Nonlinear beam dynamics

Major differences of the nonlinear betatron motion with respect to the linear one are:

Linear betatron motion

- Betatron frequency is constant and does not depend on amplitude
- Transverse phase trajectories are ellipses (or circles after Floquet transformation)
- Only integral and half-integral resonances exist
- Betatron amplitude is limited by the vacuum tube
- The motion is always regular

Nonlinear betatron motion

- Betatron frequency depends on amplitude
- Transverse phase trajectories are distorted dependently on nonlinear perturbation and initial amplitude
- All resonances $n_x Q_x + n_y Q_y = N$ exist
- Betatron amplitude is limited by Dynamic Aperture
- The motion can be regular or stochastic depending on initial conditions

First four features can be studied analytically with the nonlinear pendulum model. The fifth one (appearance of stochastic motion) manifests itself only from numerical simulation

Task assignment

The one-dimensional horizontal betatron motion with cubic nonlinearity

$$x'' + K_x(s)x = -\frac{1}{6}n(s)x^3.$$

Assume azimuthally symmetric case when $K_x(s) = const$ and $n(s) = const$. Floquet transformation helps to change nonlinear Hill equation to equation of mathematical pendulum with perturbation

$$\frac{d^2\zeta}{d\psi^2} + \nu_o^2\zeta = -\frac{1}{6}\nu_0^2 n\beta^3\zeta^3.$$

Introduction of the new notation $x = \zeta, t = \psi, \omega = \nu$ and $\alpha = -\frac{1}{6}n\beta^3$ replaces above equation with more general

$$x'' + \omega_0^2 x = \varepsilon\, \alpha\omega_0^2 x^3.$$

The task is to solve the pendulum model equation by the small parameter ε power series, and illustrate major features of nonlinear motion for cubic perturbation.

Q: Small parameter ε was inserted in the equation artificially to construct power series. What is the real small parameter of betatron motion?

Solution

It is given

$$x'' + \omega_0^2 x = \varepsilon\, \alpha \omega_0^2 x^3.$$

Assuming $x = x(u)$ is a periodic function of $u = \omega t$ with a period of 2π we expand x and ω in powers of ε,

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$
$$\omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

Zeroth order solution is $x_0 = A\cos(\omega t)$. Other initial conditions $x_0'(0) = x_1(0) = 0, x_1'(0) = 0.$

Exercise 1: Solve the above pendulum equation up to the first order of ε using the following particular solutions of the equation

$$y'' + y = a\cos\big(p(t+\alpha)\big)$$

$$y = \frac{1}{2} a\, t\, sin(t+\alpha)\text{, with } p = 1,$$
$$y = \frac{a}{1-p^2} cos\big(p(t+\alpha)\big)\text{, with } p \neq 1.$$

Study of nonlinear motion

Exercise 2: Find the first order tune shift ω_1 removing non-periodic terms from the above solution. What is the dependence of the first order tune shift on the initial amplitude for the cubic nonlinearity?

Exercise 3: Each periodic term can produce resonance at corresponding frequency. What kind of resonances can be produced according to the first order solution? Suppose which resonances will appear with increasing of the solution order?

Exercise 4: For the linear pendulum (zeroth order solution) phase trajectories are circles, with radius defined by initial amplitude

$$x^2 + x'^2 = A^2.$$

Using the first order solution find equation of the phase trajectories. Qualitatively draw the shape of the phase trajectories distorted by cubic nonlinearity in the first order.

Exercise 5: Considering phase trajectories in the form $x^2 + x'^2 = \left(R_0 + \Delta R(A)\right)^2$ study convergence of the series with respect to A (for simplicity take $\omega t = 0$) show that the convergence breaks for some initial A_{max} (dynamic aperture). What is the dependence of A_{max} on the perturbation strength α?