# Effective Theory of Stochastic Hydrodynamics and Multiplicative Noise

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13th APCTP - BLTP JINR Joint Workshop@JINR, Dubna


Confucius standing by a stream, said, “Time flows on, never creasing, night and day!”

# Outline

- Motivations
  - look at high-temperature QCD matter in heavy-ion collisions
- Baby examples of the hydro fluctuations
- Standard path integral representation of stochastic hydro
- Naive Einstein relation
- Working in multiplicative noise

# Fluid Dynamics

Fluid dynamics is the universal effective description of non-equilibrium many body systems, including relativistic QFT, with a stable equation of state and

- Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu} = 0$$

- Conservation of charge:

$$\partial_\mu J^\mu = 0$$

Local conservation of charge and energy-momentum $\Longrightarrow$ Hydrodynamics

$\Longrightarrow$ Effective theory of non-equilibrium; long-wavelength; low-frequency behaviors of many-body system.

# Interaction of Hydro Modes

Transport coefficients are shown in linearized hydro.

However,

- In hydro, there are no arbitrary coupling constants like $g$
- Interaction of modes will change hydro correlation functions
- Was known since late 1960's mode-mode coupling

# Long Time Tails

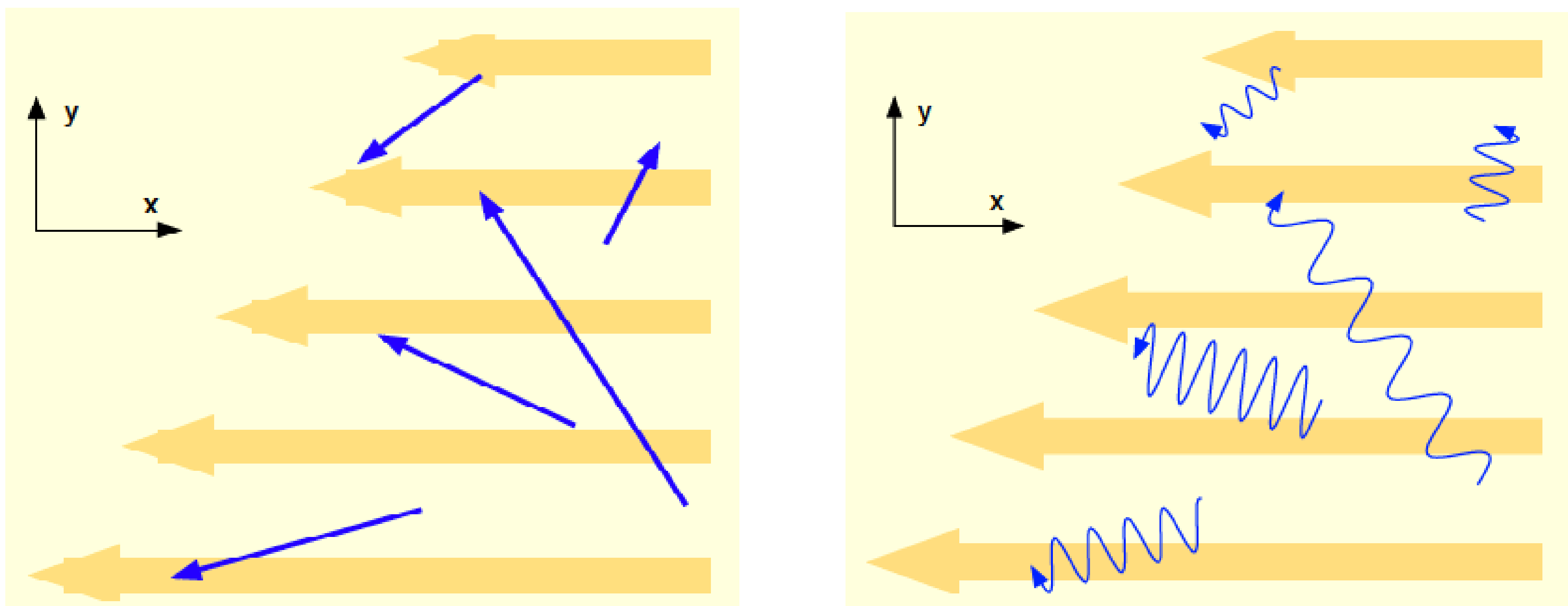
For $\mathbf{J} = -\chi_D \nabla n + n\mathbf{v}$ at $k \to 0$

$$\begin{aligned} \langle \mathbf{J}(x,t)\mathbf{J}(0,0)\rangle &= \int d^d x \, \langle n(x,t) n(0,0) \mathbf{v}(x,t) \mathbf{v}(0,0)\rangle \\ &= \int d^d x \, \langle n(x,t) n(0,0)\rangle \, \langle \mathbf{v}(x,t) \mathbf{v}(0,0)\rangle \\ &\sim \int d^d k \; e^{-\chi_D \mathbf{k}^2 t} \, e^{-\gamma_\eta \mathbf{k}^2 t} \\ &\sim t^{-\frac{d}{2}} \end{aligned}$$

hydrodynamics fails at long times

☞ Fluctuation is important!

# Effective Hydro Theory




To figure out this low-energy effective hydro theory, need both dissipation (transport coefficients) and fluctuations (thermally collective modes)

# Stochastic Hydrodynamics

$$
\begin{aligned}
T^{\mu\nu} &= \varepsilon\, u^{\mu} u^{\nu} + p\Delta^{\mu\nu} + \pi^{\mu\nu} + \xi^{\mu\nu} \\
J^{\mu} &= n\, u^{\mu} + \nu^{\mu} + \xi^{\mu}
\end{aligned}
$$

The dissipation terms are described by

$$
\begin{aligned}
\pi^{ij} &= -\eta \left( \partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij} \nabla \cdot \mathbf{u} \right) - \zeta \delta^{ij} \nabla \cdot \mathbf{u} \\
\nu^{\mu} &= -\sigma T \Delta^{\mu\nu} \partial_{\nu} \left( \frac{\mu}{T} \right) \qquad \Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}
\end{aligned}
$$

for shear viscosity $\eta$, bulk viscosity $\zeta$ and charge conductivity $\sigma$.

The induced noises:

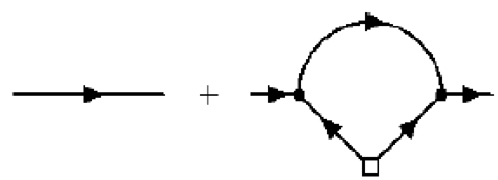
$$
\langle \xi^{\mu\nu}(x,t) \rangle = \langle \xi^{\mu}(x,t) \rangle = 0
$$

# Contribution from Hydro Loops

Navier-Stokes equation: $\partial_0 \vec{v} + \nu \nabla^2 \vec{v}$ =mode coupling+noise

Linearized propagator: $\left\langle \delta v_i^T \delta v_j^T \right\rangle_{\omega,k} = \frac{1}{\rho} \frac{\nu k^2}{i\omega - \nu k^2} P_{ij}^T \quad \nu = \frac{\eta}{\rho} \quad P_{ij}^T = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}$

Fluctuation correction:



Small $\eta$ enhance fluctuation corrections: $\eta(\omega) = \eta_0 + c_\eta \frac{T\rho\Lambda}{\eta_0} - c_\tau \sqrt{\omega} \frac{T\rho^{3/2}}{\eta_0^{3/2}}$

A bound on $\eta$ is a must; 2nd order hydro without fluctuations is inconsistent

Rather, viscosity, conductivity, etc, become scale-dependent "running masses" in the low-energy effective hydro theory

# Stochastic Hydro with Additional Noises

Firstly, coarse-grained treating for fast modes $\longrightarrow$ random noise $\xi(x,t)$

Secondly, mesoscopic equation of motion in Langevin way for slow variables $\longrightarrow$ $\psi(x,t)$

Obtaining stochastic hydro equations in terms of $\psi = (\delta n, \delta\varepsilon, \delta\pi_k)$:

$$\frac{\partial \delta n(x,t)}{\partial t} = \bar{w}\nabla^i \delta\pi_i(x,t) + \frac{\delta\pi^i \partial_i n}{\bar{h}} + \chi_D(x,t)\nabla^2 \delta n(x,t) + \sqrt{g_n}\cdot\nabla\xi_n(x,t) + \sqrt{g_\varepsilon}\cdot\nabla\xi_\varepsilon(x,t)$$

$$\frac{\partial \delta\varepsilon(x,t)}{\partial t} = \nabla^i \delta\pi_i(x,t) + \frac{\delta\pi^i \partial_i \varepsilon}{\bar{h}} + \sqrt{g_\varepsilon}\cdot\nabla\xi_\varepsilon(x,t)$$

$$\frac{\partial \delta\pi_k(x,t)}{\partial t} = \partial_k p + \frac{\pi^i \partial_i \pi_k}{\bar{h}} + \gamma_\eta(x,t)\left(\nabla^2\delta_{kj} + \frac{1}{3}\partial_k\partial_j\right)\delta\pi^j(x,t) + \sqrt{g_k}\cdot\nabla\xi_k(x,t)$$

$$\bar{h} = \varepsilon + p \quad \bar{w} = \frac{\bar{n}}{\bar{h}} \quad \chi_D \propto \sigma \quad \gamma_\eta \propto \eta$$

# MSRJD field theory representation for Langevin equations

Coarse-grained effective Langevin description with auxiliary field $\tilde{\psi}$:

$$\begin{aligned}
\langle \mathcal{O}[\psi] \rangle &= \int \mathcal{D}\psi \, P[\xi] \mathcal{O}[\psi] \delta \left( \dot{\psi} - V[\psi] - F\nabla^2[\psi] - \mathcal{G}^{\frac{1}{2}} \cdot \nabla\xi \right) \\
&\quad \cdot \det \left( \frac{\delta \left( \dot{\psi} - V[\psi] - F\nabla^2[\psi] - \mathcal{G}^{\frac{1}{2}} \cdot \nabla\xi \right)}{\delta\psi} \right) \\
&= \mathcal{N} \int \mathcal{D}\psi\mathcal{D}\tilde{\psi} \, P[\xi] \mathcal{O}[\psi] \exp \left\{ -\tilde{\psi} \left( \dot{\psi} - V[\psi] - F\nabla^2[\psi] - \mathcal{G}^{\frac{1}{2}} \cdot \nabla\xi \right) \right\} + \text{det term} \\
&= \mathcal{N}' \int \mathcal{D}\psi\mathcal{D}\tilde{\psi} \, \mathcal{O}[\psi] \exp \underbrace{\left\{ -\tilde{\psi} \left( \dot{\psi} - V[\psi] - F\nabla^2[\psi] \right) - \tilde{\psi}\mathcal{G}\nabla^2\tilde{\psi} \right\}}_{\mathcal{S}[\psi,\tilde{\psi}], \text{ desired effective action}} + \text{det term}
\end{aligned}$$

We get an effective action at low energies; in real time and near thermal equilibrium

# Time Reversal and Action Symmetry

$$\begin{aligned}
\mathcal{T} : \mathcal{S}\left[\psi, \tilde{\psi}\right] + h.c. &= -\tilde{\psi}(-t)\left(-\dot{\psi} + V[\psi] - F\nabla^2[\psi]\right) - \tilde{\psi}(-t)\mathcal{G}\nabla^2\tilde{\psi}(-t) \\
&= -\left(\mathcal{T}\tilde{\psi}\right)\left(-\dot{\psi} + V[\psi] - F\nabla^2[\psi]\right) - \left(\mathcal{T}\tilde{\psi}\right)\mathcal{G}\nabla^2\left(\mathcal{T}\tilde{\psi}\right) \\
&= -\tilde{\psi}(-t)\left(\dot{\psi}_{-t} - V[\psi](-t) - F\nabla^2[\psi](-t)\right) - \tilde{\psi}(-t)\mathcal{G}\nabla^2\tilde{\psi}(-t) \\
&\quad +F[\psi](-t)\mathcal{G}^{-1}\left(\dot{\psi}_{-t} - V[\psi](-t)\right) \\
&= \mathcal{S}\left[\psi, \tilde{\psi}\right](-t) - F[\psi]\mathcal{G}^{-1}\left(\dot{\psi} - V[\psi]\right) + h.c.
\end{aligned}$$

where $\mathcal{T}$ is the time reversal operation $t \to T - t$:

$$\mathcal{T} :\ \psi(T-t) = \psi(t) \qquad \mathcal{T} :\ \tilde{\psi}(T-t) = \tilde{\psi}(T-t) + \left(\mathcal{G}\nabla^2\right)^{-1}\left(\dot{\psi} - V[\psi]\right)$$

If requires:

$$F^{\alpha}[\psi]\mathcal{G}^{-1}_{\alpha\beta} + h.c. = \beta\frac{\partial H}{\partial \psi_\beta}$$

# Under the Help of Detailed Balance Statement

$$\Longrightarrow F^{\alpha}[\psi]\mathcal{G}^{-1}_{\alpha\beta}V^{\beta}[\psi] + h.c. = \beta Q^{\beta\gamma}\frac{\partial H}{\partial \psi_\beta}\frac{\partial H}{\partial \psi_\gamma} = 0$$

where $V^\beta = Q^{\beta\gamma}\partial H/\partial\psi_\gamma$ and $Q$ is the asymmetric matrix of Poisson bracket as definition. It is exactly know as the reversible stationary probability current being divergence-free.

The well-known stationary distribution $P_{st} \propto e^{-\beta H(x)}$: $\Downarrow$

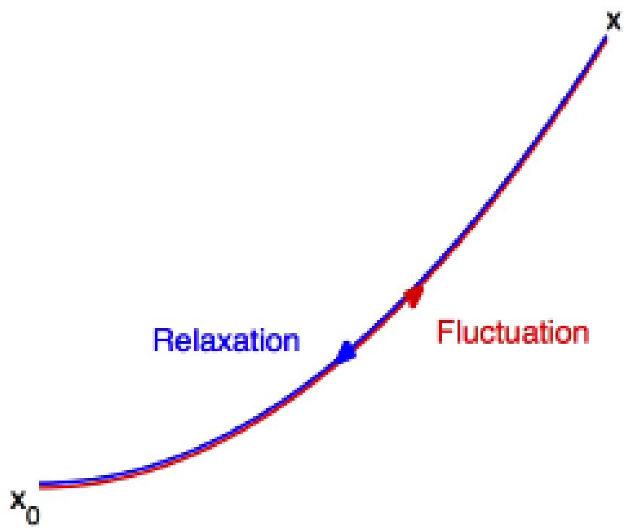
$$\int_0^T F_\alpha[\psi]\mathcal{G}^{-1}\dot{\psi}^\alpha\, d\tau + h.c. = -\ln P_{st}(T) + \ln P_{st}(0)$$

$$\Rightarrow \underbrace{e^{-S(x_T,T;x_0,0)}}_{w(x_0\to x_T)} \cdot P_{st}(x_0) = \underbrace{e^{-S(x_0,T;x_T,0)}}_{w(x_T\to x_0)} \cdot P_{st}(x_T)$$

For a equilibrium distribution, the number of transitions per time from state $x_0$ into state $x_T$ balance the number of transitions per time from state $x_T$ to $x_0$

☞ The fluctuation paths of the direct dynamics are the reversed of the relaxation paths in the language of dual dynamics, and vice versa.

# Time Reversed Relaxation Paths Minimize the Effective Action




The minimizer of the action from an attractor of the system to any point of its basin of attraction is the reversed of the relaxation path.

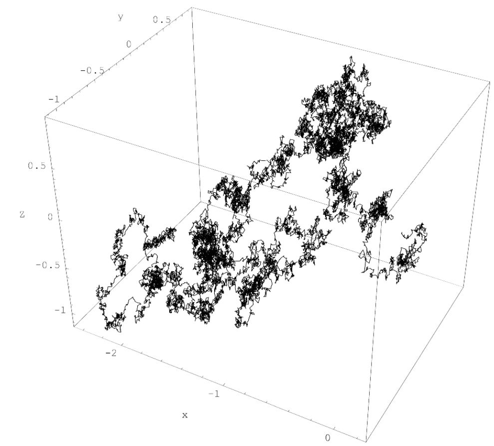
This is an extended Onsager-Machlup relation. For time reversible systems, the most probable way to get a fluctuation is through the reversal of the relaxation path from this fluctuation.

The time reversed relaxation paths also minimizes the action like the fluctuations. The full action is completed in a conjugated dynamics way

# Recall of Einstein Relation

Take the Brownian motion $\dot{\vec{v}} + \gamma \vec{v} = \vec{\xi}(t)$ giving

$$\langle v_i(t_1) v_j(t_2) \rangle = \delta_{ij} \frac{D}{2\gamma} e^{-\gamma |t_1 - t_2|}$$



for $t_{1,2} \gg \gamma$, where $\langle \xi_i(t_1) \xi_j(t_2) \rangle = D\delta(t_1 - t_2)$
What determines the noise strength D?

Assume the Brownian particle eventually equilibrates with the fluid at temperature $\left\langle v^2(t \to \infty) \right\rangle = T$ by averaging over $\xi$

The correlation functions satisfy

$$D = 2\gamma T$$

An old fashioned fluctuation-dissipation theorem!

# Hamiltonian of Fluid

The Hamiltonian $\Delta\mathcal{H}$ of grand canonical ensemble of fluid system is related to the pressure via $\ln \Xi = pV/T = -\beta\Delta\mathcal{H}$.

$$p(x,t) = p(n(x,t), \varepsilon(x,t)) = \beta\varepsilon - \frac{\mu}{T}n - s$$

Energy and particle number defined for arbitrary system:

$$\varepsilon = u_\mu T^{\mu\nu} u_\nu \quad \text{and} \quad n = J^\mu u_\mu$$

Apply equilibrium EoS: $p = p_0(\varepsilon, n) \quad \text{and} \quad s = s_0(\varepsilon, n)$

Match the system to an equilibrium system of the same $\varepsilon$ and $n$. The first derivative of $\beta\Delta\mathcal{H}$ vanish since $\delta s = \beta\delta\varepsilon - \frac{\mu}{T}\delta n$, and then

$$-\beta\Delta\mathcal{H} = \Delta s = \frac{1}{2}\frac{\partial^2 s}{\partial n^2}(\delta n)^2 + \frac{\partial^2 s}{\partial n \partial \varepsilon}\delta n \delta\varepsilon + \frac{1}{2}\frac{\partial^2 s}{\partial \varepsilon^2}(\delta\varepsilon)^2$$

# A Rigorous Fluctuation-Dissipation Theorem

It is already known that the most probable path to reach a state x (a fluctuation) is the time reversal of a relaxation path starting from x (dissipation).

The matrix of Hamiltonian is expressed as

$$\mathcal{H} = \begin{pmatrix} \theta_{nn} & \theta_{n\varepsilon} & 0 \\ \theta_{n\varepsilon} & \theta_{\varepsilon\varepsilon} & 0 \\ 0 & 0 & \frac{1}{h}\delta_{ij} \end{pmatrix}$$

It gives $\mathcal{G} = \mathcal{G}^{\mathrm{T}} = \frac{\mathcal{F}^{\mathrm{T}}\mathcal{H}^{-1}+\mathcal{H}^{-1}\mathcal{F}}{2}$, producing off diagonal noise terms, which originate in the coupled thermodynamical relation between $\varepsilon$ and $n$.

This is a generalized Onsager-Machlup relation, that explains quite easily and naturally fluctuation-dissipation relations.

Note that the E.o.S. is important because the mixed term $\propto \frac{\theta_{n\varepsilon}}{\theta_{nn}}$.

# Propagators and Vertexes

Let $\Psi = \left(\psi, \tilde{\psi}\right)$, $\psi = (\delta n, \delta\varepsilon, \delta\pi_x, \delta\pi_\perp)$, $\mathcal{L} = \Psi^T S \Psi$ and

$$S = \begin{pmatrix} 0 & \frac{\partial}{\partial t} - \mathcal{V} - \mathcal{F}\nabla^2 \\ -\frac{\partial}{\partial t} - \mathcal{V}^\dagger - \mathcal{F}^\dagger\nabla^2 & \mathcal{G}\nabla^2 \end{pmatrix}$$

with the inverse of the harmonic coupling matrix

$$S_0^{-1} = \begin{pmatrix} G_{\psi\psi} & G_{\psi\tilde{\psi}} \\ G^\dagger_{\psi\tilde{\psi}} & 0 \end{pmatrix}.$$

$$G_{\psi\psi} = G_{\psi\tilde{\psi}}\mathcal{G}_0\mathbf{k}^2 G^\dagger_{\psi\tilde{\psi}}$$

The left interaction terms are

$$\mathcal{L}_I = \tilde{\psi}V_I[\psi] + \tilde{\psi}F_I\nabla^2[\psi] + \tilde{\psi}\mathcal{G}_I\nabla^2\tilde{\psi}$$

# Not Complex Conjugate Propagators of $\psi\tilde{\psi}$

$$G_{n\tilde{n}}(\omega,\mathbf{k}) = \frac{1}{i\omega - \chi_D \mathbf{k}^2}$$

$$G_{n\tilde{\varepsilon}}(\omega,\mathbf{k}) = \frac{v_a^2 \bar{w} \mathbf{k}^2}{(i\omega - \chi_D \mathbf{k}^2)(\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2)}$$

$$G_{n\tilde{\pi}_x}(\omega,\mathbf{k}) = \frac{\bar{w}\omega k_x}{(i\omega - \chi_D \mathbf{k}^2)(\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2)}$$

$$G_{\varepsilon\tilde{\varepsilon}}(\omega,\mathbf{k}) = \frac{\gamma_s \mathbf{k}^2 - i\omega}{\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2} \qquad G_{\varepsilon\tilde{\pi}_x}(\omega,\mathbf{k}) = \frac{-ik_x}{\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2}$$

$$G_{\pi_x\tilde{\varepsilon}}(\omega,\mathbf{k}) = \frac{i v_a^2 k_x}{\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2} \qquad G_{\pi_x\tilde{\pi}_x}(\omega,\mathbf{k}) = \frac{i\omega}{\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2}$$

$$G_{\pi_i\tilde{\pi}_j}(\omega,\mathbf{k}) = \frac{1}{i\omega - \gamma_\eta \mathbf{k}^2}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)$$

1. Non-listed propagator are zero
2. The transverse momentum tensor propagator is decoupled

# Hermitian Propagators of $\psi\psi$

$$G_{nn}(\omega, \mathbf{k}) = \frac{\chi_D k^4}{\omega^2 + \chi_D^2 k^4} \qquad G_{\pi_i \pi_j}(\omega, \mathbf{k}) = \frac{\gamma_\eta k^4}{\omega^2 + \gamma_\eta^2 k^4}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)$$

$$G_{n\varepsilon}(\omega, \mathbf{k}) = G^\dagger_{\varepsilon n}(\omega, \mathbf{k}) = \frac{\chi_D v_a^2 \bar{w} k^6}{\left(\omega^2 + \chi_D^2 k^4\right)\left(\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2\right)}$$

$$G_{n\pi_x}(\omega, \mathbf{k}) = G^\dagger_{\pi_x n}(\omega, \mathbf{k}) = \frac{\chi_D k^5 \left(\bar{w}\omega + \frac{\theta_{n\varepsilon}}{2\theta_{nn}}\left(\omega + i\chi_D k^2\right)\right)}{\left(\omega^2 + \chi_D^2 k^4\right)\left(\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2\right)}$$

$$G_{\varepsilon\varepsilon}(\omega, \mathbf{k}) = \frac{\chi_D \bar{w}^2 v_a^4 k^8 + \mathcal{O}\left(k^8\right)}{\left(\omega^2 + \chi_D^2 k^4\right)\left(\left(\omega^2 - v_a^2 \mathbf{k}^2\right)^2 + \gamma_s^2 \omega^2 k^4\right)}$$

$$G_{\varepsilon\pi_x}(\omega, \mathbf{k}) = G^\dagger_{\pi_x \varepsilon}(\omega, \mathbf{k}) = \frac{\chi_D v_a^2 \bar{w}\omega k^7 \left(\bar{w}\omega + \frac{\theta_{n\varepsilon}}{2\theta_{nn}}\left(\omega + i\chi_D k^2\right)\right) + \mathcal{O}\left(k^9\right)}{\left(\omega^2 + \chi_D^2 k^4\right)\left(\omega^2 - v_a^2 \mathbf{k}^2 + i\gamma_s \omega \mathbf{k}^2\right)}$$

$$G_{\pi_x \pi_x}(\omega, \mathbf{k}) = \frac{\chi_D \bar{w}\omega^2 k^6 \left(\bar{w} + \frac{\theta_{n\varepsilon}}{\theta_{nn}}\right) + \mathcal{O}\left(k^{10}\right)}{\left(\omega^2 + \chi_D^2 k^4\right)\left(\left(\omega^2 - v_a^2 \mathbf{k}^2\right)^2 + \gamma_s^2 \omega^2 k^4\right)}$$

# Multiplicative Noise

Expanding $\mathcal{F}_I[\psi]$ up to bilinear field of $\psi$ because of the time-space dependent transport coefficients, the effect of multiplicative noise is taking into account:

$$\mathcal{F}_I = \begin{pmatrix} \lambda_1\delta n & \lambda_2\delta n & \lambda_3\delta n & \frac{4}{3}\lambda_3\delta n \\ 0 & 0 & \lambda_4\delta\varepsilon & \frac{4}{3}\lambda_4\delta\varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And one is able to get the corresponding matrix of $\mathcal{G}_I$.

Combing the TRS of detailed balance and causality, the FDT manifests as

$$\left\langle \psi(t_1)\mathcal{G}\nabla^2\tilde{\psi}(t_2)\right\rangle = \Theta(t_2,t_1)\left\langle \psi(t_1)\left(\dot{\psi}(t_2) - V[\psi](t_2)\right)\right\rangle$$

$$\Rightarrow \left\langle \psi^\alpha\dot{\psi}^\beta\right\rangle = \left\langle \psi^\alpha\mathcal{G}^{\beta\gamma}\nabla^2\tilde{\psi}_\gamma\right\rangle$$

since $\langle V[\psi]\psi\rangle = 0$. The well known result of FDT is recovered for leading order of $\mathcal{G}_0$, .

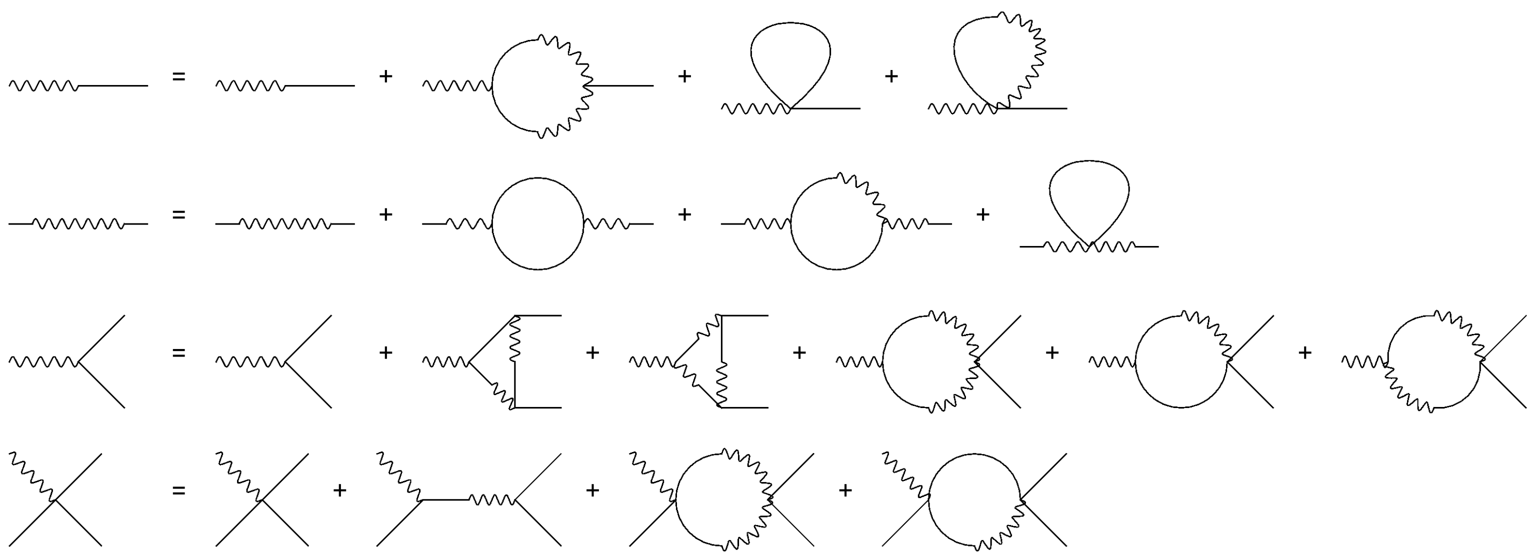
Therefore, we are going to extract the value of four unknown coefficients of $\lambda_i$ through four equations of FDT. Note: a full consideration is 6+1 coefficients.

# Feynman Diagram

Including both the vertex term of $F_I$ and $V_I$:

$$\mathcal{V}_I = \frac{1}{\hbar}\begin{pmatrix} \pi_i\partial^i & 0 & 0 \\ 0 & \pi_i\partial^i & 0 \\ 0 & 0 & \pi_i\partial^i \end{pmatrix}$$

,



# Summary and Conclusions

- Applying the lattice equation of state
- Evaluating in the two particle irreducible (2PI) effective action method
- Extending to Magnetohydrodynamics with Maxwell equation
- Investigating the critical behaviors to the second order hydro fluctuations
- Considering the gradient terms $n\nabla^2 n$ in the free energy to describe the effect of phase interface in the liquid-gas phase transition

Thank You for Your Attention!