Various properties of the elliptic gamma functions were investigated by Jackson [26], Baxter [7], Ruijsenaars (who coined its name) [36], Felder and Varchenko [22], the author [41], and Rains [31].

As a result, for $|q|, |p| < 1$ we find the general form of elliptic hypergeometric integrals as [41, 43]

$$I = \int \prod_{k=1}^m \frac{\Gamma(t_k z; p, q)}{\Gamma(w_k z; p, q)} \frac{dz}{z}, \quad \prod_{k=1}^m t_k = \prod_{k=1}^m w_k.$$

In the following we use the conventions

$$\begin{aligned} \Gamma(t_1, \dots, t_n; p, q) &:= \Gamma(t_1; p, q) \cdots \Gamma(t_n; p, q), \\ \Gamma(tz^{\pm k}; p, q) &:= \Gamma(tz^k; p, q)\Gamma(tz^{-k}; p, q), \\ \Gamma(tx^{\pm 1}z^{\pm 1}; p, q) &:= \Gamma(txz^{\pm 1}; p, q)\Gamma(tx^{-1}z^{\pm 1}; p, q). \end{aligned}$$

2. THE ELLIPTIC BETA INTEGRAL

The key theorem for elliptic hypergeometric integrals was proved in [39].

Theorem 1. *Let $|p|, |q|, |t_j| < 1$, $\prod_{j=1}^6 t_j = pq$. Then*

$$\frac{(p; p)_\infty (q; q)_\infty}{4\pi i} \int_{\mathbb{T}} \frac{\prod_{j=1}^6 \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z} = \prod_{1 \leq j < k \leq 6} \Gamma(t_j t_k; p, q), \quad (2)$$

where \mathbb{T} is the unit circle with positive orientation.

This was a fundamentally new exactly computable integral with the following properties.

- It represents the top known hypergeometric generalization of Newton's binomial theorem and its q -analogue [3], i.e. it can be called the elliptic binomial theorem.
- It describes the top known generalization of Euler's beta integral [3], including at the intermediate steps the Askey-Wilson [5] and Rahman [29] q -beta integrals.
- Formula (2) obeys the $W(E_6)$ group of symmetries – the Weyl group of the exceptional root system E_6 (see [49] for a detailed discussion of this property).
- This integral defines the orthogonality measure for two-index elliptic biorthogonal functions [41]. The latter functions represent an elliptic extension of the famous Askey-Wilson orthogonal polynomials [5] and Rahman biorthogonal rational functions [29]. They constitute also the continuous measure extension of the discrete elliptic biorthogonal rational functions of Zhedanov and the author [53] (elliptic analogues of Wilson's functions [55]).
- In a special limit this exact integral evaluation formula can be degenerated to the Frenkel-Turaev summation formula (an elliptic functions identity of hypergeometric form) [23]. The latter sum can be degenerated to the terminating Jackson ${}_8\varphi_7$ -sum and further on to the Dougall ${}_7F_6$ -sum, which was discovered also by Ramanujan. In general, elliptic hypergeometric series emerge as residue sums for particular sequences of poles of elliptic hypergeometric integral kernels. Such series play an important role [47], but we do not describe them for brevity.

- At the univariate level only one formula of such type has been found so far, but there are many multidimensional extensions to computable integrals on root systems [13, 17, 18, 32, 33, 41, 44, 50, 51, 52].

The first proof of this theorem used an elliptic extension of Askey's method described in the proceedings of a Ramanujan centennial meeting [4]. Its shortest known proof is described in [44].

3. AN ELLIPTIC ANALOGUE OF THE EULER-GAUSS HYPERGEOMETRIC FUNCTION

The following elliptic extension of the Euler-Gauss hypergeometric function ${}_2F_1(a, b; c; x)$ [3] was introduced in [41]

$$V(t_1, \dots, t_8; p, q) = \frac{(p; p)_\infty (q; q)_\infty}{4\pi i} \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(t_j x^{\pm 1}; p, q) dx}{\Gamma(x^{\pm 2}; p, q) x}, \quad \prod_{j=1}^8 t_j = (pq)^2. \quad (3)$$

Here it is assumed that $|t_j| < 1$, but by an appropriate change of the integration contour this function can be analytically continued to $t_j \in \mathbb{C}^*$.

In contrast to the elliptic beta integral (2), this V -function obeys the $W(E_7)$ -group of symmetries with the key non-trivial transformation law discovered in [41]

$$V(t; p, q) = \prod_{1 \leq j < k \leq 4} \Gamma(t_j t_k, t_{j+4} t_{k+4}; p, q) V(s; p, q), \quad (4)$$

where $|t_j|, |s_j| < 1$ and

$$\begin{cases} s_j = \varepsilon t_j, & j = 1, 2, 3, 4 \\ s_j = \varepsilon^{-1} t_j, & j = 5, 6, 7, 8 \end{cases}; \quad \varepsilon = \sqrt{\frac{pq}{t_1 t_2 t_3 t_4}} = \sqrt{\frac{t_5 t_6 t_7 t_8}{pq}}.$$

An elliptic analogue of the hypergeometric equation [3] has the form [43, 45]

$$\begin{aligned} U(t; q, p) + \mathcal{A}(t_1, t_2, \dots, t_8, q; p) \left(U(qt_1, q^{-1}t_2; p, q) - U(t; p, q) \right) \\ + \mathcal{A}(t_2, t_1, \dots, t_8, q; p) \left(U(q^{-1}t_1, qt_2; p, q) - U(t; p, q) \right) = 0, \end{aligned} \quad (5)$$

where we indicate particular scaled parameters $q^{\pm 1}t_j$ in the set $t = (t_1, \dots, t_8)$ and

$$\mathcal{A}(t_1, \dots, t_8, q; p) := \frac{\theta(t_1/qt_3, t_3t_1, t_3/t_1; p)}{\theta(t_1/t_2, t_2/qt_1, t_1t_2/q; p)} \prod_{k=4}^8 \frac{\theta(t_2t_k/q; p)}{\theta(t_3t_k; p)},$$

$$U(t; p, q) := \frac{V(t; p, q)}{\prod_{k=1}^2 \Gamma(t_k t_3^{\pm 1}; p, q)}.$$

A detailed consideration of the limiting functions obtained from the V -function by various degenerations is given in [14].

4. THE ELLIPTIC SELBERG INTEGRAL

The following exact integration formula was suggested by van Diejen and the author in [17] (for $n = 1$ it reduces to (2)):

$$\begin{aligned} & \frac{(p; p)_\infty^n (q; q)_\infty^n}{2^n n! (2\pi i)^n} \int_{\mathbb{T}^n} \prod_{1 \leq j < k \leq n} \frac{\Gamma(tz_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{m=1}^6 \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} \\ &= \prod_{j=1}^n \left(\frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \prod_{1 \leq m < s \leq 6} \Gamma(t^{j-1} t_m t_s; p, q) \right), \end{aligned} \quad (6)$$

where $|p|, |q|, |t|, |t_m| < 1$ and $t^{2n-2} \prod_{m=1}^6 t_m = pq$. A conditional proof of this relation was given in [18]. It depended on an evaluation of another elliptic hypergeometric integral which was proven in [32, 44]. In a special $p \rightarrow 0$ limit formula (6) reduces to a Gustafson multiple q -beta integral [25] which, in turn, can be degenerated to the Selberg integral (see, e.g., [3]). Integral (6) serves as the measure for a very general class of biorthogonal functions found by Rains [30, 32] who defined a multivariable extension of the author's two-index biorthogonal functions [41] and elliptic analogues of the Koornwinder-Macdonald orthogonal polynomials and their interpolation versions due to Okounkov.

There are many exact integration formulas or symmetry transformation relations for elliptic hypergeometric integrals on roots systems analogous to (6) or (4). A large list of them can be found in [50, 51] where about a half of the presented relations are formulated as conjectures.

5. AN ELLIPTIC FOURIER TRANSFORMATION

The Bailey chain technique discovered by Andrews [1] and Paule [28] is a tool for generating infinite sequences of identities for q -hypergeometric series (see, e.g. [3]). It emerged from universalization of the proofs of famous Rogers-Ramanujan identities, one of which has the following form

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}.$$

The well-poised Bailey lemma [2] provides the top known q -series tool of such type. It served as an initial step for author's elliptic generalization of this technique. First it was done for elliptic hypergeometric series (see, e.g., [40, 54]). Subsequently it was generalized to elliptic hypergeometric integrals [42], which appeared to be the very first extension of the Bailey chain technique to integrals.

Consider this construction, conditionally called an elliptic Fourier transformation, in more detail. Take two functions $\alpha(z, t)$ and $\beta(w, t)$ depending on a complex variable z and a parameter $t \in \mathbb{C}$. They are said to form an elliptic integral Bailey pair with respect

to t if they are related by the following integral transformation

$$\beta(w, t) = M(t)_{wz} \alpha(z, t) := \frac{(p; p)_\infty (q; q)_\infty}{4\pi i} \int_{\mathbb{T}} \frac{\Gamma(tw^{\pm 1} z^{\pm 1}; p, q)}{\Gamma(t^2, z^{\pm 2}; p, q)} \alpha(z, t) \frac{dz}{z}, \quad (7)$$

where $|tw^{\pm 1}| < 1$. For wider region of parameters this operator is defined by analytical continuation — it is necessary to replace the contour of integration \mathbb{T} by a contour C which separates the sequences of poles converging to zero $z = tw^{\pm 1} p^j q^k$, $j, k \in \mathbb{Z}_{\geq 0}$, from their reciprocals $z = t^{-1} w^{\pm 1} p^{-j} q^{-k}$, $j, k \in \mathbb{Z}_{\geq 0}$, diverging to infinity. Existence of such a contour is the only restriction for the definition of analytically continued operator.

An integral analogue of the Bailey lemma has the following form [42].

Theorem 2. *Let $\alpha(z, t)$ and $\beta(w, t)$ form an elliptic integral Bailey pair with respect to t . Then the functions*

$$\alpha'(w, st) = D(s; u, w) \alpha(w, t),$$

where

$$D(s; u, w) := \Gamma(\sqrt{pq} s^{-1} u^{\pm 1} w^{\pm 1}; p, q), \quad D(s; u, w) D(s^{-1}; u, w) = 1,$$

with $s, u \in \mathbb{C}$ being arbitrary new parameters, and

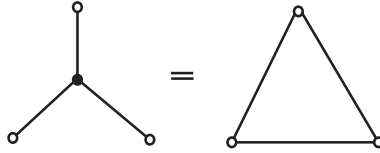
$$\beta'(w, st) = D(t^{-1}; u, w) M(s)_{wx} D(st; u, x) \beta(x, t)$$

form an elliptic integral Bailey pair with respect to the parameter st .

After substitution of explicit expressions for α' and β' into the required equality $\beta'(w, st) = M(st)_{wz} \alpha'(z, st)$, the proof boils down to the relation

$$M(s)_{wx} D(st; u, x) M(t)_{xz} = D(t; u, w) M(st)_{wz} D(s; u, z) \quad (8)$$

which is equivalent to the elliptic beta integral evaluation formula (2). Relation (8) is known as the operator form of the star-triangle relation [15] formally depicted in the figure (the black circle denotes integration over x in the left-hand side of (8)).



The main motivation for calling transformation (7) an elliptic analogue of the Fourier transformation comes from its inversion property established in [52]. Namely, up to some contour deformations the inversion relation is equivalent to the reflection $t \rightarrow t^{-1}$:

$$M(t)_{wz} M(t^{-1})_{zx} f(x) = f(w),$$

or “inversion = a sign change” (like in the Fourier transformation). One can use theorem 2 for proving various identities for elliptic hypergeometric integrals (e.g., of relation (4) or of the identity derived in [12]).

6. THE YANG-BAXTER EQUATION (YBE)

The Yang-Baxter equation (YBE) has the form [9, 21]

$$\mathbb{R}_{12}(u-v) \mathbb{R}_{13}(u) \mathbb{R}_{23}(v) = \mathbb{R}_{23}(v) \mathbb{R}_{13}(u) \mathbb{R}_{12}(u-v), \quad (9)$$

where the operator $\mathbb{R}_{jk}(u)$ acts nontrivially only in $\mathbb{V}_j \otimes \mathbb{V}_k \subset \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$ with \mathbb{V}_j being some (in general different) spaces. The variable $u \in \mathbb{C}$ is called the spectral parameter.

While investigating the eight-vertex model Baxter found a YBE solution for $\mathbb{V}_j = \mathbb{C}^2$ (i.e., $\dim \mathbb{V}_j = 2$) [7]:

$$\mathbb{R}_{12}(u) = \sum_{a=0}^3 w_a(u) \sigma_a \otimes \sigma_a, \quad w_a(u) = \frac{\theta_{a+1}(u + \eta|\tau)}{\theta_{a+1}(\eta|\tau)}, \quad (10)$$

where $\sigma_0 = \mathbb{1}$ and $\sigma_{1,2,3}$ are the Pauli matrices, $\theta_a(u|\tau) \equiv \theta_a(u)$ are the Jacobi theta functions.

Sklyanin [38] solved YBE for $\dim \mathbb{V}_1 = \dim \mathbb{V}_2 = 2, \dim \mathbb{V}_3 = \infty$, when YBE takes the form

$$\mathbb{R}_{12}(u-v) \mathbb{L}_{13}(u) \mathbb{L}_{23}(v) = \mathbb{L}_{23}(v) \mathbb{L}_{13}(u) \mathbb{R}_{12}(u-v)$$

with $\mathbb{R}_{12}(u)$ being the Baxter R-matrix (10). In this case

$$\begin{aligned} \mathbb{L}_{13}(u) &:= \mathbb{L}(u) := \sum_{a=0}^3 w_a(u) \sigma_a \otimes \mathbf{S}^a \\ &= \begin{pmatrix} w_0(u) \mathbf{S}^0 + w_3(u) \mathbf{S}^3 & w_1(u) \mathbf{S}^1 - iw_2(u) \mathbf{S}^2 \\ w_1(u) \mathbf{S}^1 + iw_2(u) \mathbf{S}^2 & w_0(u) \mathbf{S}^0 - w_3(u) \mathbf{S}^3 \end{pmatrix}, \end{aligned}$$

and the operators \mathbf{S}^a generate the Sklyanin algebra:

$$\begin{aligned} \mathbf{S}^\alpha \mathbf{S}^\beta - \mathbf{S}^\beta \mathbf{S}^\alpha &= i (\mathbf{S}^0 \mathbf{S}^\gamma + \mathbf{S}^\gamma \mathbf{S}^0), \\ \mathbf{S}^0 \mathbf{S}^\alpha - \mathbf{S}^\alpha \mathbf{S}^0 &= i \mathbf{J}_{\beta\gamma} (\mathbf{S}^\beta \mathbf{S}^\gamma + \mathbf{S}^\gamma \mathbf{S}^\beta), \end{aligned} \quad (11)$$

with (α, β, γ) a cycle of $(1, 2, 3)$ and the structure constants $\mathbf{J}_{12} = \theta_1^2(\eta)\theta_4^2(\eta)/\theta_2^2(\eta)\theta_3^2(\eta)$, etc. Relations (11) define an elliptic deformation of the $sl(2)$ -algebra of rank 1.

The generators \mathbf{S}^a can be realized explicitly as finite-difference operators [38]

$$[\mathbf{S}^a \Phi](z) = \frac{i^{\delta_{a,2}} \theta_{a+1}(\eta)}{\theta_1(2z)} \left[\theta_{a+1}(2z - 2\eta\ell) \cdot \Phi(z + \eta) - \theta_{a+1}(-2z - 2\eta\ell) \cdot \Phi(z - \eta) \right], \quad (12)$$

where the variable $\ell \in \mathbb{C}$ is called the spin.

The relation between the Sklyanin algebra and elliptic hypergeometric functions was first discussed by Rains and Rosengren [30, 35]. In [46] it was shown that the elliptic analogue of the Euler-Gauss hypergeometric function can be derived as a scalar product of solutions of generalized eigenvalue problems for linear combinations of the Sklyanin algebra generators.

7. A SOLUTION OF THE YBE FOR $\dim \mathbb{V}_j = \infty$

The logical scheme of building an infinite-dimensional YBE solution of rank 1 by Derkachov and the author in [16] consists of two steps:

- Take a defining RLL-relation for $\dim \mathbb{V}_1 = \dim \mathbb{V}_2 = \infty$, $\dim \mathbb{V}_3 = 2$ in the form

$$\mathbb{R}_{12}^?(u-v) \sigma_3 L_1(u) \sigma_3 L_2(v) = \sigma_3 L_2(v) \sigma_3 L_1(u) \mathbb{R}_{12}^?(u-v), \quad (13)$$

where $\mathbb{R}_{12}^?(u)$ is an operator (the question mark means that it is unknown) acting in the space of functions of two complex variables $\Phi(z_1, z_2)$, $L_1(u) = L(u)$ with z and ℓ replaced by z_1 and ℓ_1 , $L_2(v) = L(v)$ with z and ℓ replaced by z_2 and ℓ_2 . Solve it using some auxiliary operators $S_j, j = 1, 2, 3$, generating the permutation group \mathfrak{S}_4 , whose Coxeter relations are guaranteed by the elliptic beta integral.

- Prove that the resulting $\mathbb{R}_{12}^?(u)$ -operator obeys the general YBE for $\dim \mathbb{V}_j = \infty$.

Extract from the R-matrix the permutation operator $\mathbb{R}_{12}^?(u) := \mathbb{P}_{12} \mathbb{R}_{12}(u)$. Then relation (13) takes the form

$$\mathbb{R}_{12}(u-v) L_1(u_1, u_2) \sigma_3 L_2(v_1, v_2) = L_1(v_1, v_2) \sigma_3 L_2(u_1, u_2) \mathbb{R}_{12}(u-v), \quad (14)$$

where

$$u_1 = \frac{u}{2} + \eta \left(\ell_1 + \frac{1}{2} \right), \quad u_2 = \frac{u}{2} - \eta \left(\ell_1 + \frac{1}{2} \right), \quad v_1 = \frac{v}{2} + \eta \left(\ell_2 + \frac{1}{2} \right), \quad v_2 = \frac{v}{2} - \eta \left(\ell_2 + \frac{1}{2} \right).$$

Notice that in (14) the operator $\mathbb{R}_{12}(u-v) \equiv \mathbb{R}_{12}(u_1, u_2 | v_1, v_2)$ just permutes parameters in the product of L-operators. Denote

$$\mathbf{u} \equiv (u_1, u_2, v_1, v_2), \quad s\mathbf{u} = (v_1, v_2, u_1, u_2), \quad s = s_2 s_1 s_3 s_2,$$

where s_i are elementary permutations

$$s_1 \mathbf{u} = (u_2, u_1, v_1, v_2), \quad s_2 \mathbf{u} = (u_1, v_1, u_1, v_2), \quad s_3 \mathbf{u} = (u_1, u_2, v_2, v_1).$$

Define S_j -operators by the relations

$$\begin{aligned} S_1(\mathbf{u}) L_1(u_1, u_2) &= L_1(u_2, u_1) S_1(\mathbf{u}), & S_3(\mathbf{u}) L_2(v_1, v_2) &= L_2(v_2, v_1) S_3(\mathbf{u}), \\ S_2(\mathbf{u}) L_1(u_1, u_2) \sigma_3 L_2(v_1, v_2) &= L_1(u_1, v_1) \sigma_3 L_2(u_2, v_2) S_2(\mathbf{u}). \end{aligned}$$

These operators generate the permutation group \mathfrak{S}_4 if they satisfy the Coxeter relations:

$$S_i^2 = \mathbb{1}, \quad S_i S_j = S_j S_i, \quad |i-j| > 1, \quad S_j S_{j+1} S_j = S_{j+1} S_j S_{j+1}, \quad (15)$$

where we assume the following multiplication rule $S_j S_k := S_j(s_k \mathbf{u}) S_k(\mathbf{u})$.

Theorem 3. *The operator*

$$\mathbb{R}_{12}(\mathbf{u}) = S_2(s_1 s_3 s_2 \mathbf{u}) S_1(s_3 s_2 \mathbf{u}) S_3(s_2 \mathbf{u}) S_2(\mathbf{u})$$

solves the initial RLL = LLR relation (14).

The proof is straightforward and follows from the intertwining properties of S_j -operators.

Theorem 4. *The operator $\mathbb{R}_{12}(u) = \mathbb{P}_{12} \mathbb{R}_{12}(\mathbf{u})$ solves the YBE (9).*

The following permutation of parameters in the product of three L-operators

$$L_1(u_1, u_2) \sigma_3 L_2(v_1, v_2) \sigma_3 L_3(w_1, w_2) \rightarrow L_1(w_1, w_2) \sigma_3 L_2(v_1, v_2) \sigma_3 L_3(u_1, u_2)$$

can be realized in two different ways indicating that

$$\begin{aligned} & R_{12}(v_1, v_2|w_1, w_2) R_{23}(u_1, u_2|w_1, w_2) R_{12}(u_1, u_2|v_1, v_2) \\ &= R_{23}(u_1, u_2|v_1, v_2) R_{12}(u_1, u_2|w_1, w_2) R_{23}(v_1, v_2|w_1, w_2). \end{aligned} \quad (16)$$

This relation is proved directly with the help of cubic Coxeter relations for S_j -operators alone, i.e. it is just a word identity in the group algebra of the braid group \mathfrak{B}_4 (i.e., $S_j^2 = \mathbb{1}$ relations are not used). Multiplying relation (16) by the appropriate permutation operators \mathbb{P}_{ij} one comes to the original YBE (9).

As to the explicit construction of needed operators S_j , a miracle takes place — after demanding that $S_{1,2}$ are integral operators one comes to the elliptic Fourier transformation described above!

Denote $p = e^{2\pi i\tau}$ and $q = e^{4\pi i\eta}$. Then, for $|p|, |q| < 1$ and a special choice of periodic factors emerging from solutions of finite difference equations one obtains

$$\begin{aligned} [S_2\Phi](z_1, z_2) &= D(e^{2\pi i(v_1-u_2)}; e^{2\pi iz_1}, e^{2\pi iz_2})\Phi(z_1, z_2), \\ [S_1\Phi](z_1, z_2) &= e^{-\pi iz_1^2/\eta} M(e^{2\pi i(u_2-u_1)})_{e^{2\pi iz_1}, e^{2\pi iz_2}} e^{\pi iz^2/\eta} \Phi(z, z_2), \end{aligned}$$

where $D(t; x, y)$ and $M(t)_{xy}$ are the elliptic integral Bailey lemma entries. Notice the reduced form of the parameter dependence $S_2(\mathbf{u}) = S_2(u_2 - v_1)$ and $S_1(\mathbf{u}) = S_1(u_1 - u_2)$. The operator $S_3(\mathbf{u})$ has the same form as $S_1(\mathbf{u})$ with z_1 replaced by z_2 and $u_1 - u_2$ by $v_1 - v_2$. In this picture the Coxeter relations coincide with the identities for D and M -operators. In particular, the cubic Coxeter relation $S_1 S_2 S_1 = S_2 S_1 S_2$ is guaranteed by the elliptic beta integral since it has the explicit form

$$S_1(a) S_2(a+b) S_1(b) = S_2(b) S_1(a+b) S_2(a),$$

coinciding with equality (8). The relation $S_1^2 = \mathbb{1}$ explicitly looks as $S_1(-a)S_1(a) = \mathbb{1}$, which is the inversion relation for the elliptic Fourier transformation. Thus,

elliptic beta integral = star-triangle relation = cubic Coxeter relation.

After the similarity transformation removing the exponentials $e^{\pi iz^2/\eta}$, which is equivalent to passing to the Sklyanin algebra generators $\mathbf{S}^a \rightarrow e^{\pi iz^2/\eta} \mathbf{S}^a e^{-\pi iz^2/\eta}$, one obtains the general R-operator as a double integral operator with an elliptic hypergeometric kernel of the form

$$\begin{aligned} [R_{12}(\mathbf{u})f](x_1, x_2) &= \frac{(p; p)_\infty^2 (q; q)_\infty^2}{(4\pi i)^2} \Gamma(\sqrt{pq} x_1^{\pm 1} x_2^{\pm 1} e^{2\pi i(v_2-u_1)}; p, q) \\ &\times \int_{\mathbb{T}^2} \frac{\Gamma(e^{2\pi i(v_1-u_1)} x_2^{\pm 1} x^{\pm 1}, e^{2\pi i(v_2-u_2)} x_1^{\pm 1} y^{\pm 1}; p, q)}{\Gamma(e^{4\pi i(v_1-u_1)}, e^{4\pi i(v_2-u_2)}, x^{\pm 2}, y^{\pm 2}; p, q)} \\ &\times \Gamma(\sqrt{pq} e^{2\pi i(v_1-u_2)} x^{\pm 1} y^{\pm 1}; p, q) f(x, y) \frac{dx dy}{x y}, \end{aligned} \quad (17)$$

where one should impose some mild constraints on the parameter values in order to satisfy the intertwining relations with the taken integration contour \mathbb{T} [16].

8. THE ELLIPTIC MODULAR DOUBLE

The derived R-operator (17) is symmetric in p and q . Hence there exists a second RLL-relation obtained from the first one (13) by permutation of p and q :

$$\mathbb{R}_{12}(u-v) \sigma_3 L'_1(u) \sigma_3 L'_2(v) = \sigma_3 L'_2(v) \sigma_3 L'_1(u) \mathbb{R}_{12}(u-v),$$

where $L'(u) = L(\text{fixed } u, \text{ fixed } g = \eta(2\ell + 1), 2\eta \leftrightarrow \tau)$. This means that there is a second copy of the Sklyanin algebra generated by the operators

$$\begin{aligned} \tilde{\mathbf{S}}^a = & e^{\frac{2\pi i}{\tau} z^2} \frac{i^{\delta_{a,2}} \theta_{a+1}(\frac{\tau}{2}|2\eta)}{\theta_1(2z|2\eta)} \left[\theta_{a+1} \left(2z - g + \frac{\tau}{2}|2\eta \right) e^{\frac{1}{2}\tau\partial_z} \right. \\ & \left. - \theta_{a+1} \left(-2z - g + \frac{\tau}{2}|2\eta \right) e^{-\frac{1}{2}\tau\partial_z} \right] e^{-\frac{2\pi i}{\tau} z^2}, \quad e^{\alpha\partial_z} f(z) = f(z + \alpha). \end{aligned}$$

The direct product of two such Sklyanin algebras was introduced by the author in [46] under the name ‘‘elliptic modular double’’. It represents an elliptic generalization of the Faddeev modular double for the $sl_q(2)$ quantum algebra [20]. Vice versa, demanding existence of the elliptic modular double together with the meromorphy of functions in the variable $e^{2\pi iz}$ removes periodic factors in solutions of difference equations and determines the operators S_j uniquely.

9. THE SUPERCONFORMAL INDEX

The first physical interpretation of elliptic hypergeometric integrals has been found by the author in the context of Calogero-Sutherland type models [45]. The most remarkable application of such integrals in physics has been discovered by Dolan and Osborn [19].

The superconformal index [27, 34] is a topological index of four-dimensional ($4d$) supersymmetric gauge field theories with local gauge invariance group G and global flavor symmetry group F and some set of fields described by irreducible representations of these groups $R_{G,j}$ and $R_{F,j}$. A heuristic derivation of this object resulted in the following matrix integral

$$I(y; p, q) = \int_G d\mu(z) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \text{ind}(p^n, q^n, z^n, y^n) \right) \quad (18)$$

with Haar measure $d\mu(z)$ and

$$\begin{aligned} \text{ind}(p, q, z, y) = & \frac{2pq - p - q}{(1-p)(1-q)} \chi_{\text{adj}_G}(z) \\ & + \sum_j \frac{(pq)^{R_j/2} \chi_{r_{F,j}}(y) \chi_{r_{G,j}}(z) - (pq)^{1-R_j/2} \chi_{\bar{r}_{F,j}}(y) \chi_{\bar{r}_{G,j}}(z)}{(1-p)(1-q)}. \end{aligned} \quad (19)$$

Here $\chi_{R_F,j}(y)$ and $\chi_{R_G,j}(z)$ are characters of the respective representations, and R_j are some fractional numbers (R-charges of the fields). For instance, for the unitary group $SU(N)$, $z = (z_1, \dots, z_N)$, $\prod_{a=1}^N z_a = 1$, one has

$$\int_{SU(N)} d\mu(z) = \frac{1}{N!} \int_{\mathbb{T}^{N-1}} \Delta(z) \Delta(z^{-1}) \prod_{a=1}^{N-1} \frac{dz_a}{2\pi i z_a},$$

where $\Delta(z) = \prod_{1 \leq a < b \leq N} (z_a - z_b)$ is the Vandermonde determinant.

Where is the elliptic beta integral here? Let us take $G = SU(2)$, $F = SU(6)$, and the representations (“adj”=adjoint, “f”=fundamental)

$$\begin{aligned} 1) \text{ “vector superfield”}: & (adj, 1), \quad \chi_{SU(2),adj}(z) = z^2 + z^{-2} + 1, \\ 2) \text{ “chiral superfield”}: & (f, f), \quad \chi_{SU(2),f}(z) = z + z^{-1}, \quad R_f = 1/3, \\ \chi_{SU(6),f}(y) = \sum_{k=1}^6 y_k, & \quad \chi_{SU(6),\bar{f}}(y) = \sum_{k=1}^6 y_k^{-1}, \quad \prod_{k=1}^6 y_k = 1, \end{aligned}$$

Then after denoting $t_k = (pq)^{1/6} y_k$, $k = 1, \dots, 6$, the superconformal index formula reproduces identically the left-hand side of the elliptic beta integral (2) [19].

In order to generate the right-hand side expression in (2) one should take $G = 1$, $F = SU(6)$ with the single “chiral superfield” described by antisymmetric tensor of the second rank T_A : $\Phi_{ij} = -\Phi_{ji}$,

$$\chi_{SU(6),T_A}(y) = \sum_{1 \leq i < j \leq 6} y_i y_j, \quad R_{T_A} = 2/3.$$

So, the elliptic beta integral evaluation formula shows that two functions on characters (18) for different sets of representations of two different $G \times F$ -groups coincide. This equality of superconformal indices describes the confinement phenomenon in the simplest supersymmetric quantum chromodynamics model or the Seiberg duality [37]. An almost complete list of such dualities for simple gauge groups and corresponding elliptic hypergeometric integral identities (about half of which are conjectures) emerging as equalities of dual indices is given in [50, 51]. In general, elliptic hypergeometric integrals define new matrix models and describe the most complicated known class of computable nonperturbative path integrals in four-dimensional quantum field theories.

10. A 4d/2d CORRESPONDENCE

Solutions of the YBE are related to integrable spin chain models and 2d Ising type spin systems [21]. Let us replace in the Coxeter/Bailey relation (8) $u, x, z \rightarrow e^{iu}, e^{ix}, e^{iz}$ with real u, x, z and act by it on a localized “continuous spin” state function $(\delta(z - y) + \delta(z + y))/2$, where $\delta(z)$ is the Dirac delta function. This yields the elliptic beta integral rewritten in the form

$$\int_0^{2\pi} \rho(u) D_{\xi-\alpha}(x, u) D_{\alpha+\gamma}(y, u) D_{\xi-\gamma}(w, u) du = \chi D_{\alpha}(y, w) D_{\xi-\alpha-\gamma}(x, w) D_{\gamma}(x, y),$$

where

$$D_\alpha(x, y) = D(e^{-\alpha}; e^{ix}, e^{iy}), \quad \rho(u) = \frac{(p; p)_\infty (q; q)_\infty}{4\pi} \theta(e^{2iu}; p) \theta(e^{-2iu}; q),$$

$$\chi = \Gamma(e^{-2\alpha}, e^{-2\gamma}, e^{2\alpha+2\gamma-2\xi}; p, q), \quad e^{-\xi} = \sqrt{pq},$$

which coincides with the functional star-triangle relation considered by Bazhanov and Sergeev [10].

Now one can interpret the figure given earlier as a transformation of the elementary cell partition function: circles carry spins u, w, \dots with the self-energy $\rho(u)$, edges carry Boltzmann weights D_α , the black circle contains the integration (summation) over u -spin values. Using such a relation one can map the honeycomb, triangular, and square lattice spin models onto each other and their partition functions become equal to some (multiple) elliptic hypergeometric integrals.

The same figure can be interpreted in the context of quiver gauge theories: black circles denote the gauge group G (in superconformal indices this corresponds to the contribution of vector superfields in the adjoint representation and integration over the gauge group), white circles denote some external flavor groups, the edges denote the bifundamental fields (their contributions to superconformal indices yield the Boltzmann weights D_α), etc. The presence of more than one black circle would indicate that the gauge group is not simple. Evidently one can construct in this way lattice-type $4d$ quiver field theories so that their superconformal indices coincide with partition functions of various statistical mechanics models with the continuous spin values [48] since both are described by the same elliptic hypergeometric integrals.

As shown in [48], one can go further and interpret the symmetry transformations for elliptic hypergeometric integrals on root systems as the star-star relations [8]. Conjecturally, to each such non-trivial relation one can associate an elliptic integrable system. As a consequence, the Seiberg duality of $4d$ field theories becomes related to the Kramers-Wannier type duality transformations for partition functions of $2d$ spin systems.

11. CONCLUSION

We can conclude that the elliptic hypergeometric functions are universal objects with wide applications. This brief survey does not cover all their known instances and, in particular, the list of given references is incomplete (for its extensions, see [47, 50, 51]). We finish by indicating various fields where elliptic hypergeometric functions have found their applications and more is expected to lie ahead.

In mathematics: theory of analytic finite-difference equations (e.g., the elliptic hypergeometric equation), harmonic analysis on root systems, representation theory, theory of $SL(3, \mathbb{Z})$ automorphic forms, approximation theory, continued fractions, combinatorics, topology, etc.

In theoretical and mathematical physics: $4d$ supersymmetric dualities, integrable N -particle quantum mechanical systems, $2d$ topological field theories [24], $2d$ solvable models of statistical mechanics and noncompact spin chains, random matrices and determinantal point processes [11], etc.

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