Chapter 4. Power series method

4.1 Cubic nonlinearity

A widely used approach to analyze nonlinear problems is perturbation theory, which replaces the original nonlinear problem with the resembling problem, but with known solution and small perturbation. Representing the difference between the problems by small parameter ε and using unperturbed solution, we try to find the perturbed solution in the form of ε power series.

We start from the simplest case — the one-dimensional horizontal motion described by equation (4.1.1)

$$x'' + K_x(s)x = -\frac{1}{6}n(s)x^3. \tag{4.1.1}$$

We left only cubic nonlinearity on the right-hand side, because it is easier to study than the leading quadratic coefficient. Also, we assume azimuthally symmetric case when $K_x(s) = const$ and $n(s) = const$. Floquet transformation helps to change nonlinear Hill equation to equation of mathematical pendulum with perturbation

$$\frac{d^2\zeta}{d\psi^2} + \nu_o^2\zeta = -\frac{1}{6}\nu_0^2 n\beta^3\zeta^3. \tag{4.1.2}$$

Introduction of the new notation $x = \zeta, t = \psi, \omega = \nu$ and $\alpha = -\frac{1}{6}n\beta^3$ replaces equation (4.1.2) with more general

$$x'' + \omega_0^2 x = \varepsilon\, \alpha\omega_0^2 x^3. \tag{4.1.3}$$

Similar equations appear in the study of the synchrotron motion in an accelerator or in the motion of a nonlinear pendulum. On the right-hand side of (4.1.3) we introduced dimensionless parameter ε, to explicitly isolate the cubic term as a perturbation and for the ease of calculations. In the obtained solution we will assign $\varepsilon = 1$. Note that the actual small (implicit) parameter of the problem is betatron oscillations amplitude. We are looking for the solution of (4.1.3) expanded in power series in ε

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{4.1.4}$$

Substituting this series in (4.1.3) and gathering coefficients of the same powers in ε yields harmonic oscillator equation in the zeroth order, and the solution is

$$x_0 = A\cos\omega_0 t, \tag{4.1.5}$$

where initial conditions are $x_0(0) = A, x_0'(0) = 0$. The first order in ε gives equation

$$x_1'' + \omega_0^2 x_1 = \alpha\omega_0^2 x_0^3 = \alpha\omega_0^2 A^3\cos^3(\omega_0 t), \tag{4.1.6}$$

and $\cos^3(\omega_0 t)$ expands as the sum of two terms

$$\cos^3(\omega_0 t) = \frac{1}{4}(\cos(3\omega_0 t) + 3\cos(\omega_0 t)). \tag{4.1.7}$$

General solution of the inhomogeneous differential equation is a sum of homogeneous equation fundamental solution and particular solution of the inhomogeneous equation. Equation

$$y'' + y = a\cos\big(p(t+\alpha)\big) \tag{4.1.8}$$

has two types of particular solutions

$$\begin{aligned} y &= \frac{1}{2} a\, t\, sin(t+\alpha), \quad with\, p = 1, \\ y &= \frac{a}{1-p^2} cos\big(p(t+\alpha)\big), with\, p \neq 1. \end{aligned} \tag{4.1.9}$$

Applying (4.1.9) to equation (4.1.6) with the right-hand side (4.1.7) yields particular solution for the first term

$$x_{11} = -\frac{1}{32}\alpha A^3\cos(3\omega_0 t) \tag{4.1.10}$$

and for the second

$$x_{12} = \frac{3}{8}\alpha A^3 \omega_0 t \sin(\omega_0 t). \tag{4.1.11}$$

The second solution grows linearly in time regardless of the initial amplitude A, whereas intuitive assumption (confirmed by experience) is that motion should be close to linear and bounded for small amplitudes (which are the true smallness parameters).

Solution (4.1.11) is called secular, and it appeared from incorrect choice of approximate solution (4.1.4). Correct approach developed by Lindstedt (Poincaré-Lindstedt method) consists of simultaneous expansion of deviation x and frequency ω. Assuming $x = x(u)$ is a periodic function of $u = \omega t$ with a period of 2π we expand x and ω in powers of ε, so that along with (4.1.4) we have

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \tag{4.1.12}$$

Substituting (4.1.4) and (4.1.12) in (4.1.3) yields

$$\begin{gathered}(\omega_0^2 + 2\varepsilon\omega_0\omega_1 + \cdots)(x_0'' + \varepsilon x_1'' + \cdots) + \omega_0^2(x_0 + \varepsilon x_1 + \cdots) = \\ = \alpha\varepsilon\omega_0^2(x_0 + \varepsilon x_1 + \cdots)^3\end{gathered} \tag{4.1.13}$$

where prime is differentiation with respect to u. The zeroth order equation is

$$x_0'' + x_0 = 0, \tag{4.1.14}$$

with solution $x_0 = A\cos(u) = A\cos(\omega t)$. First order equation is

$$\omega_0^2 x_1'' + 2\omega_0\omega_1 x_0'' + \omega_0^2 x_1 = \alpha\omega_0^2 x_0^3, \tag{4.1.15}$$

or

$$x_1'' + x_1 = -2\frac{\omega_1}{\omega_0}x_0'' + \alpha x_0^3. \tag{4.1.16}$$

Substituting zeroth order solution $x_0 = A\cos(\omega t)$ into (4.1.16) gives

$$x_1'' + x_1 = 2\frac{\omega_1}{\omega_0} A\cos(\omega t) + \frac{\alpha}{4} A^3(\cos(3\omega t) + 3\cos(\omega t)). \qquad (4.1.16)$$

Periodicity of $x = x(\omega t)$ requires cancellation of $\cos(\omega t)$ terms, otherwise, solution will have secular terms. Therefore, we have to choose

$$\omega_1 = -\frac{3}{8}\alpha\omega_0 A^2, \qquad (4.1.17)$$

and solution of equation (4.1.16) is

$$x_1 = \left(C_1 - \frac{\alpha}{16}A^3\right)\cos\omega t + C_2\sin\omega t - \frac{\alpha}{32}A^3\cos 3\omega t. \qquad (4.1.18)$$

With initial conditions $x_1(0) = 0, x_1'(0) = 0$ (we already used initial oscillation amplitude in the zeroth order solution) the first order solution is

$$x_1 = \frac{\alpha}{32}A^3(\cos\omega t - \cos 3\omega t), \qquad (4.1.19)$$

and solution of (4.1.3) up to the first order is

$$x = A\cos\omega t + \frac{\alpha}{32}A^3(\cos\omega t - \cos 3\omega t). \qquad (4.1.20)$$

Solution of simple problem (4.1.3) revealed several key differences between linear and nonlinear motions. Nonlinear perturbation is the reason that oscillation frequency is not the same for different particles but depends on theirs oscillation amplitude squared (4.1.17). Such systems are called non-isochronous. In addition, solution includes several harmonics of the fundamental frequency (first and third in our case); therefore, it is anharmonic.

Now we will discuss anharmonicity influence on the phase trajectories. Differentiating (4.1.20) with respect to $u = \omega t$ yields

$$x' = -A\sin\omega t - \frac{\alpha}{32}A^3(\sin\omega t - 3\sin\omega t). \qquad (4.1.21)$$

Phase trajectories of the linear motion are circles, with radius defined by initial amplitude

$$x^2 + x'^2 = A^2. \tag{4.1.22}$$

In our case, adding the squares of (4.1.20) and (4.1.21) results

$$\begin{aligned} x^2 + x'^2 = A^2 & \\ &+\frac{\alpha}{16}A^4\big(1 - 2\cos(2\omega t) + \cos(4\omega t)\big) \\ &+\frac{1}{2}\left(\frac{\alpha}{16}\right)^2 A^6(3 - 2\cos(2\omega t) + \cos(4\omega t) - 2\cos(6\omega t)). \end{aligned} \tag{4.1.23}$$

The phase trajectory of (4.1.23) is a circle with radius $R_0 = A$ distorted by the oscillation frequency harmonics with $\Delta R \sim A^3 + A^5 + \cdots$. The sketch of the phase trajectory is on Fig.4.1.1.

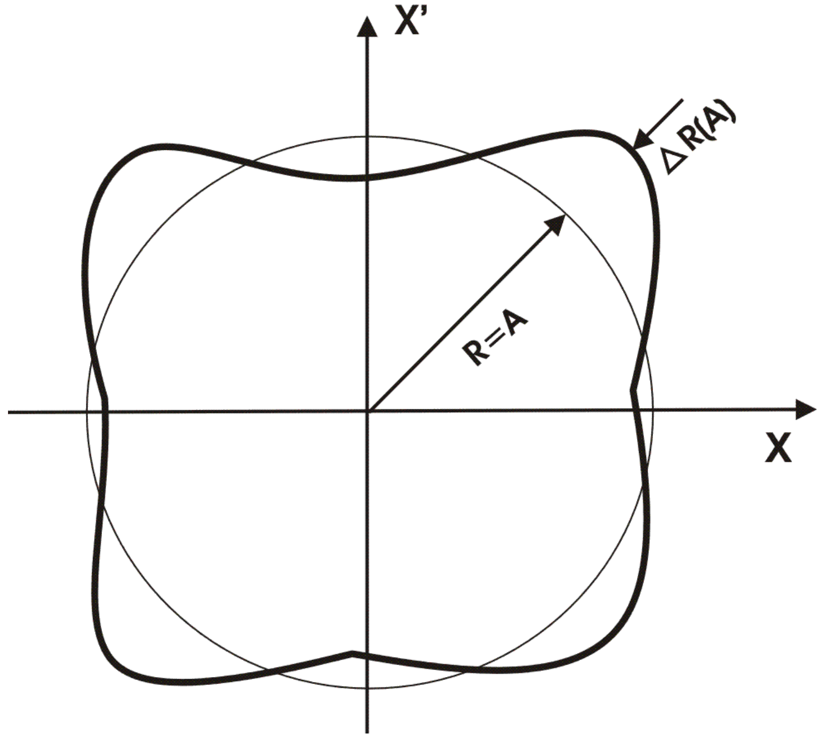

Fig.4.1.1 Nonlinear distortion of phase trajectories

The depth of the phase circle modulation increases with initial amplitude A.

Considering phase trajectories in the form

$$x^2 + x'^2 = \big(R_0 + \Delta R(A)\big)^2, \tag{4.1.24}$$

where $R_0 = A$, and ΔR is a function of initial deviation, we estimate for maximum distortion

$$(R_0 + \Delta R_{max})^2 = A^2 \left(1 + \frac{\alpha}{16} A^2 + \left(\frac{\alpha}{16}\right)^2 A^4 + \cdots\right). \quad (4.1.25)$$

Naturally, we want to demand convergence of the series with addition of solution's higher orders. Using Cauchy's radical test to analyze convergence of the series

$$S = \sum_n a_n = \sum_n \left(\frac{\alpha}{16}\right)^n A^{2n}, \quad (4.1.26)$$

we obtain

$$\sqrt[n]{a_n} = \frac{\alpha}{16} A^2 < 1. \quad (4.1.27)$$

In other words, proposed solution expanded in series exists only for initial oscillation amplitude

$$A < \frac{4}{\sqrt{\alpha}}. \quad (4.1.28)$$

Our derivation was rough; therefore, obtained result is only an illustration of the nonlinear dynamics important property. Contrarily to unperturbed equation of motion, where oscillation amplitude is limited only by external geometrical factors (vacuum chamber walls), solution of perturbed equation of motion has a limitation of the oscillation amplitude. Motion with the greater amplitude is unstable. The maximum allowable initial amplitude (4.1.28) defines an area called dynamic aperture. Also, from (4.1.28) we conclude that with larger perturbation α dynamic aperture is smaller.

At the end of the discussion we switch back to original variables $A = A_x/\sqrt{\beta}$, $\omega = \nu$ and $\alpha = -n\,\beta^3/6$, and obtain betatron oscillation frequency correction from the cubic perturbation in the case of an azimuthally symmetric magnetic field:

$$\Delta\nu = \frac{1}{16} n\, A_x^2 \beta_x^2 \nu_0. \tag{4.1.29}$$

4.2 Quadratic nonlinearity

The power series method allows searching of the higher orders solutions. However, the search procedure becomes more and more complicated with each step: the number of terms in the power series and the number of harmonics increase. Hence, the fair question arises: what order of the expansion is sufficient, so that deriving it was technically possible, and obtained solution plausibly described behavior of the system.

The question is not simple, and we will return to it repeatedly. To illustrate complexity of the problem, we consider a quadratic perturbation of the particle motion in the circular accelerator (the first term on the right-hand side of equation (3.1.5))

$$x'' + K_x(s)x = -\frac{1}{2} m(s) x^2. \tag{4.2.1}$$

After Floquet transformation our equation is

$$\frac{d^2\zeta}{d\psi^2} + \nu_0^2 \zeta = -\frac{1}{2}\nu_0^2 m \beta^{\frac{5}{2}} \zeta^2, \tag{4.2.2}$$

and simplification of the notation similarly to (4.1.3) results

$$x'' + \omega_0^2 x = \varepsilon \alpha \omega_0^2 x^2, \tag{4.2.3}$$

where $\alpha = -\frac{1}{2}m\beta^{5/2}$. Substituting solution in the form of series (4.1.4) and (4.1.12), proceeding with calculations as in (4.1.13-16), we obtain equation of the first order approximation

$$\begin{aligned} x_1'' + x_1 &= 2A\frac{\omega_1}{\omega_0}\cos\omega t + \alpha A^2\cos^2\omega t \\ &= 2A\frac{\omega_1}{\omega_0}\cos\omega t + \frac{\alpha A^2}{2}(1+\cos 2\omega t) \end{aligned} \tag{4.2.4}$$

Avoiding secular terms yields $\omega_1 = 0$. It means that oscillation frequency does not depend on the amplitude in the first order approximation for quadratic nonlinearity. But does it mean that this type of perturbation does not lead to amplitude dependence at all? Continue to solve in the next order.

Solution of equation (4.2.4) with the initial conditions given in the previous chapter is

$$x_1 = \frac{1}{6}\alpha A^2(3 - 2\cos\omega t - \cos 2\omega t). \tag{4.2.5}$$

Equation of the second order approximation ($\sim \varepsilon^2$) is

$$x_2'' + x_2 = 2\alpha x_0 x_1 - 2x_0''\frac{\omega_2}{\omega_0}. \tag{4.2.6}$$

Substituting the zero and the first order solutions in the right hand side gives

$$x_2'' + x_2 = -\frac{1}{6}\alpha^2A^3 + \left(\frac{5}{6}\alpha^2A^3 + 2A\frac{\omega_2}{\omega_0}\right)\cos\omega t - \frac{1}{6}\alpha^2A^3(\cos 2\omega t + \cos 3\omega t). \tag{4.2.7}$$

To obtain regular motion in the second order, we have to define

$$\omega_2 = -\frac{5}{12}\alpha^2\omega_0 A^2, \tag{4.2.8}$$

thus the second order solution for a sextupole perturbation leads to the same quadratic dependence of the oscillation frequency on the amplitude as the first order of the octupole perturbation (4.1.17). If we stopped solving only at the first order, then the result would have been incorrect (no amplitude dependence).

Returning to the notation of the original problem yields

$$\Delta\nu = -\frac{5}{48}m^2 A_x^2 \beta_x^4 \nu_0. \tag{4.2.9}$$

Derivation of the second order solution was quite laborious and did not give fundamentally new results (origin of various harmonics is seen from (4.2.7)).

4.3 Resonances

So far we were assuming that equation (4.1.3) right hand side is independent of time (in our notation time is phase function of betatron oscillations $\psi(\theta)$). However, equation nonlinear coefficient and beta function depend on the azimuthal angle $\theta = s/R$. Hence, in the general case the inhomogeneous equation is

$$x'' + \omega_0^2 x = g(t), \tag{4.3.1}$$

where perturbation $g(t)$ is a periodic function, which expands in Fourier series (for simplicity we assume that the function is even)

$$g(t) = \frac{a_0}{2} + \sum_1^\infty a_n \cos n\Omega t. \tag{4.3.2}$$

Substituting this expression into (4.3.1) and searching for particular solution in the form of a Fourier series yields

$$x_n(t) = \frac{a_n}{\omega_0^2 - n^2\Omega^2} \cos n\Omega t. \tag{4.3.3}$$

Resonance condition $\omega_0 \approx \pm n\Omega$ creates infinitely large terms in the solution series and prevents convergence of the series.

Obtained result shows that described method is not suitable for studying system behavior in the vicinity of the nonlinear resonance. It also reveals that this behavior (in particular, topology of the phase space) substantially differs from the behavior of the non-resonance system.

Suggested reading

1. A. Lichtenberg and M. Lieberman. Regular and stochastic dynamics. "The World", Moscow, 1984.

2. G.E.O. Jakallya. Methods of perturbation theory for nonlinear systems. "Science", Moscow, 1979.