Space Dimension Dynamics and Modified Coulomb Potential of Quarks - Dubna Potential

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Introduction

- Quarkonium spectroscopy indicates that between valence quarks inside hadrons, the potential on small scales has $D = 3$ Coulomb form and at hadronic scales has $D = 1$ Coulomb one.
- We may form an effective potential in which at small scales dominates $D = 3$ component and at hadronic scale - $D = 1$, the Coulomb-plus-linear potential (the "Cornell potential"):

$$V(r) = -\frac{k}{r} + \frac{r}{a^2} = \mu(x - \frac{k}{x}),\ \mu = 1/a = 0.427 GeV,\ x = \mu r,$$

where $k = \frac{4}{3}\alpha_s = 0.52 = x_0^2,\ x_0 = 0.72$ and $a = 2.34 GeV^{-1}$ were chosen to fit the quarkonium spectra [Eichten et al 1978].
- We consider the dimension $D(r)$ of space of hadronic matter dynamically changing with r and corresponding Coulomb potential

$$V_D(r) \sim r^{2-D(r)},$$

where effective dimension of space $D(r)$ changes from 3 at small r to 1 at hadronic scales $\sim$ 1fm.

Coulomb problem in D-dimensions

Poisson equation with point-like source in D-dimensional space, $\Delta\varphi = e\delta^D(x)$, has the solution

$$\varphi(D,r) = -\frac{\Gamma(D/2)}{2(D-2)\pi^{D/2}} e r^{2-D},$$

$$V(D,r) = e\varphi(D,r) = -\alpha(D) r^{2-D},\ \alpha(D) = \frac{e^2\Gamma(D/2)}{2(D-2)\pi^{D/2}},$$

$$V(3,r) = -\frac{\alpha(3)}{r} = -\frac{e^2}{4\pi r},\ V(4,r) = -\frac{\alpha(4)}{r^2} = -\frac{e^2}{4\pi^2 r^2}\,.$$

Indeed,

$$\int d^D x \Delta\varphi = \Omega_D r^{D-1} \frac{d}{dr}\frac{a_D}{r^{D-2}} = -(D-2)\Omega_D a_D = e,\ a_D = -\frac{e}{(D-2)\Omega_D},\ a_3 = -\frac{e}{4\pi},$$

$$\int dx^D e^{-x^2} = (2\pi \int_0^\infty dr r e^{-r^2})^{D/2} = \pi^{D/2} = \Omega_D \int_0^\infty dr r^{D-1} e^{-r^2} = \frac{\Omega_D}{2}\Gamma(D/2),\ \Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,.$$

Coulomb problem in D-dimensions

- As defined so far, the coupling constant has a mass dimension $d_e = (D-3)/2 = -\varepsilon$. To work with a dimensionless coupling constant e, we introduce the mass scale μ.
- Then, the potential energy takes the following form

$$\begin{aligned} V(D,r) &= -\frac{\Gamma(D/2)}{2(D-2)\pi^{D/2}} e^2 \mu^{2\varepsilon} r^{2-D} \\ &= -\alpha(D)(\mu r)^{2\varepsilon}/r \\ &= -\alpha(D) x^{2-D} \mu. \end{aligned}$$

Dimension dynamics from Cornell potential

- Cornell potential contains QCD dynamics. We may compare it with Coulomb potential with dynamical dimension. Let us define dimension of space from the equality of $V(r) = \mu(x - \frac{k}{x})$ and $V(D, r) = -\alpha(D) r^{2-D}$:

$$\frac{k - x^2}{x^{3-D}} = \alpha(D) = \frac{e^2 \Gamma(D/2)}{2(D-2)\pi^{D/2}} = \alpha_s \frac{2\Gamma(D/2)}{(D-2)\pi^{(D-2)/2}}, \ \alpha_s = \frac{e^2}{4\pi}.$$

- For any values of x and D

$$\alpha_s(D, x) = \frac{\pi^{(D-2)/2}}{2\Gamma(D/2)}(D-2)\alpha, \ \alpha = \frac{k - x^2}{x^{3-D}} = (k - x^2)x^{D-3}.$$

- At the point $D = 1, \ x = x_1$,

$$\alpha_s(1, x_1) = \frac{1}{2\pi}\left(1 - \frac{k}{x_1^2}\right), \ x_1^2 > x_0^2 = k.$$

Hamiltonian formulation of space dimension dynamics

- Let us consider simplest Hamiltonian dynamics

$$\dot{x}_1 = \{H, x_1\},$$
$$\dot{x}_2 = \{H, x_2\},$$

for dynamical variables (phase space) (x_1, x_2), Hamiltonian H

$$H = \frac{p^2}{2m} + V(x) = \frac{x_1^2}{2m} + V(x_2)$$

and Poisson structure

$$\{A, B\} = f_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = f_{12} \left(\frac{\partial A}{\partial x_1} \frac{\partial B}{\partial x_2} - \frac{\partial A}{\partial x_2} \frac{\partial B}{\partial x_1} \right).$$

- Instead of solving the system of motion equations, we may solve them in a semi-algebraic way: having one integral of motion - Hamiltonian, we may find x_1 from the Hamiltonian, insert it in the motion equation for x_2 and solve it.

- The variables x, D and α are nonnegative, so it is natural to introduce, free from this restriction, variables:

$$t = \ln x$$

$$x_1 = \ln \alpha_s$$

$$x_2 = \ln D$$

- Then we obtain the following Hamiltonian and motion equations

$$\begin{aligned}
&H(x_1, x_2, t) = x_1 - V(x_2, t) \Rightarrow x_1 = V(x_2, t),\\
&\dot{x}_1 = f_{12}\frac{\partial V}{\partial x_2},\\
&\dot{x}_2 = -f_{12},\ \ V(x_2, t) = \ln(\frac{\pi^{(D-2)/2}}{2\Gamma(D/2)}(D-2)\frac{k-x^2}{x^{3-D}}).
\end{aligned}$$

- We may also take $x_1 = \alpha$, then

$$x_1 = V(t, x_2) = (k - x^2)x^{D-3} = (k - x^2)x^{\exp(x_2)-3} = (k - e^{2t})e^{t(e^{-t}-3)},$$

$$\dot{x}_1 = \frac{\partial V}{\partial x_2} = (k - x^2)x^{e^{x_2}-3}\ln x e^{x_2} = (k - e^{2t})te^{t(e^{-t}-3)}e^{-t},\ f_{12} = 1,$$

$$\dot{\alpha} = \beta = te^{-t}\alpha = \beta_1\alpha,\ \beta_1 = \ln\frac{\alpha e^{3t}}{k - e^{2t}}$$

$$\dot{x}_2 = -1 \Rightarrow x_2 = -t,\ D = 1/x$$

$$\alpha_s(D, x) = \frac{\pi^{(D-2)/2}}{2\Gamma(D/2)}(D-2)\frac{k - x^2}{x^{3-D}} = \frac{\pi^{(1/x-2)/2}}{2\Gamma(1/2x)}(1/x-2)\frac{k - x^2}{x^{3-1/x}}$$

$$= \frac{\pi^{(1/x-2)/2}}{2\Gamma(1/2x)}(1/x-2)(\sqrt{k}-x)\frac{\sqrt{k}+x}{x^{3-1/x}}.$$

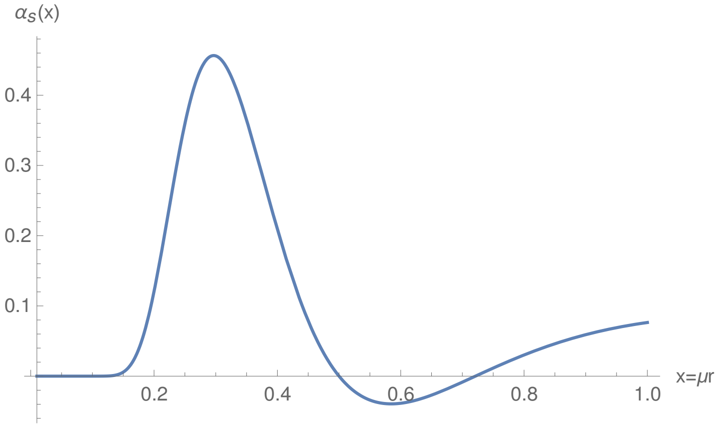

Figure: α_s as a function of $x = \mu r \in (0.01, 1.0)$

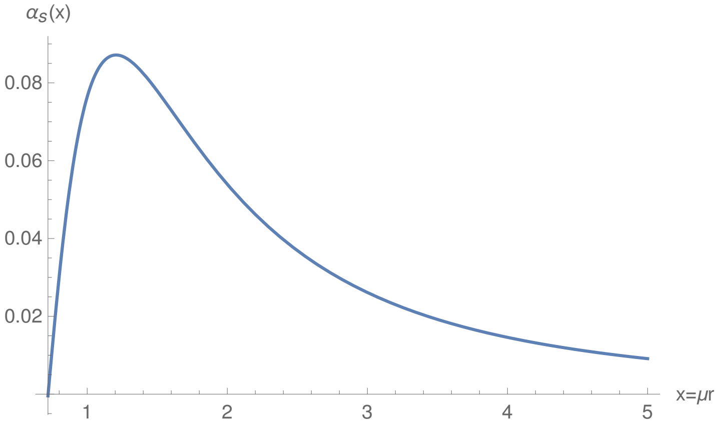

Figure: α_s as a function of $x = \mu r \in (0.72, 5)$

- Note that $x > 0$ and $\alpha_s \geq 0$ when $x < \min(1/2, \sqrt{k}) = 1/2$ or $x > \max(1/2, \sqrt{k}) = \sqrt{k} = 0.72$ and for $0.5 < x < 0.72, \alpha_s < 0$, see figures 1 and 2.
- For $x_1 = 1$, we have from $\alpha_s(1, x_1)$

$$\alpha_s = \frac{1}{2\pi}(1 - k) = \frac{0.48}{2\pi} = 0.0764.$$

- We may exclude the negative values by using different values of μ: $x_1 = r\mu_1 = 1/2,\ x_2 = r\mu_2 = 0.72,\ \mu_2/\mu_1 = 1.44.$

Compactification and Dimension dynamics

Let us take one of the dimensions y as circle with radius R. This corresponds to a periodic structure with a point charge sources at each point $y_n = y + 2\pi R n, n = 0, \pm 1, \pm 2, ...$

$$\Delta\varphi = e \sum_n \delta^D(x)\delta(y_n), \varphi(D, r, y) = \sum_n \varphi(D, r, y_n),$$

$$V(D, r, y) = -\alpha(D+1) \sum_{n=-\infty}^{\infty} (r^2 + (2\pi R n + y)^2)^{(1-D)/2}.$$

When $D = 3$, the potential can be writen in a closed form [Bures, Siegl 2014]

$$V(3, r, y) = -\frac{\alpha(4)}{2Rr} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(y/R)} = \begin{cases} -\alpha(4)/(2Rr), & r \gg R \\ -\alpha(4)/(r^2 + y^2), & r,\ y \ll R \end{cases},$$

where $\alpha(4)/(2R) = \alpha(3)$.

Compactification and Dimension dynamics

Alternatively, we can rewrite the potential as

$$V(3,r,y) = -\frac{\alpha(4)}{4Rr}\left[\coth\left(\frac{r+\mathrm{i}y}{2R}\right) + \coth\left(\frac{r-\mathrm{i}y}{2R}\right)\right],$$

or, using

$$A^{-\alpha} = 1/\Gamma(\alpha)\int_0^\infty dt t^{\alpha-1} e^{-tA},$$

by means of the Theta function as

$$V(3,r,y) = -\alpha(4)\int_0^\infty dt e^{-tr^2}\sum_{-\infty}^{\infty} e^{-t(2\pi Rn+y)^2}$$

$$= -\alpha(4)\int_0^\infty dt e^{-tr^2}\frac{\theta\left(\frac{\mathrm{i}y}{2\pi R}, e^{\frac{\mathrm{i}}{4R^2 t}}\right)}{2R\sqrt{\pi}\sqrt{t}}.$$

Compactification and Dimension dynamics

- For $y = 0$, the potential takes the following simple form

$$V(3, r, y = 0) = -\frac{\alpha(4)}{2Rr} \coth \frac{r}{2R}.$$

- From $V(3, r, y)$, we see that for big r, the effective dimension of space is 3 and for small r is 4.
- For intermediate scales, the effective dimension might change smoothly from 3 to 4. Integrating $V(3, r, y)$ by coordinate y, we define mean potential depending only on the variable r, [Bures, Siegl 2014]

$$\bar{V}(3, r) = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \, V_3(r, \vartheta) = -\frac{\alpha(4)}{2Rr} = -\frac{\alpha(3)}{r}.$$

Compactification and Dimension dynamics

- As in the Cornell potential case, we define the dimension dynamics from equality between the corresponding Coulomb potentials:

$$\frac{\alpha(4)}{2r}\frac{\sinh(r/R)}{\cosh(r/R)-\cos(y/R)} = \alpha(D)(x)^{2-D},$$
$$\mu = 1/R,\ x = \mu r,\ r^2 = x_1^2 + x_2^2 + x_3^2.$$

- From this equality, the dynamical dimension of space $D(y, r)$ is defined as implicit function and needs numerical solution.
- Alternatively, we may define y as an explicit function of x and D as

$$y = R\arccos(\cosh x - A(D)x^{D-3}\sinh x),$$
$$A(D) = \frac{\mu\alpha(4)}{2\alpha(D)},\ \alpha(D) = \frac{e^2\Gamma(D/2)}{2(D-2)\pi^{D/2}}.$$

Compactification and Dimension dynamics

If we have two circlular coordinates - a torus, then

$$\Delta\varphi = e\sum_{n,m}\delta^D(x)\delta(y_n)\delta(z_m),$$

$$\varphi(D,r,y,z) = \sum_{n,m}\varphi(D,r,y_n,z_m),$$

$$V(D,r,y,z) = -\alpha(D+2)\sum_{n,m=-\infty}^{\infty}(r^2+(2\pi R_1 n+y)^2+(2\pi R_2 m+z)^2)^{-D/2}.$$

Compactification and Dimension dynamics

General expression for Coulomb potential in $(D+d)$-dimensional space $\mathbb{R}^D \times \mathbb{T}^d$ where $\mathbb{T}^d = S^1 \times \cdots \times S^1$ (d-times) is the d-dimensional torus. D refer to the "big" dimensions $\mathbf{x} = (x_1, \ldots x_D)$, whereas d to the "small-compactified" ones $\mathbf{y} = (y_1, \ldots y_d)$. Then

$$
\begin{aligned}
&\Delta\varphi = e \sum_{n_1,\ldots,n_d} \delta^D(\mathbf{x})\delta(y_{1,n_1})\ldots\delta(y_{d,n_d}),\\
&\varphi(D,d,r,y_1,\ldots,y_d) = \sum_{n_1,\ldots,n_d} \varphi(D,d,r,y_{1,n_1},\ldots,y_{d,n_d}),\\
&V(D,d)(r,y_1,\ldots,y_d)\\
&= -\alpha(D+d) \sum_{n_1,\ldots,n_d=-\infty}^{\infty} (r^2 + (2\pi R_1 n_1 + y_1)^2 + \cdots + (2\pi R_d + y_d)^2)^{-(D+d-2)/2}\\
&= -\frac{\alpha(D+d)}{\Gamma\left(\frac{D+d-2}{2}\right)} \int_0^\infty dt t^{\frac{D+d-4}{2}} e^{-tr^2} e^{-t(2\pi R_1 n_1 + y_1)^2} \ldots e^{-t(2\pi R_d n_d + y_d)^2}\\
&= -\frac{\alpha(D+d)}{\Gamma\left(\frac{D+d-2}{2}\right)} \int_0^\infty dt t^{\frac{D+d-4}{2}} e^{-tr^2} \Pi_{i=1}^d e^{-ty_i^2} B_i(t,y_i),\\
&B_i(t,y_i) = \sum_{n_i=-\infty}^{\infty} e^{-t(2\pi R_i n_i + y_i)^2} = e^{-ty_i^2}\theta(2\mathrm{i}R_i y_i t, 4\pi\mathrm{i}R_i^2 t),
\end{aligned}
$$

where the sums in B_i's were written by means of the Theta function.

Compactification and Dimension dynamics

For a point quark inside hadron of size R at a temperature T we have

$$\Delta\varphi = e \sum_{k,l,n,m} \delta(\tau_k)\delta(x_l)\delta(y_n)\delta(z_m),$$

$$\varphi(0,\tau,x,y,z) = \sum_{k,l,n,m} \varphi(0,\tau_k,x_l,y_n,z_m),$$

$$V(0,\tau,x,y,z) = -\alpha(4) \sum_{k,l,n,m=-\infty}^{\infty} ((2\pi k/T+\tau)^2$$

$$+(2\pi R_1 l + x)^2 + (2\pi R_2 n + y)^2 + (2\pi R_3 m + z)^2)^{-1}$$

$$= -\alpha(4)\int_0^\infty dt t B_0(t,\tau)B_1(t,x)B_2(t,y)B_3(t,z),$$

$$B_1(t,x) = \sum_{n=-\infty}^{\infty} e^{-t(2\pi R_1 n + x)^2} = e^{-tx^2}\theta(2\mathrm{i}R_1 xt, 4\pi\mathrm{i}R_1^2 t), \ldots,\ R_0 = 1/T,$$

where we have written the sums by means of the Theta function.

Theta functions

Theta functions is the analytic function $\theta(z, \tau)$ in 2 variables defined by

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp[\mathrm{i}\pi(\tau n^2 + 2nz)] = 1 + 2\sum_{n\geq 1} \exp(\mathrm{i}\pi\tau n^2)\cos(2\pi nz),$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, the upper half plane Im $\tau > 0$. The series converges absolutely and uniformly on compact sets.

Integrals

Let us calculate the following integral

$$I(a) = \int_0^{\pi} \frac{d\vartheta}{a^2 + 1 - 2a\cos\vartheta} = \frac{\pi}{|a^2 - 1|} = \begin{cases} \pi/a^2, & a^2 \gg 1 \\ \pi, & a^2 \ll 1 \end{cases}.$$

Obviously, $I(1) = \infty$, but

$$I(1) = \frac{1}{2}\int_0^{\pi} \frac{d\vartheta}{1 - \cos(\vartheta)} = \frac{1}{4}\int_0^{\pi} \frac{d\vartheta}{\sin^2\frac{\vartheta}{2}} = \frac{1}{2}\int_0^1 \frac{dx}{(1 - x^2)^{3/2}}$$

$$= \frac{1}{4}\int_0^1 \frac{dy}{y^{1/2}(1 - y)^{3/2}} = B(1/2, -1/2) = \frac{1}{4}\frac{\Gamma(1/2)\Gamma(-1/2)}{\Gamma(0)} = 0, ?!$$

$$B(\alpha, \beta) = \int_0^1 dx x^{\alpha - 1}(1 - x)^{\beta - 1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, Real \quad \alpha, \beta > 0.$$

Integrals

In our case, $a = \exp(r/R) > 1$ and the corresponding integral is

$$I = \frac{1}{a}\int \frac{d\theta}{b + 2\cos\theta} = \frac{1}{\mathrm{i}a}\int \frac{dz}{z^2 + bz + 1}$$
$$= I(z,a) = \frac{1}{\mathrm{i}a(a - 1/a)} \ln\frac{z + a}{z + 1/a},$$
$$I(a) = I(-1,a) - I(1,a) = \frac{1}{\mathrm{i}a(a - 1/a)} \ln\frac{(-1+a)(1+1/a)}{(-1+1/a)(1+a)} = \frac{\pi}{a^2 - 1},$$
$$b = a + 1/a,\ \ z = e^{\mathrm{i}\theta}.$$

Now,

$$I = \int_0^{2\pi} \frac{d\vartheta}{a^2 + 1 - 2a\cos(\vartheta)} = I(a) + I(-a) = \frac{2\pi}{|a^2 - 1|} = \frac{\pi\exp(-r/R)}{\sinh(r/R)},$$
$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\vartheta}{\cosh(r/R) - \cos(\vartheta)} = \frac{2a}{a^2 - 1} = \frac{1}{\sinh(r/R)},\ \ a = \exp(r/R).$$

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