Calculation of Feynman diagrams

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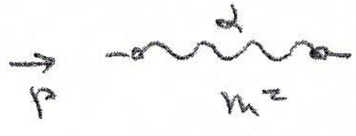
I Definitions:

$$\equiv \frac{1}{[p^2]^\alpha}$$



$$\equiv \frac{1}{[p^2+m^2]^\alpha}$$

Euclidean space:

$D = 4 - 2\varepsilon$

$\lambda = D/2 - 1 = 1 - \varepsilon$

dimensional regularization

$d^4k \to d^Dk \, (\mu^2)^\varepsilon$

(1.3)

(II) Tadpoles

1.  $\equiv \displaystyle\int \frac{d^D k}{(k^2)^\beta\,[k^2+m^2]^\alpha} = (*)$ $\qquad [k^2 = m^2 z]$

Polyar coordinates: $\boxed{d^D k \equiv \frac{1}{2}(k^2)^{\frac{D}{2}-1}\, dk^2\, d\Omega}$

$\displaystyle\int d\Omega = S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is the surface of the unit hypersphere in R_D

Euler Γ-function

$$(*) = \frac{S_D}{2}\underbrace{\int_0^\infty \frac{dz\cdot z^{\lambda-\beta}}{(z+1)^\alpha}}_{\text{Euler } \beta\text{-function}} \cdot \frac{1}{(m^2)^{\alpha+\beta-D/2}} = (**)$$

(1.4)

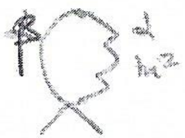
$$\int_0^\infty \frac{dz\, z^\gamma}{(1+z)^\alpha} = \int_1^\infty \frac{dt\,(t-1)^\gamma}{t^\alpha} = \int_0^1 \underbrace{\frac{dp\,(1-p)^\gamma\, p^\alpha}{p^2 \cdot p^\gamma}}_{dp\cdot p^{\alpha-\gamma-2}(1-p)^\gamma} = \underbrace{\frac{\Gamma(\alpha-\gamma-1)\,\Gamma(\gamma+1)}{\Gamma(\alpha)}}_{B(\alpha-\gamma-1,\ \gamma+1)}$$

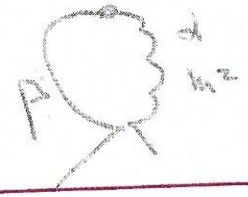
$\underbrace{z+1=t,\ t=\frac{1}{p}}$

$\alpha-\gamma-1>0,\ \gamma>-1$

$$(*) = \frac{\pi^{D/2}}{\Gamma(D/2)} \cdot \underbrace{\frac{\Gamma(\frac{D}{2}-\beta)\,\Gamma(\alpha+\beta-\frac{D}{2})}{\Gamma(\alpha)}}_{\boxed{B(\beta,\alpha)}} \frac{1}{(m^2)^{\alpha+\beta-D/2}} \qquad (1)$$

$\boxed{B(0,\alpha) \equiv B(\alpha)}$

2. If $\alpha+\beta < D/2$  $\xrightarrow[m^2\to 0]{} 0$

Dimensional regularization (convension):  $\underset{m^2=0}{\equiv} 0$ for any α and β

3. Really  $\xrightarrow[m^2\to 0]{} \delta(\alpha+\beta-\frac{D}{2})$ but the case $\alpha+\beta=\frac{D}{2}$ is not relevant in dimensional regalarization (it is possible to add small Δ always)

4. Consider the integral (1.5)

$$\int \frac{d^D k}{[k^2 - 2pk + m^2]^\alpha} = (*)$$

We introduce $k_1 = k - p$, then $[k^2 - 2pk + m^2] = [k_1^2 + m^2 - p^2]$

Moreover, $d^D k = d^D k_1$, then

$$(*) = \int \frac{d^D k_1}{[k_1^2 + \underbrace{m^2 - p^2}_{m_1^2}]^\alpha} = \pi^{D/2} \underbrace{B(0,\alpha)}_{\frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)}} \frac{1}{(\underbrace{m_1^2}_{m^2 - p^2})^{\alpha - D/2}} \qquad (2)$$

III Feynman parameterization

$[k_i^2+m_i^2] \equiv a_i$

$$\frac{1}{\prod_{i=1}^{N} \underbrace{[k_i^2+m_i^2]^{\alpha_i}}_{a_i^{\alpha_i}}} = \frac{\Gamma\left(\sum_i^N \alpha_i\right)}{\prod_i^N \Gamma(\alpha_i)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \dots \int_0^{1-\sum_i^{N-2} x_i} dx_{N-1} \cdot \left.\frac{\prod_{i=1}^{N} x_i^{\alpha_i - 1}}{\left(\sum_i^N x_i \cdot a_i\right)^{\sum_i^N \alpha_i}}\right|_{x_N = 1-\sum_i^{N-1} x_i} \qquad (3)$$

If $N=2$

$$\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2}} = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx_1 \frac{x_1^{\alpha_1-1} (1-x_1)^{\alpha_2-1}}{(x_1 a_1 + (1-x_1) a_2)^{\alpha_1+\alpha_2}} \qquad (3a)$$

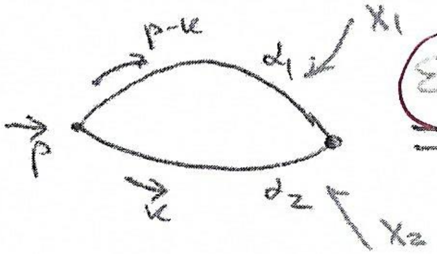
Feynman parameterization transforms products of propagators to a new propagator with effective mass, which is dependent on x_i.
With this new propagator, we can apply formulas for tadpoles (which were done already)

I. Massless diagrams

A.V.K., S. Teber "Multi-loop technique for massless Feynman diagram calculations", Phys. Part. Nucl. **50** (2019) 1. [arXiv: 1805.05109 [hep-th]]

IV One Loop

1) Using Feynman parameters



$$\underset{\text{Eq. (3a)}}{=} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0 dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1} \int \frac{d^D k}{[\quad]_1^{\alpha_1+\alpha_2}} = (*)$$

$$[\,]_1 = x\underbrace{(k-p)^2}_{k^2+p^2-2(pk)} + (1-x)k^2 = k^2 + \underbrace{p^2 x}_{m^2_{eff}} - 2(pk)\cdot x \Rightarrow m^2_{1,eff} = m^2_{eff} - p^2x^2 = p^2x(1-x)$$

$$\int \frac{d^D k}{[\,]_1^{\alpha_1+\alpha_2}} \underset{\text{Eq. (2)}}{=} \pi^{D/2} \underbrace{\frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1+\alpha_2)}}_{B(0,\alpha_1+\alpha_2)} \frac{1}{[\underbrace{p^2x(1-x)}_{m^2_{1,eff}}]^{\alpha_1+\alpha_2-D/2}}$$

$$(*) = \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1+\alpha_2)} \frac{1}{(p^2)^{\alpha_1+\alpha_2-D/2}} \int_0^1 dx \cdot \overbrace{\frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[x(1-x)]^{\alpha_1+\alpha_2-D/2}}}^{x^{D/2-\alpha_2-1}(1-x)^{D/2-\alpha_1-1}} = \pi^{D/2} A(\alpha_1,\alpha_2) \frac{1}{(p^2)^{\alpha_1+\alpha_2-D/2}} \quad (4)$$

$$\int_0^1 dx\, x^{D/2-\alpha_2-1}(1-x)^{D/2-\alpha_1-1} = \frac{\Gamma(D/2-\alpha_1)\Gamma(D/2-\alpha_2)}{\Gamma(D-\alpha_1-\alpha_2)}$$

$A(\alpha_1,\alpha_2)$

$$A(\alpha_1,\alpha_2) = \frac{a(\alpha_1)\,a(\alpha_2)}{a(\alpha_1+\alpha_2-D/2)}, \quad a(\alpha) = \frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}, \quad \tilde{\alpha} = D/2 - \alpha$$

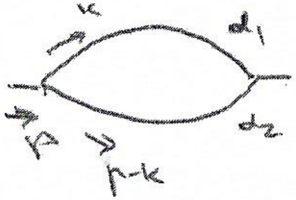
2) Recover Eq(4) using Fourier transform

(1.8)

$$\int \frac{d^D p\, e^{ipx}}{p^{2\alpha}} = \pi^{D/2} \frac{2^{\overbrace{D-2\alpha}^{2\tilde{\alpha}}}\, a(\alpha)}{x^{2\tilde{\alpha}}} \qquad \boxed{\tilde{\alpha} = D/2 - \alpha}$$

$$\int \frac{d^D x\, e^{-ipx}}{x^{2\alpha}} = \pi^{D/2} \frac{2^{+2\tilde{\alpha}}\, a(\alpha)}{p^{2\tilde{\alpha}}} \Rightarrow \frac{1}{p^{2\alpha}} = \frac{1}{\pi^{D/2}}\frac{a(\alpha)}{2^{2\alpha}\,\cancel{a(\tilde{\alpha})}} \int \frac{d^D x\, e^{-ipx}}{x^{2\tilde{\alpha}}}$$

Then



$$\equiv \int \frac{d^D k}{k^{2\alpha_1}(p-k)^{2\alpha_2}} = \int \frac{d^D k}{} \frac{1}{\pi^D} \frac{a(\alpha_1)\, a(\alpha_2)}{2^{2(\alpha_1+\alpha_2)}} \int \frac{d^D x_1\, e^{-ikx_1}}{x_1^{2\tilde{\alpha}_1}} \int \frac{d^D x_2\, e^{-i(p-k)x_2}}{x_2^{2\tilde{\alpha}_2}} = (*)$$

Note that $\int d^D k\, e^{ik(x_2-x_1)} = (2\pi)^D \delta^D(x_2-x_1)$

(4)

$$(*) = \frac{2^D\, a(\alpha_1)\, a(\alpha_2)}{2^{2(\alpha_1+\alpha_2)}} \underbrace{\int \frac{d^D x_1\, e^{-ipx_1}}{x_1^{2(\tilde{\alpha}_1+\tilde{\alpha}_2)}}}_{\pi^{D/2}\, 2^{D-(\tilde{\alpha}_1+\tilde{\alpha}_2)} \frac{a(\tilde{\alpha}_1+\tilde{\alpha}_2)}{(p^2)^{\underbrace{D/2-\tilde{\alpha}_1-\tilde{\alpha}_2}_{\gamma}}}} = \pi^{D/2} \underbrace{\cancel{2^K}}_{=1} \underbrace{\frac{a(\alpha_1)\, a(\alpha_2)\, a(\cancel{-\alpha_1-\alpha_2}+D)}{a(\alpha_1+\alpha_2-D/2)}}_{A(\alpha_1,\alpha_2)} \frac{1}{(p^2)^{\alpha_1+\alpha_2-D/2}}$$

$\gamma = \alpha_1+\alpha_2-D/2$

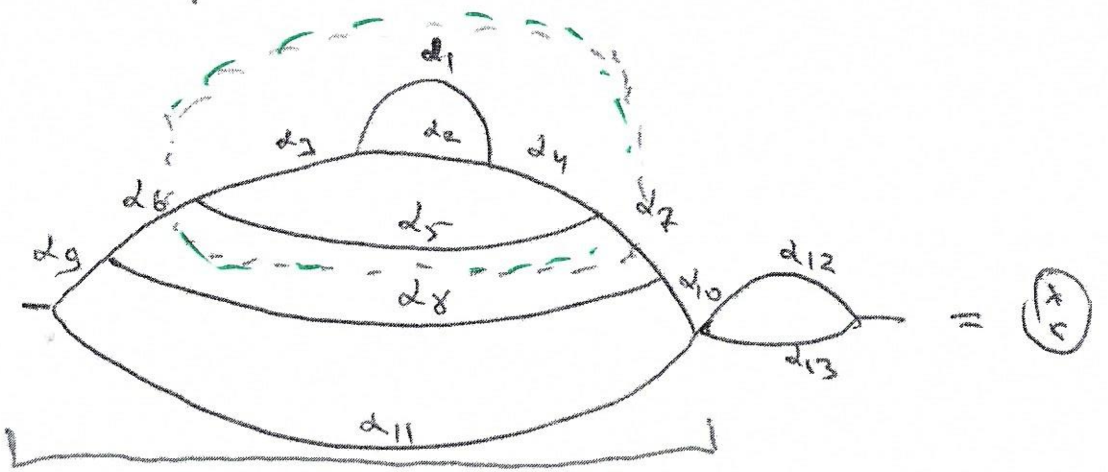
$K = D - 2(\alpha_1+\alpha_2) + D - 2(\overbrace{\tilde{\alpha}_1+\tilde{\alpha}_2}^{D-\alpha_1-\alpha_2}) = 0$

(1.9)

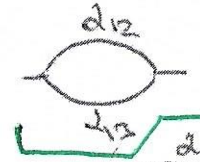
3) Chain

$$\underset{\vec{p}}{\bullet\!\xrightarrow{\;\alpha_1\;}\!\bullet\!\xrightarrow{\;\alpha_2\;}\!\bullet} \equiv \frac{1}{(p^2)^{\alpha_1}}\,\frac{1}{(p^2)^{\alpha_2}} = \frac{1}{(p^2)^{\alpha_1+\alpha_2}} = \bullet\!\xrightarrow{\;\alpha_1+\alpha_2\;}\!\bullet \qquad (5)$$

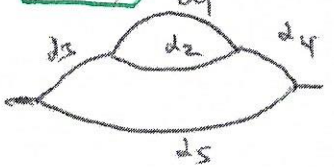
V) So, Equations (4) and (5) give a possibility to calculate any diagram, which is combination of loops and chains.

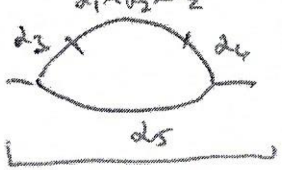


$= ?$

1)


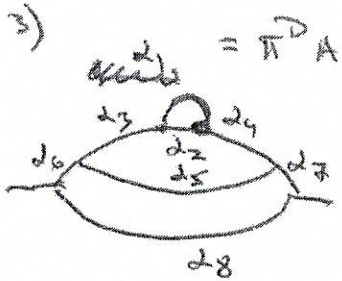
$= \pi^{D/2} A(\alpha_{12}, \alpha_{13}) \xrightarrow{\;\alpha_{12}+\alpha_{13}-D/2\;}$

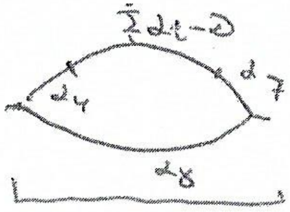
2)


$= \pi^{D/2} A(\alpha_1,\alpha_2)$


$= \pi^{D} A(\alpha_1,\alpha_2)\, A\left(\sum_1^4 \alpha_i - \frac{D}{2}, \alpha_5\right) \xrightarrow{\;\sum_1^5 \alpha_i - D\;}$

$\sum_1^4 \alpha_i - \frac{D}{2}$


3) 

$= \pi^{D} A(\alpha_1,\alpha_2)\, A\left(\sum_1^4 \alpha_i - \frac{D}{2}, \alpha_5\right)$  $= \pi^{\frac{3D}{2}} A(\alpha_1,\alpha_2)\, A\left(\sum_1^4 \alpha_i - \frac{D}{2}, \alpha_5\right) A\left(\sum_1^7 \alpha_i - D, \alpha_8\right) \xrightarrow{\sum_1^8 \alpha_i - \frac{3D}{2}}$ (2.20)

$\sum_1^7 \alpha_i - D$

$(\ast) = \pi^{\frac{5D}{2}} A(\alpha_1,\alpha_2)\, A\left(\sum_1^4 \alpha_i - \frac{D}{2}, \alpha_5\right) A\left(\sum_1^7 \alpha_i - D, \alpha_8\right) A\left(\sum_1^{10} \alpha_i - \frac{3D}{2}, \alpha_{11}\right) \cdot A(\alpha_{12},\alpha_{13}) \xrightarrow{\sum_1^{13} \alpha_i - \frac{5D}{2}}$

number of loops

4) To find "numbers" it is useful to use the relations $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and the expansion for $\Gamma(1+a\varepsilon)$: $\Gamma(1+a\varepsilon) = \exp\left[-\gamma_E a\varepsilon + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} (-1)^k a^k \varepsilon^k\right]$

number of integrations

<u>Note</u>: The terms $\pi^{D/2 \cdot k}$ should be not considered really in results, because every integration gives:

from measure of the integral

$$\frac{\pi^{\frac{D}{2}}\Gamma(1+\varepsilon)}{(2\pi)^D} \cdot g^2 (\mu^2)^\varepsilon = \frac{g^2}{(4\pi)^{\frac{D}{2}}} \Gamma(1+\varepsilon)(\mu^2)^\varepsilon \equiv \overbrace{\frac{g^2}{16\pi^2}}^{\alpha} \boxed{\underbrace{(4\pi)^\varepsilon \Gamma(1+\varepsilon)(\mu^2)^\varepsilon}_{(\bar{\mu}^2)^\varepsilon \;\leftarrow\; \overline{MS}\text{-scheme}}}$$

charge squared

dimensional regularization

3) Let $\alpha_i = 1$

1.11

So, our result has the following form of arguments (of Γ-function)

$$\sum_1^4 \alpha_i - D/2 = 2+\varepsilon, \quad \sum_1^7 \alpha_i - D = 3+2\varepsilon, \quad \sum_1^{10} \alpha_i - \frac{3D}{2} = 4+3\varepsilon$$

$$A^2(1,1)\, A(2+\varepsilon,1)\, A(3+2\varepsilon,1)\, A(4+3\varepsilon,1) = (*)$$

$$A(1,1) = \frac{\Gamma^2(1-\varepsilon)\overbrace{\Gamma(\varepsilon)}^{\Gamma(1+\varepsilon)/\varepsilon}}{\Gamma^2(1)\underbrace{\Gamma(2-2\varepsilon)}_{(1-2\varepsilon)\Gamma(1-2\varepsilon)}} = \frac{1}{(1-2\varepsilon)\varepsilon}\,\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$A(2+\varepsilon,1) = \frac{\overbrace{\Gamma(-2\varepsilon)}^{\Gamma(1-2\varepsilon)/(-2\varepsilon)}\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)}{\underbrace{\Gamma(2+\varepsilon)}_{(1+\varepsilon)\Gamma(1+\varepsilon)}\Gamma(1)\Gamma(1-3\varepsilon)} = \frac{1}{(-2\varepsilon)(1+\varepsilon)}\,\frac{\Gamma(1-2\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-3\varepsilon)}$$

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \Rightarrow \Gamma(\alpha-1) = \frac{\Gamma(\alpha)}{(\alpha-1)}$$

$$A(3+2\varepsilon,1) = \frac{\overbrace{\Gamma(-1-3\varepsilon)}^{\Gamma(1-3\varepsilon)/[3\varepsilon(1+3\varepsilon)]}\Gamma(1-\varepsilon)\overbrace{\Gamma(2+3\varepsilon)}^{(1+3\varepsilon)\Gamma(1+3\varepsilon)}}{\underbrace{\Gamma(3+2\varepsilon)}_{2(1+\varepsilon)(1+2\varepsilon)\Gamma(1+2\varepsilon)}\Gamma(1)\underbrace{\Gamma(-4\varepsilon)}_{\Gamma(1-4\varepsilon)/(-4\varepsilon)}} = -\frac{3}{3}\,\frac{1}{(1+\varepsilon)(1+2\varepsilon)}\,\frac{\Gamma(1-3\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma(1+2\varepsilon)\Gamma(1-4\varepsilon)}$$

$$A(4+3\varepsilon,1) = \frac{\overbrace{\Gamma(-2-4\varepsilon)}^{\Gamma(1-4\varepsilon)/(2(1+2\varepsilon)(1+4\varepsilon)4\varepsilon)}\Gamma(1-\varepsilon)\overbrace{\Gamma(3+4\varepsilon)}^{2(1+2\varepsilon)(1+4\varepsilon)\Gamma(1+4\varepsilon)}}{\underbrace{\Gamma(4+3\varepsilon)}_{3(1+\varepsilon)(2+3\varepsilon)(1+3\varepsilon)\Gamma(1+3\varepsilon)}\Gamma(1)\underbrace{\Gamma(-1-5\varepsilon)}_{\frac{1}{(5\varepsilon)(1+5\varepsilon)}\cdot\Gamma(1-5\varepsilon)}} = -\frac{5(1+5\varepsilon)}{24(1+\varepsilon)(1+\frac{3}{2}\varepsilon)(1+3\varepsilon)}\,\frac{\Gamma(1-4\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+4\varepsilon)}{\Gamma(1+3\varepsilon)\Gamma(1-5\varepsilon)}$$

$$(*) = -\frac{5}{36\varepsilon^2}\,\frac{(1+5\varepsilon)}{(1-2\varepsilon)(1+\varepsilon)^3(1+2\varepsilon)(2+3\varepsilon)(1+3\varepsilon)} \cdot \underbrace{\frac{\Gamma^7(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(1+4\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1-5\varepsilon)}}_{\Gamma^5(1-\varepsilon)K_1}$$

K_1: Euler constant $\quad \Gamma(1+a\varepsilon) = \exp\left[a\gamma\varepsilon + \sum_{k=2}^{\infty} \frac{(-a\varepsilon)^k}{k} \zeta(k) \right]$ Euler ζ-function ~~constant~~

(1.12)

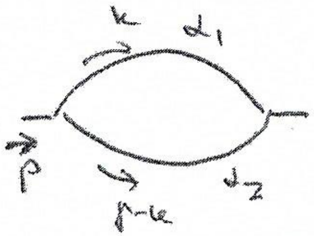
$$K_1 = \frac{\Gamma^7(1-\varepsilon)\,\Gamma(1+4\varepsilon)}{\Gamma(1-2\varepsilon)\,\Gamma(1-5\varepsilon)\,\Gamma^4(1+\varepsilon)} = \frac{\exp\left[\cancel{-3\gamma_E\varepsilon} + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}\left\{7 + (-)^k 4^k\right\}\varepsilon^k\right]}{\exp\left[\cancel{-3\gamma_E\varepsilon} + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}\left\{2^k + 5^k + (-)^k \cdot 4\right\}\varepsilon^k\right]} = \exp\left[\sum_{k=2}^{\infty} \frac{\zeta(k)}{k} Q_1(k)\varepsilon^k\right] = ⓔ$$

$$Q_1(k) = 7 - 2^k - 5^k + \underbrace{(-)^k(4^k - 4)}_{4(-)^k(4^{k-1}-1)}, \quad Q_1(k=2) = -10, \quad Q_1(k=3) = -186$$

$$ⓔ = \exp\left[-5\zeta(2)\varepsilon^2 - 62\zeta(3)\varepsilon^3 + O(\varepsilon^4)\right]$$

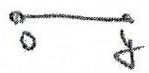
But already at two-loop 

(VI) Dual representation: all $p \to x \Rightarrow$ (1.13)



$$= \int \frac{d^D k}{k^{2\alpha_1}\left[(p-k)^2\right]^{\alpha_2}} \xrightarrow{D} \int \frac{d^D x}{x^{2\alpha_1}\left[(y-x)^2\right]^{\alpha_2}} \equiv$$

$=$

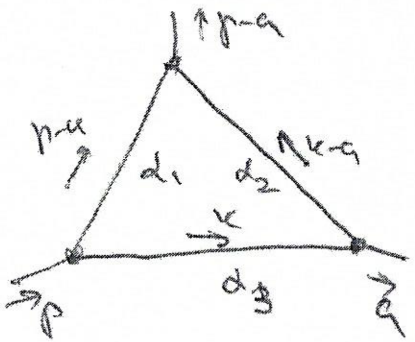
$= \pi^{D/2} A(\alpha_1, \alpha_2)$ 

$\Rightarrow$ same integral but another graphic representation:
in partial, loops $\leftrightarrow$ chains

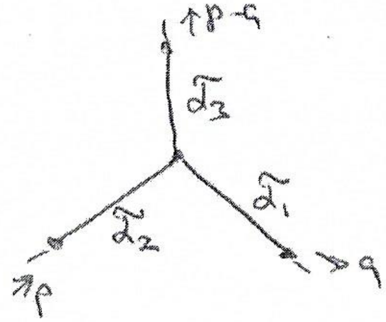
VII Uniquness relation

1.14

$$\boxed{\tilde{\alpha}_i := \tfrac{D}{2} - \alpha_i}$$



$$\underset{\sum_i \alpha_i = D}{\equiv} \pi^{D/2} A(\alpha_1, \alpha_2)$$



$$\int \frac{d^D k}{k^{2\alpha_3} \left[(p-k)^2\right]^{\alpha_1} \left[(k-q)^2\right]^{\alpha_2}}$$

$$\frac{1}{\left[(p-q)^2\right]^{\tilde{\alpha}_3} p^{2\tilde{\alpha}_2} q^{2\tilde{\alpha}_1}}$$

from conformal invariance

Conformal transformations save angles between vectors (really ~~...~~ we will use inversion)

1) To prove uniqueness relation it is convenient to perform the inversion:

$$k = \frac{1}{k_1},\ p = \frac{1}{p_1},\ q = \frac{1}{q_1}$$

$$k^2 = \frac{1}{k_1^2}, \quad (p-k)^2 = p^2 - 2(pk) + k^2 \xrightarrow[\text{transform}]{\text{Conformal}} \frac{1}{p_1^2} - \frac{2(p_1 k_1)}{p_1^2 k_1^2} + \frac{1}{k_1^2} = \frac{(p_1-k_1)^2}{p_1^2 k_1^2} \qquad (1.15)$$

$$d^D k = \tfrac{1}{2}(k^2)^{\overbrace{\frac{D}{2}-1}^{\lambda}} dk^2\, d\Omega$$

$$\int_0^\infty \tfrac{1}{2} k^{2\lambda} dk^2 = \int_\infty^0 \tfrac{1}{2} k_1^{-2\lambda} (-1) \frac{dk_1^2}{(k_1^2)^2} = \int_0^\infty \tfrac{1}{2} k_1^{2\lambda} dk_1^2 \frac{1}{(k_1^2)^D}, \quad \text{So} \quad d^D k = \frac{d^D k_1}{(k_1^2)^D}$$

Then

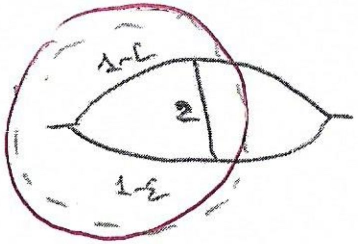
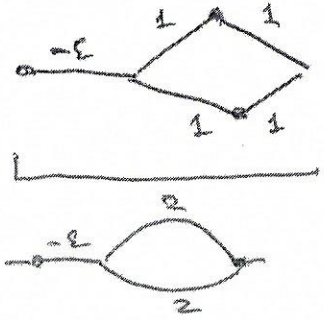
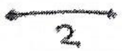
$$\int \frac{d^D k}{k^{2\alpha_3}[(p-k)^2]^{\alpha_1}[(k-q)^2]^{\alpha_2}} = \underbrace{\int \frac{d^D k_1 \cdot \cancel{k_1^{2\alpha_3}} (\cancel{k_1^2} p_1^2)^{\alpha_1} (\cancel{k_1^2} q_1^2)^{\alpha_2}}{\cancel{(k_1^2)^D} [(p_1-k_1)^2]^{\alpha_1} [(q_1-k_1)^2]^{\alpha_2}}}_{=\text{loop}} = \pi^{D/2} A(\alpha_1, \alpha_2).$$

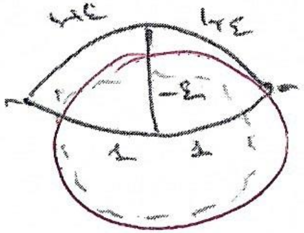
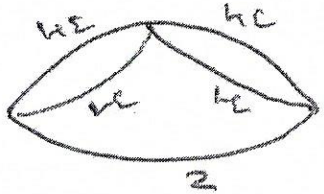
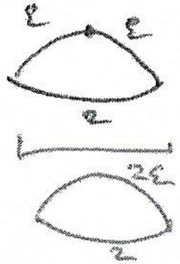
$$\underbrace{\cdot \frac{(p_1^2)^{\alpha_1}(q_1^2)^{\alpha_2}}{[(p_1-q_1)^2]^{\alpha_1+\alpha_2-\frac{D}{2}}}} = (*) \qquad \pi^{D/2} A(\alpha_1,\alpha_2) \frac{1}{p^{2\tilde{\alpha}_2} q^{2\tilde{\alpha}_1} [(p^2-q^2)]^{\tilde{\alpha}_3}}$$

$$\llcorner\!\lrcorner = \frac{1}{p^{2\alpha_1}} \frac{1}{q^{2\alpha_2}} \frac{(p^2 q^2)^{\alpha_1+\alpha_2-\frac{D}{2}}}{[(p^2-q^2)]^{\alpha_1+\alpha_2-\frac{D}{2}}} = \frac{1}{p^{2(\underbrace{\frac{D}{2}-\alpha_2}_{\tilde{\alpha}_2})}} \frac{1}{q^{2(\underbrace{\frac{D}{2}-\alpha_1}_{\tilde{\alpha}_1})}} \cdot \frac{1}{[(p^2-q^2)]^{\underbrace{\alpha_1+\alpha_2-\frac{D}{2}}_{=\frac{D}{2}-\alpha_3=\tilde{\alpha}_3}}}$$

2) Examples

(1.16)

 $= \cancel{\pi^{D/2}}\, A(1-\varepsilon, 1-\varepsilon)$  $= \cancel{\pi^{D}}\, A(1-\varepsilon, 1-\varepsilon)\, A(2,2)$ 

 $=$  $\dfrac{1}{A(1-\varepsilon, 1-\varepsilon)} = \dfrac{A^2(1-\varepsilon, 1+\varepsilon)}{A(1-\varepsilon, 1-\varepsilon)}$  $= \dfrac{A^2(1-\varepsilon, 1+\varepsilon)}{A(1-\varepsilon, 1-\varepsilon)}\, A(2, 2\varepsilon)\,\cdot$ 

we introduce the additional integration

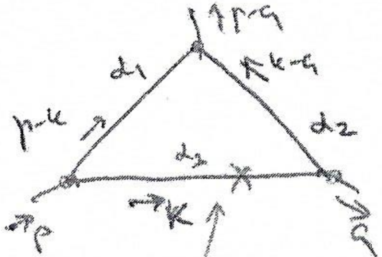
(VIII)

IBP (integration by parts) procedure

Chetyrkin, Tkachov, 1981 (117)
Vassiliev et al., 1981

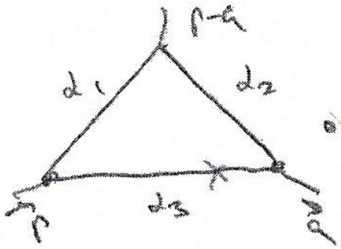
$$\int \frac{d^D k \overbrace{\frac{d}{dk_\mu} k_\mu}^{D}}{\left[k^{2\alpha_3} (k-q)^{2\alpha_2} (k-p)^{2\alpha_1} \right]} = \int d^D k \left[\underbrace{\frac{d}{dk_\mu} \left(\frac{k^\mu}{[\;]} \right)}_{=0 \text{ Stokes theorem}} - k^\mu \frac{d}{dk^\mu} \frac{1}{[\;]} \right]$$

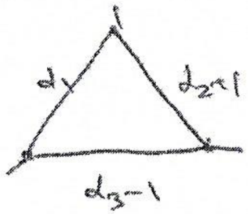
$\equiv$

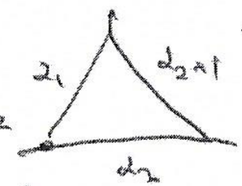


distinguished line

$$\left\{ \begin{aligned} & -k^\mu \frac{d}{dk_\mu} \frac{1}{k^{2\alpha_3}} = \frac{2\alpha_3}{k^{2\alpha_3}} \\ & -k^\mu \frac{d}{dk_\mu} \frac{1}{(k-q)^{2\alpha_2}} = 2\alpha_2 \frac{(k, k-q)}{(k-q)^{2(\alpha_2+1)}} = \alpha_2 \frac{k^2 + (k-q)^2 - q^2}{\left[(k-q)^2 \right]^{\alpha_2+1}} \end{aligned} \right\}$$

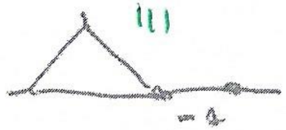


$\cdot (D - 2\alpha_3 - \alpha_1 - \alpha_2) = \alpha_2 \Big[$


$- q^2$


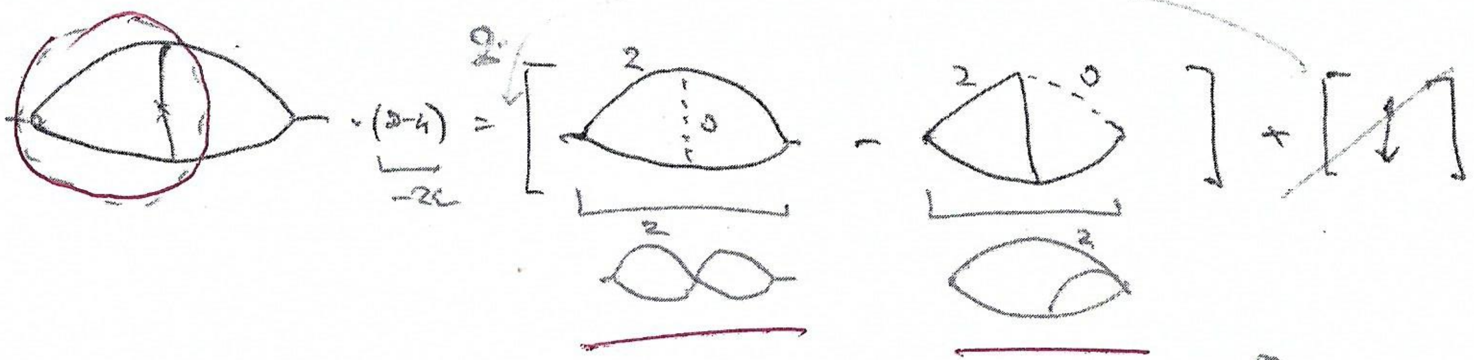
$\Big] + \alpha_1 \Big[\longleftrightarrow \Big]$

$\equiv$

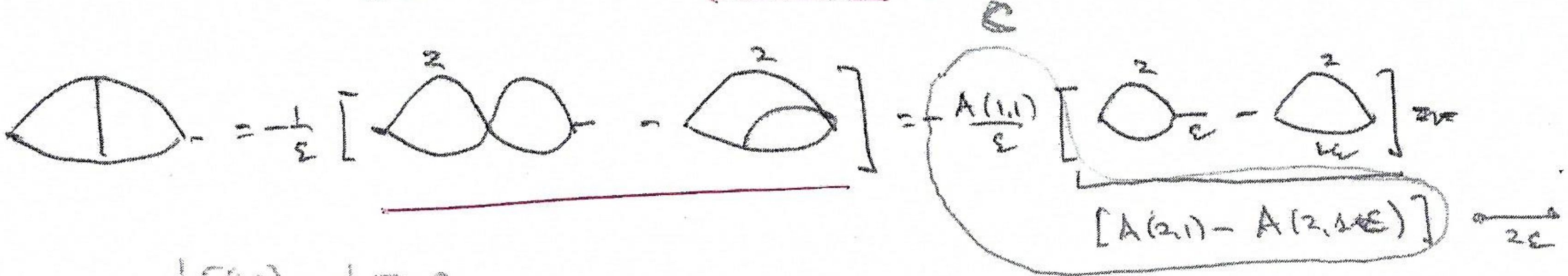


Example:

(1.18)



So



$$C = -\frac{1}{\varepsilon}\,\frac{\Gamma^2(1+\varepsilon)\overbrace{\Gamma(\varepsilon)}^{\frac{1}{\varepsilon}\Gamma(1+\varepsilon)}}{\Gamma^2(1)\underbrace{\Gamma(2-2\varepsilon)}_{(1-2\varepsilon)\Gamma(1-2\varepsilon)}}\left[\frac{\overbrace{\Gamma(-\varepsilon)}^{-\frac{1}{\varepsilon}\Gamma(1-\varepsilon)}\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2)\Gamma(1)\Gamma(1-2\varepsilon)} - \frac{\Gamma(-\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(2)\Gamma(1+\varepsilon)\Gamma(1-3\varepsilon)}\right] = \frac{1}{\varepsilon^3}\,\frac{\Gamma^2(1+\varepsilon)}{(1-2\varepsilon)}\underbrace{\frac{\Gamma^4(1-\varepsilon)}{\Gamma^2(1-2\varepsilon)}}_{K_1}\left[1 - \underbrace{\frac{\Gamma^2(1-2\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-3\varepsilon)\Gamma(1-\varepsilon)}}_{K_2}\right]$$

$$K_1 = \frac{\Gamma^4(1-\varepsilon)}{\Gamma^2(1-2\varepsilon)} = \frac{\exp\left[4\gamma\varepsilon + 4\sum_{k\geq 2}^{\infty}\frac{\zeta(k)}{k}\varepsilon^k\right]}{\exp\left[4\gamma\varepsilon + 2\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}2^k\varepsilon^k\right]} = \exp\left[4\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}\underbrace{(1-2^{k-1})}_{q_1(k)}\varepsilon^k\right]$$

$q_1(2)=0,\ q_1(3)=-3,\ q_1(4)=-7,\ q_1(5)=-15,\ q_1(6)=-31$ homework

(1.19)

$$K_2 = \frac{\Gamma^2(1-2\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-3\varepsilon)\Gamma(1-\varepsilon)} = \frac{\exp\left[2\gamma\varepsilon + \sum_{k=2}^{\infty}\frac{\zeta(k)}{k}\left[2^{k+1}+(-2)^k\right]\varepsilon^k\right]}{\exp\left[2\gamma\varepsilon + \sum_{k=2}^{\infty}\frac{\zeta(k)}{k}\left[2(-1)^k+3^k+1\right]\varepsilon^k\right]} = \exp\left[\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}\underbrace{\left[2^{k+1}-1-3^k+2(-1)^k(2^{k-1}-1)\right]}_{q_2(k)}\varepsilon^k\right] \quad (+)$$

$q_2(2)=0$, $q_2(3)=-18$, $q_2(4)=-36$, $q_2(5)=-240$, $q_2(6)=-630$] homework

$$(5) = \exp\left[-6\zeta(3)\varepsilon^3 - 9\zeta(4)\varepsilon^4 - 48\zeta(5)\varepsilon^5 - 105\zeta(6)\varepsilon^6 + O(\varepsilon^7)\right] \simeq \left(1-6\zeta(3)\varepsilon^3+18\zeta^2(3)\varepsilon^6\right).$$

$$\left(1-9\zeta(4)\varepsilon^4-48\zeta(5)\varepsilon^5-105\zeta(6)\varepsilon^6\right) + O(\varepsilon^7) = 1-6\zeta(3)\varepsilon^3-9\zeta(4)\varepsilon^4-48\zeta(5)\varepsilon^5 -$$

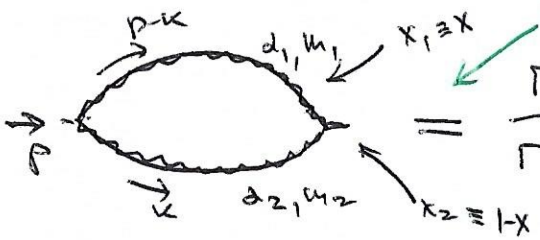
$$-\left(105\zeta(6)-18\zeta^2(3)\right)\varepsilon^6\Big]$$

$$C = \frac{\Gamma^2(1+\varepsilon)}{(1-2\varepsilon)}\left[6\zeta(3)+9\zeta(4)\varepsilon+48\zeta(5)\varepsilon^2+\underbrace{\left(105\zeta(6)-\cancel{18}^{\,42}\zeta^2(3)\right)\varepsilon^3}_{21\left(5\zeta(6)-2\zeta^2(3)\right)\varepsilon^3}\right]\left(1-4\cancel{\zeta(3)}\varepsilon^3\right)$$

II Evaluating diagrams with masses

One-loop

Using Feynman parameters



Eq. (3a)

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1} \overbrace{\int \frac{d^D k}{[\ ]_1^{\alpha_1+\alpha_2}}} = \circledast$$

Eq. (2) $\pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1+\alpha_2)} \frac{1}{(m^2_{1,eff})^{\alpha_1+\alpha_2-D/2}}$

$$[\ ]_1 = x\{\underbrace{(p-k)^2}_{p^2-2pk+k^2}+m_1^2\} + (1-x)\{k^2+m_2^2\} = k^2 + p^2 x - 2(pk)x + \underbrace{m_1^2 x + m_2^2(1-x)}_{m^2_{eff}}$$

$$m^2_{1,eff} = m^2_{eff} - p^2x^2 = p^2 x(1-x) + m_1^2 x + m_2^2(1-x) = x(1-x)\Big[p^2 + \underbrace{\frac{m_1^2}{1-x} + \frac{m_2^2}{x}}_{\mu^2}\Big]$$

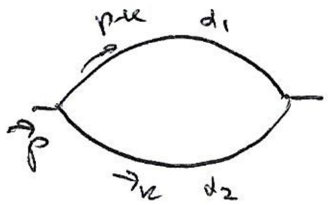
$$\frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[x(1-x)]^{\alpha_1+\alpha_2-D/2}}$$

$$\circledast = \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx \cdot \frac{x^{\overbrace{D/2-1-\alpha_2}^{\lambda}} (1-x)^{\overbrace{D/2-1-\alpha_1}^{\lambda}}}{(p^2+\mu^2)^{\alpha_1+\alpha_2-D/2}} = \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot \int_0^1 dx \cdot \overbrace{x^{D/2-1-\alpha_2} (1-x)^{D/2-1-\alpha_1}} \cdot$$

$$\cdot \underbrace{\quad}^{\alpha_1+\alpha_2-D/2} \qquad \mu^2 = \frac{m_1^2}{1-x} + \frac{m_2^2}{x}$$

|| Loop as the integral from the ~~tadpole~~ propagator with the effective mass μ;
$\Rightarrow$ L-loop as integral from (L-1) loop with the effective mass μ.

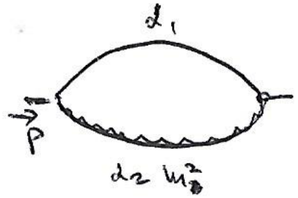
1) massless case: $\mu^2=0$ (we considered already)



$$= \pi^{D/2}\,\frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\,\frac{1}{(p^2)^{\alpha_1+\alpha_2-\frac{D}{2}}}\underbrace{\int_0^1 dx\cdot x^{\overbrace{\frac{D}{2}-1-\alpha_2}^{\tilde{\alpha}_2-1}}(1-x)^{\overbrace{\frac{D}{2}-1-\alpha_1}^{\tilde{\alpha}_1-1}}}_{\frac{\Gamma(\frac{D}{2}-\alpha_1)\Gamma(\frac{D}{2}-\alpha_2)}{\Gamma(D-\alpha_1-\alpha_2)}} = \pi^{D/2} A(\alpha_1,\alpha_2)\,\frac{1}{(p^2)^{\alpha_1+\alpha_2-\frac{D}{2}}} \qquad (4)$$

$$A(\alpha_1,\alpha_2)=\frac{a(\alpha_1)a(\alpha_2)}{a(\alpha_1+\alpha_2-\frac{D}{2})}\,,\quad a(\alpha)=\frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}\,,\quad \tilde{\alpha}=\frac{D}{2}-\alpha$$

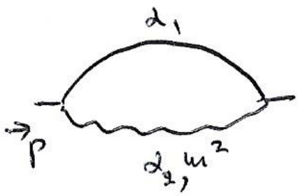
2) the case $m_1^2=0$, $m_2^2=m^2$



$$= \pi^{D/2}\,\frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\overbrace{\int_0^1 dx\cdot\frac{x^{\frac{D}{2}-1-\alpha_2}(1-x)^{\frac{D}{2}-1-\alpha_1}}{[p^2+\frac{m^2}{x}]^{\alpha_1+\alpha_2-\frac{D}{2}}}}^{\int_0^1 dx\cdot\frac{x^{\alpha_1-1}(1-x)^{\frac{D}{2}-1-\alpha_1}}{[p^2x+m^2]^{\alpha_1+\alpha_2-\frac{D}{2}}}}$$

~~Important on-shell case: $p^2=-m^2 \Rightarrow [p^2x+m^2]=m^2(1-x)$~~

(2a) Important on-shell case: $p^2 = -m^2 \Rightarrow [p^2 x + m^2] = m^2(1-x)$



$$= \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\cancel{\Gamma(\alpha_1)}\Gamma(\alpha_2)} \frac{1}{(m^2)^{\alpha_1+\alpha_2-D/2}} \underbrace{\int_0^1 dx \cdot x^{\alpha_1-1}(1-x)^{D-1-2\alpha_1-\alpha_2}}_{\frac{\cancel{\Gamma(\alpha_1)}\Gamma(D-2\alpha_1-\alpha_2)}{\Gamma(D-\alpha_1-\alpha_2)}} =$$

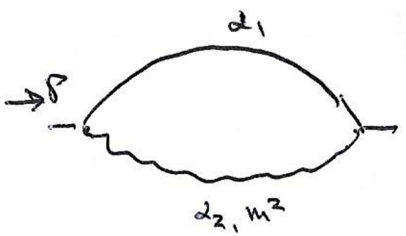
$$= \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-D/2)\,\Gamma(D-2\alpha_1-\alpha_2)}{\Gamma(\alpha_2)\,\Gamma(D-\alpha_1-\alpha_2)} \frac{1}{(m^2)^{\alpha_1+\alpha_2-D/2}}$$

(2b) Inverse mass expansion: $\frac{1}{[p^2x+m^2]^{\alpha_1+\alpha_2-D/2}} = \frac{1}{(m^2)^{\alpha_1+\alpha_2-D/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha_1+\alpha_2-D/2)}{n!\,\Gamma(\alpha_1+\alpha_2-D/2)} (-1)^n \left(\frac{p^2 x}{m^2}\right)^n$

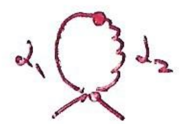
$$\int_0^1 \frac{dx\, x^{\alpha_1-1}(1-x)^{D/2-\alpha_2-\alpha_1}}{[p^2x+m^2]^{\alpha_1+\alpha_2-D/2}} = \frac{1}{(m^2)^{\alpha_1+\alpha_2-D/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha_1+\alpha_2-D/2)}{n!\,\Gamma(\alpha_1+\alpha_2-D/2)} \frac{\Gamma(\alpha_1+n)\Gamma(D/2-\alpha_1)}{\Gamma(D/2+n)} (-1)^n \left(\frac{p^2}{m^2}\right)^n =$$

$$= \frac{1}{(m^2)^{\alpha_1+\alpha_2-D/2}} \underbrace{{}_2F_1\left[\begin{matrix}\alpha_1, \alpha_1+\alpha_2-D/2 \\ D/2\end{matrix} \middle| -\frac{p^2}{m^2}\right]}_{{}_2F_1 \text{ - hypergeometric function}} \cdot \frac{\Gamma(\alpha_1)\Gamma(D/2-\alpha_1)}{\Gamma(D/2)}$$

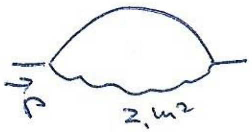
2.5



$$= \pi^{D/2} \overbrace{\underbrace{\frac{\Gamma(\frac{D}{2}-\alpha_1)\,\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_2)\,\Gamma(\frac{D}{2})}}_{B(\alpha_1,\alpha_2)} \frac{1}{(m^2)^{\alpha_1+\alpha_2-\frac{D}{2}}}} \; {}_2F_1\left[\begin{matrix}\alpha_1,\ \alpha_1+\alpha_2-\frac{D}{2}\\ \frac{D}{2}\end{matrix}\ \middle|\ -\frac{p^2}{m^2}\right]$$

$B(\alpha_1,\alpha_2) \leftarrow$ (see tadpole )

Let $\alpha_1 = 1,\ \alpha_2 = 1$



$$= \frac{\pi^{D/2}}{(1-\varepsilon)} \frac{\Gamma(1+\varepsilon)}{(m^2)^{1+\varepsilon}} \; {}_2F_1\left[\begin{matrix}1,\ 1+\varepsilon\\ 2-\varepsilon\end{matrix}\ \middle|\ \underbrace{-\frac{p^2}{m^2}}_{x}\right] = (*)$$

$${}_2F_1\left[\begin{matrix}1,\ 1+\varepsilon\\ 2-\varepsilon\end{matrix}\ \middle|\ -x\right] = \sum_{k=0}^{\infty} \frac{\Gamma(k+1+\varepsilon)}{k!\,\Gamma(1+\varepsilon)} \frac{\Gamma(2-\varepsilon)}{\Gamma(k+2-\varepsilon)} (-x)^k = (1-\varepsilon) \underbrace{\sum_{k=0}^{\infty} \frac{(-x)^k}{k+1-\varepsilon} \frac{\Gamma(k+1+\varepsilon)\,\Gamma(1-\varepsilon)}{\Gamma(k+1-\varepsilon)\,\Gamma(1+\varepsilon)}}$$

$$\frac{(1-\varepsilon)}{(-x)} \sum_{k=1}^{\infty} \frac{(-x)^k}{k-\varepsilon} \frac{\Gamma(k+\varepsilon)\,\Gamma(1-\varepsilon)}{\Gamma(k-\varepsilon)\,\Gamma(1+\varepsilon)} = (*) = \frac{(1-\varepsilon)}{(-x)} \sum_{m=0}^{\infty} \varepsilon^m \varphi_m(-x)$$

$$(*) = \frac{\pi^{D/2}}{(m^2)^{\varepsilon}} \frac{\Gamma(1+\varepsilon)}{(-p^2)} \sum_{m=0}^{\infty} \varepsilon^m \varphi_m(-x)$$

2.6

$$\frac{\Gamma(k+\varepsilon)}{\Gamma(k)\Gamma(1+\varepsilon)} = \exp\Big[-\sum_{m=1}^{\infty}\frac{(-\varepsilon)^m}{m}\underbrace{S_m(k-1)}_{\equiv S_m}\Big]$$

harmonic sums

$$S_{\pm m}(k) = \sum_{\ell=1}^{k}\frac{(\pm 1)^{\ell}}{\ell^m}\,,\quad \underbrace{S_{\pm m_1,\pm m_2,\dots}(k)}_{\pm m_1,\pm m_2,\dots} = \sum_{\ell=1}^{k}\frac{(\pm 1)^{\ell}}{\ell^{m_1}}\underbrace{S_{\pm m_2,\dots}(\ell)}_{\pm m_2,\dots}$$

nested sums

$$\frac{\Gamma(k+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(k-\varepsilon)\Gamma(1+\varepsilon)} = \frac{\exp\Big[-\sum\limits_{m=1}^{\infty}\frac{(-\varepsilon)^m}{m}S_m\Big]}{\exp\Big[-\sum\limits_{m=1}^{\infty}\frac{\varepsilon^m}{m}S_m\Big]} = \exp\Big[\sum_{m=1}^{\infty}\frac{(1-(-)^m)}{m}\varepsilon^m S_m\Big] \underset{m=2\ell+1}{=} \exp\Big[2\sum_{\ell=0}^{\infty}\frac{\varepsilon^{2\ell+1}}{2\ell+1}S_{2\ell+1}\Big] =$$

$$= \exp\Big[2\varepsilon S_1 + \tfrac{2}{3}\varepsilon^3 S_3 + O(\varepsilon^4)\Big] = \Big(1 + 2\varepsilon S_1 + \underbrace{\tfrac{\varepsilon^2}{2}2^2 S_1^2}_{2\varepsilon^2 S_1^2} + \underbrace{\tfrac{\varepsilon^3}{3!}2^3 S_1^3}_{\frac{4}{3}S_1^3\varepsilon^3}\Big)\Big(1 + \tfrac{2}{3}\varepsilon^3 S_3\Big) \overset{+O(\varepsilon^4)}{=} \Big(1 + 2\varepsilon S_1 + 2\varepsilon^2 S_1^2 + \tfrac{2}{3}\varepsilon^3(2S_1^3 + S_3)\Big) + O(\varepsilon^4)$$

$$\sum_{k=1}^{\infty}\frac{z^k}{k-\varepsilon}\frac{\Gamma(k+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(k-\varepsilon)\Gamma(1+\varepsilon)} = \sum_{k=1}^{\infty}\frac{z^k}{k}\Big(1 + \varepsilon\big(2S_1 + \tfrac{1}{k}\big) + \varepsilon^2\big(2S_1^2 + \tfrac{2S_1}{k} + \tfrac{1}{k^2}\big) + \varepsilon^3\Big(\tfrac{2}{3}(2S_1^3 + S_3) + \tfrac{2S_1^2}{k} + \tfrac{2S_1}{k^2} + \tfrac{1}{k^3}\Big)\Big) + O(\varepsilon^4) =$$

$$\frac{1}{k-\varepsilon} = \frac{1}{k(1-\varepsilon/k)} = \frac{1}{k}\Big(1 + \frac{\varepsilon}{k} + \frac{\varepsilon^2}{k^2} + \frac{\varepsilon^3}{k^3}\Big)$$

$$= \sum_{m=0}^{\infty}\varepsilon^m \varphi_m(z)$$

Fleischer, A.V.K., Veretin, 1998

1.11

2.7

$$\sum_{k=1}^{\infty} \frac{z^k}{k^a} = Li_a(z)\,, \quad \sum_{k=1}^{\infty} \frac{S_1}{k^a} z^k = S_{a-1,2}(z)$$

$$\underbrace{Li_1(z) = -\ln(1-z)}_{\equiv L_1(z)} \qquad S_0(z) = \tfrac{1}{2}\ln^2(1-z) = \tfrac{1}{2}\underbrace{Li_1^2(z)}_{\equiv L_1^2(z)}$$

Nielsen

$$S_{a+1,b}(z) = \frac{(-)^{a+b}}{a!\,b!}\int_0^1 \frac{\ln^a(t)\ln^b(1-zt)}{t}\,dt\,, \quad \overset{\overset{L_a(z)}{=}}{Li_a}(z) = S_{a-1,1}(z)$$

— Polylogarithmic functions

$$\sum_{k=1}^{\infty} \frac{S_1^2}{k} z^k = L_1L_2 + \tfrac{1}{3}L_1^3 - 2S_{1,2}\,, \quad \sum_{k=1}^{\infty} \frac{S_1^2}{k^2} z^k = \tfrac{1}{2}L_2^2 + 2S_{1,3} - 2S_{2,2}\,, \quad \sum_{k=1}^{\infty} \frac{S_3}{k} z^k = L_1L_3 - \tfrac{1}{2}L_2^2\,,$$

$$\sum_{k=1}^{\infty} \frac{S_1^3}{k} z^k = \tfrac{1}{4}L_1^4 + L_1L_3 - \tfrac{1}{2}L_2^2 + \tfrac{3}{2}L_1^2L_2 - 3L_1S_{1,2}$$

$$\sum_{k=1}^{\infty} \frac{z^k}{k} = L_1(z) = -\ln(1-z) \equiv \varphi_0(z)$$

$$\sum_{k=1}^{\infty} \frac{z^k}{k}\left(2S_1 + \frac{1}{k}\right) = L_1^2(z) + L_2(z) \equiv \varphi_1(z)$$

$$\sum_{k=1}^{\infty} \frac{z^k}{k}\left(2S_1^2 + \frac{2S_1}{k} + \frac{1}{k^2}\right) = 2\left[L_1L_2 + \tfrac{1}{3}L_1^3 - 2S_{1,2}\right] + 2S_{1,2} + L_4 = 2\left[\tfrac{1}{3}L_1^3 + L_1L_2 - S_{1,2}\right] + L_4 \equiv \varphi_2(z)$$

$$\sum_{k=1} \frac{z^k}{k}\left[\tfrac{2}{3}(2S_1^3 + S_3) + \frac{2S_1^2}{k} + \frac{2S_1}{k^2} + \frac{1}{k^3}\right] = \tfrac{2}{3}\Big[2\left(\tfrac{1}{4}L_1^4 + L_1L_3 - \tfrac{1}{2}L_2^2 + \tfrac{3}{2}L_1^2L_2 - 3L_1S_{1,2}\right) + L_1L_3 - \tfrac{1}{2}L_2^2\Big] +$$

$$+ 2\left\{\tfrac{1}{2}L_2^2 + 2S_{1,3} - 2S_{2,2}\right\} + 2S_{2,2} + L_4 = \tfrac{2}{3}\left[\tfrac{1}{2}L_1^4 + 3L_1^2L_2 + 3L_1L_3 - 6L_1S_{1,2} - \cancel{\tfrac{3}{2}L_2^2}\right] + 2\left[\cancel{\tfrac{1}{2}L_2^2} + 2S_{1,3} - S_{2,2}\right] + L_4$$

$$\equiv \varphi_3(z)$$

Now there are popular the Remiddi-Vermaseren (RV) Polynomials and
the Gonzharov (G) Polynomials

RV polynomials :

$$H_0(z) = \int^z \frac{dz}{z} \equiv \ln z \, ; \quad H_1(z) = \int_0^z \frac{dz'}{1-z'} \equiv -\ln(1-z) \, , \quad H_{-1}(z) = \int_0^z \frac{dz'}{1+z'} = \ln(1+z)$$

$$H_{0,b,\dots}(z) = \int_0^z \frac{dz}{z} H_{b,\dots}(z) \, ; \quad H_{1,b,\dots}(z) = \int_0^z \frac{dz'}{1-z'} H_{b,\dots}(z)$$

$$H_{-1,b,\dots}(z) = \int_0^z \frac{dz'}{1+z'} H_{b,\dots}(z) \, ;$$

Properties $\quad H_{\underbrace{1,\dots,1}_{m}}(z) = \frac{(-1)^m}{m!} \ln^m(1-z), \quad H_{\underbrace{-1,\dots,-1}_{m}} = \frac{1}{m!} \ln^m(1+z)$

G polylogarithms $\quad G_{a,b,\dots}(z) = \int_0^z \frac{dz'}{z'-a} G_{b,\dots}(z')$ - generalization of RV polylogarithms

$$L_1 \equiv Li_1(z) = -\ln(1-z) = H_1(z) \quad , \quad L_2 \equiv Li_2(z) = -\int_0^1 \frac{dt}{t}\ln(1-zt) = -\int_0^z \frac{dz'}{z'}\ln(1-z') = \int_0^z \frac{dz'}{z'} H_1(z') = H_{0,1}(z)$$

$$\| \int_0^z \frac{dt}{t} \underbrace{Li_1(t)}_{H_1(z)} = H_{0,1}(z)$$

$$\begin{array}{c} Li_3(z) \\ \| \\ L_3 \end{array} = \int_0^z \frac{dt}{t} L_2(t) = \int_0^z \frac{dt}{t} H_{0,1}(t) = H_{0,0,1}(z) \quad , \quad \begin{array}{c} L_4 \\ \| \\ Li_4(z) \end{array} = H_{0,0,0,1}(z) \qquad \Rightarrow L_k = H_{\underbrace{0,\dots,0}_{k-1},1}(z)$$

$$S_{0,2}(z) = \frac{1}{2}\ln^2(1-z) = -\int_0^z \frac{dz'}{1-z'}\ln(1-z') = \int_0^z \frac{dz'}{1-z'} H_1(z') = H_{1,1}(z) \quad (\text{Ok, property})$$

$$\int_0^z \frac{dz'}{1-z'}\ln(1-z') = \int_{1-z}^1 \frac{dt}{t}\ln t = \frac{1}{2}\ln^2 t\Big|_{1-z}^1 = -\frac{1}{2}\ln^2(1-z)$$

$$S_{1,2}(z) = \int_0^z \frac{dt}{t} S_{0,2}(t) = H_{0,1,1}(z) \quad , \quad S_{2,2}(z) = \int_0^z \frac{dt}{t} S_{1,2}(t) = H_{0,0,1,1}(z)$$

$$S_{0,3}(z) = -\frac{1}{6}\ln^3(1-z) = \frac{1}{2}\int_0^z \frac{dz'}{1-z'}\ln^2(1-z') = \int_0^z \frac{dz'}{1-z'} H_{1,1}(z') = H_{1,1,1}(z) \quad (\text{Ok})$$

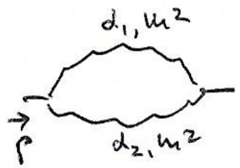
$$\int_0^z \frac{dz'}{1-z'}\ln^2(1-z') = \int_{1-z}^1 \frac{dt}{t}\ln^2 t = \frac{1}{3}\ln^3 t\Big|_{1-z}^1 = -\frac{1}{3}\ln^3(1-z)$$

$$S_{1,3}(z) = \int_0^z \frac{dt}{t} S_{0,3}(t) = H_{0,1,1,1}(z)$$

So, we expressed all elements of our results through RV polylogarithms !!!

3) the case $m_1^2 = m_2^2 = m^2$

$$\mu^2 = \frac{m^2}{x} + \frac{m^2}{1-x} = \frac{m^2}{x(1-x)} \qquad p^2 + \mu^2 = \frac{1}{x(1-x)}\left[p^2 x(1-x) + m^2\right]$$



$$= \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \frac{dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{\left[p^2 x(1-x)+m^2\right]^{\alpha_1+\alpha_2-\frac{D}{2}}} = (*)$$

$$\frac{1}{\left[p^2 x(1-x)+m^2\right]^{\alpha_1+\alpha_2+\frac{D}{2}}} = \frac{1}{(m^2)^{\alpha_1+\alpha_2-\frac{D}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha_1+\alpha_2-\frac{D}{2})}{n!\,\Gamma(\alpha_1+\alpha_2-\frac{D}{2})} (-)^n \left[\frac{p^2 x(1-x)}{m^2}\right]^n$$

$$\alpha_1 + \alpha_2 \equiv \alpha$$

$$\int_0^1 \frac{dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{\left[p^2 x(1-x)+m^2\right]^{\alpha_1+\alpha_2-\frac{D}{2}}} = \frac{1}{(m^2)^{\alpha_1+\alpha_2-D/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha_1+\alpha_2-\frac{D}{2})}{n!\,\Gamma(\alpha_1+\alpha_2-\frac{D}{2})} \frac{\Gamma(\alpha_1+n)\Gamma(\alpha_2+n)}{\underbrace{\Gamma(\overbrace{\alpha_1+\alpha_2}^{\alpha}+2n)}_{\frac{2^{(\alpha_1+\alpha_2+2n)-1}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha_1+\alpha_2}{2}+n\right)\Gamma\left(\frac{\alpha+1}{2}+n\right)}} (-)^n \left(\frac{p^2}{m^2}\right)^n =$$

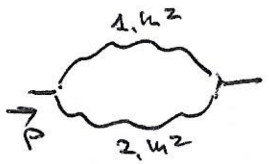
$$\Gamma(2\alpha) = \frac{2^{2\alpha-1}}{\sqrt{\pi}}\Gamma(\alpha)\Gamma(\alpha+\tfrac{1}{2})$$

$$= \frac{1}{(m^2)^{\alpha_1+\alpha_2-\frac{D}{2}}} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\sqrt{\pi}}{2^{\alpha-1}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+1}{2})} {}_3F_2\left[\begin{matrix}\frac{\alpha_1}{2}, \frac{\alpha_2}{2}, \alpha_1+\alpha_2-\frac{D}{2} \\ \frac{\alpha}{2}, \frac{\alpha+1}{2}\end{matrix}\middle| -\frac{p^2}{4m^2}\right]$$

$$(*) = \frac{\pi^{D/2}}{(m^2)^{\alpha_1+\alpha_2-\frac{D}{2}}} \frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})\sqrt{\pi}}{2^{\alpha-1}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+1}{2})} {}_3F_2\left[\begin{matrix}\alpha_1, \alpha_2, \alpha_1+\alpha_2-\frac{D}{2} \\ \frac{\alpha}{2}, \frac{\alpha+1}{2}\end{matrix}\middle| -\frac{p^2}{4m^2}\right]$$

(2.11)

$\alpha_1 = 1\ ,\ \alpha_2 = 2 \Rightarrow \alpha = 3$



$$= \frac{\pi^{3/2}}{(m^2)^{1+\varepsilon}} \underbrace{\frac{\Gamma(1+\varepsilon)\sqrt{\pi}}{2^2\,\Gamma(3/2)\,\Gamma(2)}}_{\Gamma(1+\varepsilon)/2}\ {}_{\cancel{3}2}F_{\cancel{2}1}\left[\begin{matrix}1,\ \cancel{2},\ 1+\varepsilon \\ 3/2,\ \cancel{2}\end{matrix}\ \Big|\ -\frac{p^2}{4m^2}\right] = (*) = \frac{\Gamma(1+\varepsilon)\,\pi^{3/2}}{(m^2)^{1+\varepsilon}}\frac{1}{(-p^2)} \sum_{m=0}^{\infty} \varepsilon^m \bar{\varphi}_m \cdot \rho$$

$${}_2F_1\left[\begin{matrix}1,\ 1+\varepsilon \\ 3/2\end{matrix}\ \Big|\ -\frac{x}{4}\right] = \sum_{k=0}^{\infty} \frac{\Gamma(k+1+\varepsilon)}{\Gamma(1+\varepsilon)} \frac{\Gamma(3/2)}{\Gamma(k+3/2)}\left(-\frac{x}{4}\right)^k = (*) = \sum_{k=0}^{\infty}\frac{\Gamma(k+1+\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\Gamma(k+1)}{\Gamma(2k+2)}(-x)^k = \frac{1}{(-x)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\Gamma(k)}{\Gamma(2k)}(-x)^k =$$

$$\Gamma(2k+2) = \frac{2^{2k+1}}{\sqrt{\pi}}\Gamma(k+1)\Gamma(k+3/2) \Rightarrow \frac{1}{\Gamma(k+3/2)} = \frac{2^{2k+1}}{\sqrt{\pi}}\frac{\Gamma(k+1)}{\Gamma(2k+2)} \Rightarrow \frac{\Gamma(3/2)}{\Gamma(k+3/2)} = \frac{2^{2k}\,\Gamma(k+1)}{\Gamma(2k+2)}$$

$$\binom{k}{2k}^{-1} = \frac{(k!)^2}{(2k)!} = \frac{\Gamma^2(k+1)}{\Gamma(2k+1)} = \frac{\Gamma^2(k)}{\Gamma(2k)}\cdot\frac{k}{2} \qquad \Big| = \frac{2}{(-x)}\sum_{k=1}^{\infty}\binom{k}{2k}^{-1}\frac{1}{k}\frac{\Gamma(k+\varepsilon)}{\Gamma(k)\Gamma(1+\varepsilon)}(-x)^k = \frac{2}{(-x)}\sum_{m=0}^{\infty}\varepsilon^m\bar{\varphi}_m\cdot\rho$$

$$\frac{\Gamma(k+\varepsilon)}{\Gamma(k)\Gamma(1+\varepsilon)} = \exp\Big[-\sum_{m=1}^{\infty}\frac{(-\varepsilon)^m}{m}\underbrace{S_m(k-1)}_{\equiv S_m}\Big] = \exp\Big[S_1\varepsilon - \frac{\varepsilon^2}{2}S_2 + O(\varepsilon^3)\Big] = 1 + S_1\varepsilon + \frac{\varepsilon^2}{2}\left[S_1 - S_2\right] + O(\varepsilon^3)$$

$$y = \frac{1-\sqrt{\frac{x}{x+4}}}{1+\sqrt{\frac{x}{x+4}}}\ ,\ \rho = \frac{1-y}{1+y}$$

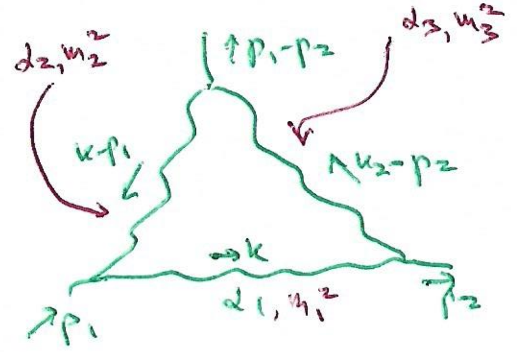
$$\bar{\varphi}_0 = \ln y \equiv \ell\ ,\ \bar{\varphi}_1 = \frac{1}{2}\ell^2 - \zeta_2 + 2\ell\bar{L}_1 - 2\bar{L}_2\ ,\ \bar{L}_2 \equiv \mathrm{Li}_2(-y)\ ,\ \bar{L}_1 \equiv \mathrm{Li}_1(-y) = -\ln(1+y)$$

$$\bar{\varphi}_2 = \frac{1}{3}\ell^3 + 2L_1\ell^2 + 2\ell\left[2L_1^2 - \zeta_2\right] - 4L_1\left[\zeta_2 + L_2\right] - 4\zeta_3 - 4L_3 + 8S_{1,2}(-y)\ ,\ L_3 \equiv \mathrm{Li}_3(-y)$$

Davydychev, Kalmykov, 2004

IBP relation:

Consider the triangle:



$$\int \frac{d^D k}{\{\ \}} \cdot \left[D = \frac{\partial}{\partial k_\mu} k_\mu \right]$$

$$\{\ \} = (k^2+m_1^2)^{\alpha_1} \left((k-p_1)^2+m_2^2\right)^{\alpha_2} \left((k-p_2)^2+m_3^2\right)^{\alpha_3}$$

IBP relation: $$\frac{1}{\{\ \}} \frac{\partial}{\partial k_\mu} k_\mu = \underbrace{\frac{\partial}{\partial k_\mu} \left\{ \frac{k_\mu}{\{\ \}} \right\}}_{\text{boundary integral} = 0} - k_\mu \frac{\partial}{\partial k_\mu} \frac{1}{\{\ \}}$$

$\to \infty$

$$k_\mu \frac{\partial}{\partial k_\mu} \frac{1}{(k^2+m^2)^\alpha} = -2\alpha\, k_\mu \frac{k_\mu}{(\ )^{\alpha+1}} = \frac{-2\alpha\, k^2}{(\ )^{\alpha+1}} = -2\alpha \left[\frac{1}{(\ )^\alpha} - \frac{m^2}{(\ )^{\alpha+1}} \right]$$

$$k_\mu \frac{\partial}{\partial k_\mu} \frac{1}{((k-p)^2+m^2)^\alpha} = -2\alpha \frac{(k, k-p)}{(\ )^{\alpha+1}} = -\alpha \left[\frac{1}{(\ )^\alpha} - \underline{\frac{m^2}{(\ )^{\alpha+1}}} + \frac{k^2+\mu^2}{(\ )^{\alpha+1}} - \underline{\frac{(p^2+\mu^2)}{(\ )^{\alpha+1}}} \right]$$

$2(k, k-p) = k^2 + (k-p)^2 - p^2$

$$D\, \triangle^{\alpha_2\ \alpha_3}_{\alpha_1;\ \vec p_1\ \vec p_2} = 2\alpha_1 \left[\underline{\triangle} - m_1^2\, \triangle_{+} \right] + \alpha_2 \left[\underline{\triangle} + {}^{+}\triangle_{-} - (p_1^2+m_1^2+m_2^2)\, {}^{+}\triangle \right] +$$

$$+\ \alpha_3 \left[\leftrightarrow \right]$$

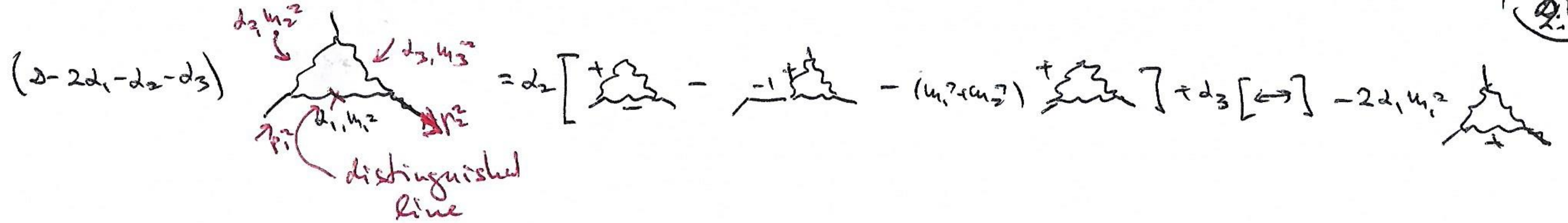
distinguished line

$$(D - 2\alpha_1 - \alpha_2 - \alpha_3)\, \triangle_{\times} = \alpha_2 \left[{}^{+}\triangle_{-} - {}^{-1\ +}\triangle - \underline{(m_1^2+m_2^2)\, {}^{+}\triangle} \right]$$

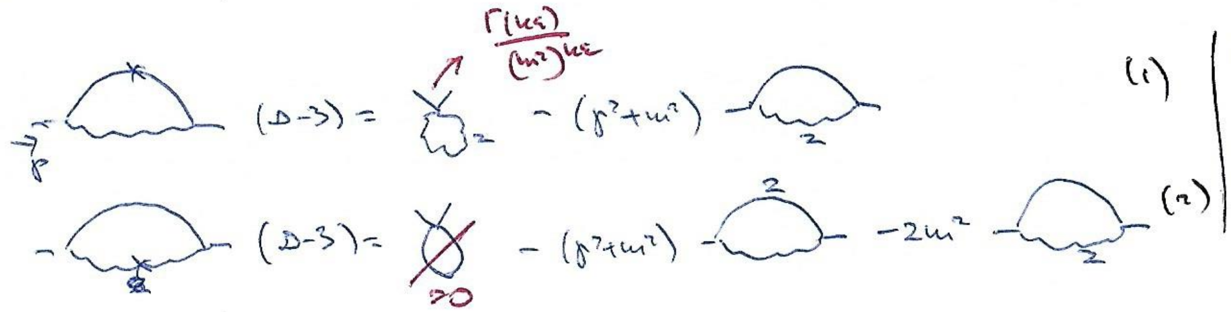
$$+\ \alpha_3 \left[\leftrightarrow \right] - \underline{2\alpha_1 m_1^2\, \triangle_{+}}$$

massive propagators

additional terms in massive case !!!



Ⓐ One-loop with $m_1^2=0$, and $m_2^2=m^2$

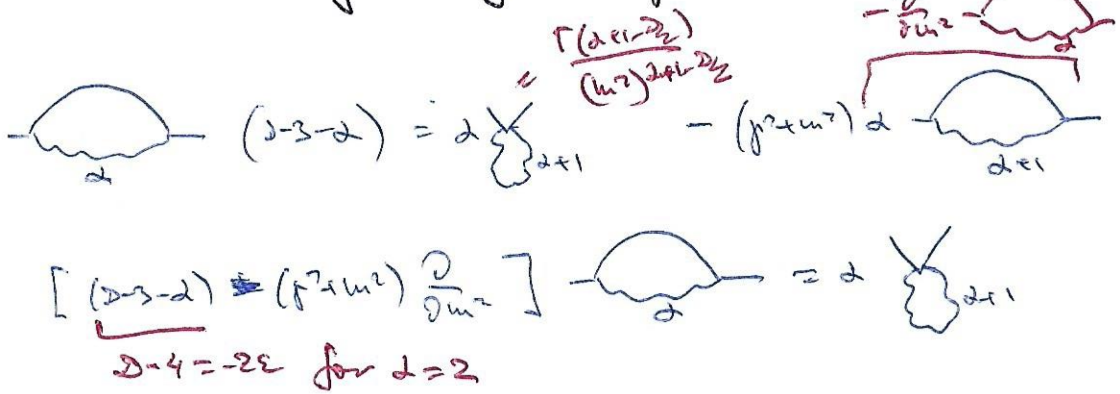


Only one one-loop diagram is independent

It is master diagram

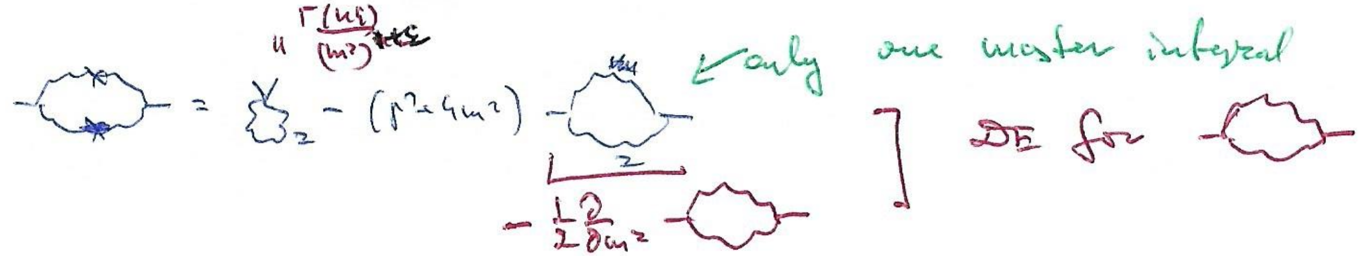
We calculate  ← finite

For master-integral IBP gives differential equation (DE) with the inhomogeneous term containing only tadpole



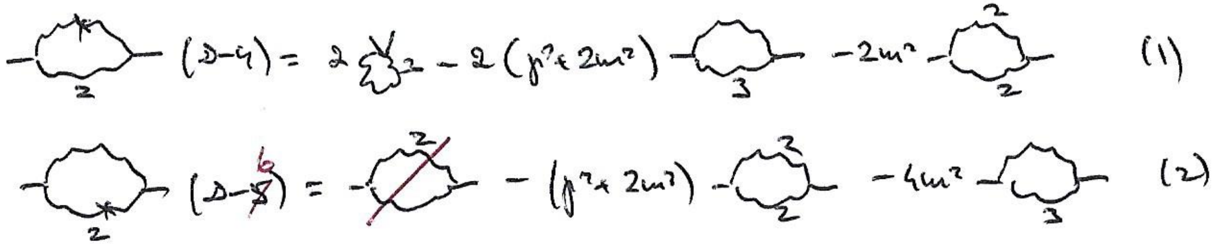
, because $\frac{\alpha}{(k^2+m^2)^{\alpha+1}} = -\frac{\partial}{\partial m^2}\frac{1}{(k^2+m^2)^{\alpha}}$

(B) One-loop with $m_1^2 = m_2^2 = m^2$

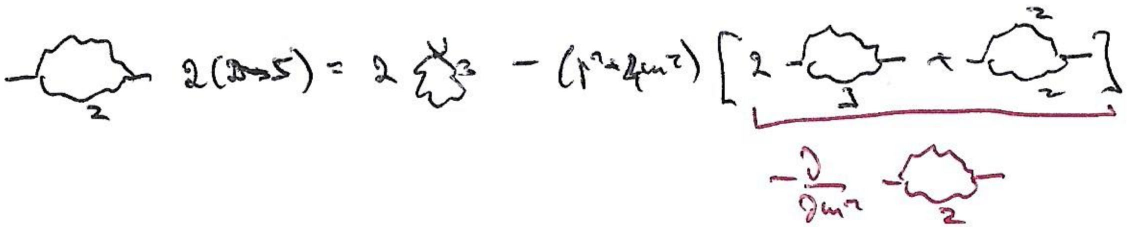


If we live  as master-integral :

(finite)



Sum of (1) and (2)



J. Henn, 2013

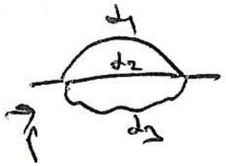
Put All master integrals in a column and solve the DE system

the most popular method now.

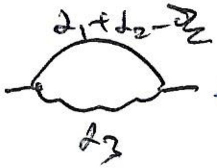
Two-loops with masses (only sunsets, for simplicity)

 ← sunset topology

Ⓐ two massless lines

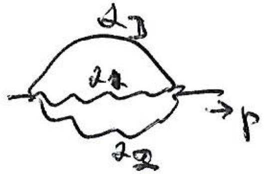


$= \pi^{D/2} A(\alpha_1,\alpha_2)$



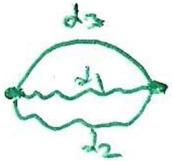
← one loop

Ⓑ one massless line and $m_1^2 = m_2^2 = m^2$

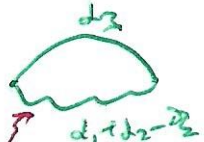


$\bar{\alpha} = \alpha_1+\alpha_2+\alpha_3$

Tadpole
$p^2=0$



$$= \pi^{D/2} \frac{\cancel{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx \cdot \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[x(1-x)]^{\alpha_1+\alpha_2-\frac{D}{2}}}$$



$=$

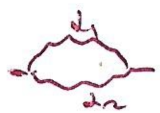
$\mu^2 = \frac{m^2}{x(1-x)}$

$$\pi^{D/2} \frac{\Gamma(\frac{D}{2}-\alpha_3)\Gamma(\bar{\alpha}-D)}{\Gamma(\frac{D}{2})\cancel{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}} \frac{1}{(\mu^2)^{\bar{\alpha}-D}}$$

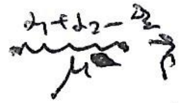
$$\left[\frac{x(1-x)}{m^2}\right]^{\bar{\alpha}-D}$$

$$\int_0^1 dx \cdot \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{\cancel{[x(1-x)]^{\alpha_1+\alpha_2-\frac{D}{2}}}} [x(1-x)]^{\cancel{\bar{\alpha}-D}\ \alpha_3-\frac{D}{2}} = \frac{\Gamma(\alpha_1+\alpha_3-\frac{D}{2})\Gamma(\alpha_2+\alpha_3-\frac{D}{2})}{\Gamma(\alpha_1+\alpha_2+2\alpha_3-D)}$$

$$= \pi^D \frac{\Gamma(\frac{D}{2}-\alpha_3)\Gamma(\alpha_1+\alpha_2+\alpha_3-D)\Gamma(\alpha_1+\alpha_3-\frac{D}{2})\Gamma(\alpha_2+\alpha_3-\frac{D}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\frac{D}{2})\ \Gamma(\alpha_1+\alpha_2+2\alpha_3-D)\ (m^2)^{\bar{\alpha}-D}}$$



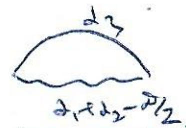
$$= \pi^{D/2} \frac{\Gamma(\alpha_1+\alpha_2-\frac{D}{2})}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx \cdot x^{\alpha_1-1}(1-x)^{\alpha_2-1} \underbrace{\frac{1}{[p^2x(1-x)+m^2]^{\alpha_1+\alpha_2-\frac{D}{2}}}}_{\frac{1}{[x(1-x)]^{\alpha_1+\alpha_2-\frac{D}{2}}}}$$



$\mu^2 = \frac{m^2}{x(1-x)}$

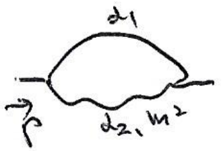
General case (i.e. $p \neq 0$) $\qquad \bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$

$$= \pi^{D/2} \frac{\cancel{\Gamma(\alpha_1+\alpha_2-D/2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dx \cdot \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[x(1-x)]^{\alpha_1+\alpha_2-D/2}}$$



$$= \boxed{*}$$

$$\underbrace{\qquad}_{\pi^{D/2} \frac{\Gamma(\bar{\alpha}-D)}{\Gamma(\alpha_3)\cancel{\Gamma(\alpha_1+\alpha_2-D/2)}} \int_0^1 dy \frac{y^{\alpha_3-1}(1-y)^{D/2-1-\alpha_3}}{[p^2 y + \mu^2]^{\bar{\alpha}-D}}}$$



$$= \pi^{D/2} \cdot \frac{\Gamma(\alpha_1+\alpha_2-D/2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \underbrace{\int_0^1 dx \cdot \frac{x^{\alpha_1-1}(1-x)^{D/2-1-\alpha_1}}{[p^2 x + m^2]^{\alpha_1+\alpha_2-D/2}}}_{\int_0^1 dx \frac{x^{\alpha_1-1}(1-x)^{D/2-1-\alpha_1}}{x^{\alpha_1+\alpha_2-D/2}\left[p^2 + \frac{m^2}{x}\right]^{\alpha_1+\alpha_2-D/2}}}$$

$$\frac{m^2}{x(1-x)}$$

$$\frac{1}{(\mu^2)^{\bar{\alpha}-D}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\bar{\alpha}-D)}{n!\,\Gamma(\bar{\alpha}-D)} \cdot (-1)^n \cdot \frac{(p^2 y)^n}{(\mu^2)^n}$$

$$\left[\frac{x(1-x)}{m^2}\right]^{\bar{\alpha}-D} \qquad \frac{(p^2 y)^n}{(m^2)^n}[x(1-x)]^n$$

$$\frac{1}{\mu^{2\gamma}}$$

$$[p^2 y + \mu^2]^{-\gamma} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\gamma)}{n!\,\Gamma(\gamma)} (-1)^n \frac{(p^2 y)^n}{(\mu^2)}$$

$$\int_0^1 dx \cdot \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{\cancel{[x(1-x)]^{\alpha_1+\alpha_2-D/2}}} \int_0^1 dy \cdot y^{\alpha_3-1}(1-y)^{D/2-1-\alpha_3} [x(1-x)]^{\cancel{\alpha_1+\alpha_2-D+\alpha_3}\ \alpha_3-D/2}$$

$$y^n (x(1-x))^n = \frac{\Gamma(\alpha_1+\alpha_3-D/2+n)\Gamma(\alpha_2+\alpha_3-D/2+n)}{\Gamma(\alpha_1+\alpha_2+2\alpha_3-D+2n)} \cdot \frac{\Gamma(\alpha_3+n)\Gamma(D/2-\alpha_3)}{\Gamma(D/2+n)}$$

$$\boxed{*} = \pi^D \frac{\Gamma(\bar{\alpha}-D)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cancel{\Gamma(\alpha_3)}} \frac{1}{(m^2)^{\bar{\alpha}-D}} \underbrace{\sum_{n=0}^{\infty} \frac{\Gamma(n+\bar{\alpha}-D)}{n!\,\Gamma(\bar{\alpha}-D)} \frac{\Gamma(\alpha_3+n)\Gamma(D/2-\alpha_3)}{\Gamma(D/2+n)} \frac{\Gamma(\alpha_1+\alpha_3-D/2+n)\Gamma(\alpha_2+\alpha_3-D/2+n)}{\underbrace{\Gamma(\overbrace{\alpha_1+\alpha_2+2\alpha_3-D}^{2\gamma}+2n)}_{\frac{2^{2\gamma+2n-1}}{\sqrt{\pi}}\Gamma(\gamma+n)\Gamma(\gamma+\frac{1}{2}+n)}}}$$

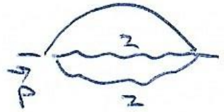
$$\Gamma(2\alpha) = \frac{2^{2\alpha-1}}{\sqrt{\pi}} \Gamma(\alpha)\Gamma(\alpha+\tfrac{1}{2})$$

$$\frac{\cancel{\Gamma(\alpha_3)}\Gamma(D/2-\alpha_3)}{\Gamma(D/2)} \cdot \frac{\sqrt{\pi}}{2^{2\gamma-1}} \frac{\Gamma(\alpha_1+\alpha_3-D/2)\Gamma(\alpha_2+\alpha_3-D/2)}{\Gamma(\gamma)\Gamma(\gamma+\frac{1}{2})} \, {}_4F_3\left[\begin{matrix}\bar{\alpha}-D,\ \alpha_3,\ \alpha_1+\alpha_3-D/2,\ \alpha_2+\alpha_3-D/2 \\ D/2,\ \gamma,\ \gamma+\frac{1}{2}\end{matrix} \,\middle|\, \frac{-p^2}{4m^2}\right]$$

(2.18)

Let $\alpha_3=1,\ \alpha_1=\alpha_2=2 \Rightarrow \alpha=5$ $\quad 2\gamma = 6-D = 2+2\varepsilon \Rightarrow \gamma = 1+\varepsilon$

[IBP gives relations between diagrams with various indices α_i]



$$= \pi^D \frac{\Gamma(1+2\varepsilon)}{(m^2)^{1+2\varepsilon}} \frac{\Gamma(\cancel{1+\varepsilon})}{\underbrace{\Gamma(2-\varepsilon)}_{(1-\varepsilon)}} \cdot \frac{\sqrt{\pi}}{2^{1+2\varepsilon}} \frac{\Gamma(1+\varepsilon)}{\cancel{\Gamma(1+\varepsilon)}\Gamma(3/2+\varepsilon)} \ {}_{\cancel{4}\cancel{3}\,3\,2}F \left[\begin{matrix} 1+2\varepsilon,\ 1,\ 1+\varepsilon,\ \cancel{1+\varepsilon} \\ 2-\varepsilon,\ \cancel{1+\varepsilon},\ 3/2+\varepsilon \end{matrix} \middle| -\frac{p^2}{4m^2} \right] =$$

$$\Gamma(1+2\varepsilon) = \frac{2^{2\varepsilon}}{\sqrt{\pi}} \Gamma(1/2+\varepsilon)\Gamma(1+\varepsilon)$$

$$\frac{\Gamma(1+2\varepsilon)\sqrt{\pi}}{2^{1+2\varepsilon}\Gamma(3/2+\varepsilon)} = \frac{1}{2}\frac{\Gamma(1/2+\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(3/2+\varepsilon)} = \frac{\Gamma(1+\varepsilon)}{(1+2\varepsilon)}$$

$$y = \frac{1-\sqrt{\frac{x}{x+4}}}{1+\sqrt{\frac{x}{x+4}}}$$

$$= \frac{\pi^D}{(m^2)^{1+2\varepsilon}} \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1+2\varepsilon)} \ {}_3F_2 \left[\begin{matrix} 1+2\varepsilon,\ 1,\ 1+\varepsilon \\ 2-\varepsilon,\ 3/2+\varepsilon \end{matrix} \middle| -\frac{p^2}{4m^2} \right] \overset{(*)}{=} \frac{\pi^D}{(-p^2)} \ln^2 y + O(\varepsilon)$$

$$ {}_3F_2 \left[\begin{matrix} 1,1,1 \\ 2,\ 3/2 \end{matrix} \middle| -\frac{x}{4} \right] = \sum_{k=0}^{\infty} \frac{\Gamma^2(k+1)}{\underbrace{\Gamma(k+2)}_{\Gamma(k+1)/(k+1)}} \frac{\overbrace{\Gamma(3/2)}^{\Gamma^2(k+1)\cancel{2^{2k}}/\Gamma(2k+2)}}{\Gamma(k+3/2)} \left(-\frac{x}{\cancel{4}}\right)^k = \sum_{k=1}^{\infty} \frac{\Gamma^{\cancel{3}\,2}(k)}{\cancel{\Gamma(k+1)}\,\cancel{(k)}} \frac{1}{\Gamma(2k)} (-x)^k \frac{1}{(-x)} = \frac{1}{(-x)} \int_0^x \frac{dx_1}{x_1} \sum_{k=1}^{\infty} \overbrace{\frac{\Gamma^2(k)}{\Gamma(2k)}}^{\binom{k}{2k}^{-1}\cdot\frac{2}{k}} (-x_1)^k = \frac{2}{(-x)} \int_0^x \frac{dx_1}{x_1} \cdot \rho_1 \ln y_1$$

$$y_1 = \frac{1-\sqrt{\frac{x_1}{x_1+4}}}{1+\sqrt{\frac{x_1}{x_1+4}}}, \quad \rho_1 = \frac{1-y_1}{1+y_1}$$

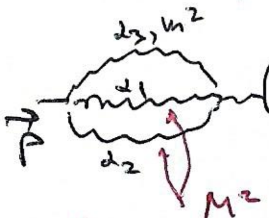
$$\hookrightarrow \sqrt{\frac{x_1}{x_1+4}} = \frac{1-y_1}{1+y_1} \Rightarrow \frac{x_1}{x_1+4} = \frac{(1-y_1)^2}{(1+y_1)^2} \Rightarrow \frac{x_1+4}{x_1} = \frac{(1+y_1)^2}{(1-y_1)^2} \Rightarrow \frac{4}{x_1} = \frac{(1+y_1)^2}{(1-y_1)^2} - 1 = \frac{4y_1}{(1-y_1)^2} \Rightarrow x_1 = \frac{(1-y_1)^2}{y_1} = \frac{1}{y_1} - 2 + y_1 \Rightarrow$$

$$\Rightarrow \frac{dx_1}{dy_1} = -\frac{1}{y_1^2} + 1 = \frac{1-y_1^2}{y_1^2} \Rightarrow \frac{dx_1}{dy_1} \cdot \frac{1}{x_1} = \frac{(1-y_1^2)}{y_1^2} \cdot \frac{y_1}{(1-y_1)^2} = \frac{1+y_1}{y_1(1-y_1)}$$

$x_1 = 0 \quad y_1 = 1$

$$\int_0^x \frac{dx_1}{x_1} \rho_1 \ln y_1 = -\int_y^1 dy_1 \cdot \frac{\cancel{(1+y_1)}}{y_1\cancel{(1-y_1)}} \frac{\cancel{(1-y_1)}}{\cancel{(1+y_1)}} \ln y_1 = -\ln^2 y_1 \cdot \frac{1}{2}\Big|_y^1 = \frac{1}{2}\ln^2 y \Rightarrow {}_3F_2 \left[\begin{matrix} 1,1,1 \\ 2,\ 3/2 \end{matrix} \middle| -\frac{x}{4} \right] = \frac{1}{(-x)} \ln^2 y$$

Last example



$(p^2 = m^2)$ (on-shell $p^2 = -m^2$)

$$\underline{\underline{\quad d_3 = 1,\ d_2 = d_1 = 2 \quad}} \quad J_{122}$$

$$S_a(n) = \sum_{i=1}^{n} \frac{1}{i^a}$$

$S_1 = S_1(n-1)$, $\bar{S}_1 = S_1(2n-1)$, $\bar{\bar{S}}_1 = S_1(4n-1)$

$$M^2 J_{122} = \frac{1}{x} \sum_{n=1}^{\infty} (-x^2)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left(-\frac{1}{2n} \ln x + \frac{S_1}{2n} - \frac{7}{2} \frac{\bar{S}_1}{n} + \frac{\bar{\bar{S}}_1}{n} + \frac{1}{4n^2} \right) +$$

$$x = \frac{m^2}{M^2}$$

$$+ \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{(-16x^2)^n}{\binom{2n}{n}\binom{4n}{2n}} \left(-\frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{4n^2} \right) + O(\varepsilon) =$$

(Kniehl, A.V.K., Onishchenko, Veretin 2005)

$$= \frac{d}{d\delta} \left\{ \frac{x^{1+\delta}}{(3+2\delta)(2+\delta)} \, {}_4F_3 \left[\begin{matrix} 1, \frac{3}{2}, \frac{7}{2}+\delta, 1+\frac{\delta}{2} \\ 2+\frac{\delta}{2}, \frac{5}{4}+\frac{\delta}{2}, \frac{7}{4}+\frac{\delta}{2} \end{matrix} \middle| -\frac{x^2}{4} \right] \right\} \Bigg|_{\delta=0} +$$

$$+ \, {}_4F_3 \left[\begin{matrix} 1, 1, 1, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{4}, \frac{5}{4} \end{matrix} \middle| -\frac{x^2}{4} \right]$$

NRQED, NRQCD

Compact form of the integral representation:

$$M^2 J_{122} = -\frac{1}{2x}\int_0^1 \frac{dt}{t\sqrt{1-t}}\left(\underbrace{\frac{1}{\sqrt{1+4A^2}}}_{\sim t^3} L(A) - 2\ln A\right) \to \text{Integrals of Elliptic integrals}$$

$\sim t^3 \Rightarrow$ Elliptic integral (in the case if $L(A)=1$)

$A = \frac{xt}{4}$,

$$L_1(A) = \ln\frac{\sqrt{1+4A^2}-1}{\sqrt{}+1}, \quad L_2^{(A)} = \ln\frac{\sqrt{}-2A}{\sqrt{}+2A}, \quad L(A) = L_1(A) + L_2(A)$$

Elliptic polilogarithms

- Paris group [Bloch, Vanhove, 2013]
- Mainz group [S. Weinzierl et all, started 2014]
- Brodel, Duhr, Dulat, Penante, Tancredi [started 2017]

(A. Levin, 2007) ← Mathematics

Conclusion

I have shown rather effective methods of calculation of massless Feynman diagrams and also Feynman integrals containg massive propagators