Strong Field QED in perturbative and non-perturbative regimes

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Quantum Field Theory at the Limits:
from Strong Fields to Heavy Quarks (2019)

Dubna International Advanced School of Theoretical Physics

About our group

- Beginning of **1960** — **Vladimir Ritus** and **Anatoly Nikishov** start to study Strong Field QED at Lebedev Physical Institute (FIAN) (led by the Nobel Prize winner Igor Tamm (1958))
- End of **1960** — **Nikolay Narozhny** joins their group while his PhD at MEPhI
- Until **1985** (later with Morozov D.A.) they conduct fundamental studies of elementary processes in SFQED
- End of **1990**-s — **Alexander Fedotov** joins Prof. Nikolay Narozhny at MEPhI and they initiate a new wave in investigations of SFQED effects (Kasimir effect, e^-e^+ vacuum pair creation, harmonic generation, QED cascades and other)
- Mid **2000** and later — **Evgeny Gelfer**, Arseny Mironov, Konstantin Krylov
- Students: Renat Gallyamov, Arseny Berezin, Egor Sozinov

Our group

Prof. Nikolay Narozhny
06.11.1940 – 15.02.2016

Alexander Fedotov
and
Evgeny Gelfer

Overview

External field concept . . . depends on the topic!

Assume strong field = laser field (though other cases, e.g. , channeled electrons in crystals, heavy ion/electron bunches collisions, magnetars may be also of interest). Examples:

- External field concept: # absorbed/emitted photons ($\simeq 1$) in a mode $\ll$ their total number N_γ

$$N_\gamma \simeq \frac{(E^2/4\pi)V}{\hbar\omega} \gg 1 \quad \Longrightarrow \quad \boxed{E \gg \sqrt{\hbar\omega/V}}$$

$$\text{for tight focusing} \quad V \sim \lambda^3 \quad \Longrightarrow \quad \boxed{E \gg \omega^2\sqrt{\hbar/c^3}}$$

  is satisfied for $I \gtrsim 10^5$W/cm^2 (c.f. 1.4×10^{-4}W/cm^2 coming from Sun).
- Atomic physics: $E \gtrsim E_{\rm at} = e/a_B^2 = m^2e^5/\hbar^4 = 5 \times 10^9$V/cm, corresponds to $I \gtrsim cE_{\rm at}^2/4\pi = 3 \times 10^{16}$W/cm^2.

Relativistic intensity

Assume e^- in plane wave

$$\mathbf{E} = \{E \sin\varphi,\ 0,\ 0\},\ \mathbf{H} = \{0,\ E\sin\varphi,\ 0\}, \quad \varphi = \omega(t - z/c)$$

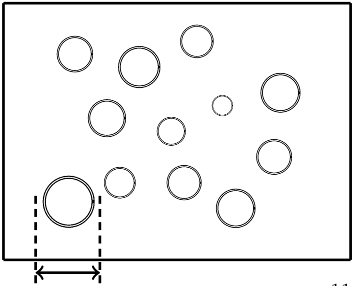
Equations of motion can be solved exactly ($\mathbf{p}_0 = 0$, $\varphi_0 = 0$):

$$\varepsilon = mc^2\left[1 + \frac{1}{2}\left(\frac{eE}{m\omega c}\right)^2 (\cos\varphi - 1)^2\right], \quad p_\perp \simeq mc\frac{eE}{m\omega c}$$

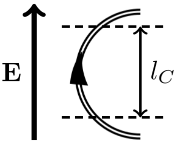
- $\boxed{\xi \equiv a_0 = \frac{eE}{m\omega c}}$ — classical non-linearity parameter
- $\xi \gtrsim 1 \Longrightarrow e^-$ becomes ultrarelativistic during $t < T$
- $\xi \sim eE\lambda/mc^2$ — work done by the field at distance λ
- For lasers: $a_0 = eE/m\omega c = 6 \times 10^{-10}\lambda[\mu\text{m}]\sqrt{I[\text{W/cm}^2]} \gtrsim 1$, or $I \gtrsim 3 \times 10^{18}\text{W/cm}^2$.

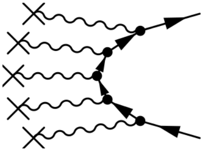
As we will see, this has also important implications in QED.

SFQED characteristic field



$l_C = \hbar/mc \approx 3.9 \times 10^{-11}$ cm





Let $\mathbf{E} = const$, $\mathbf{H} = 0$, pair is created if $eEl_C = mc^2$

$$E_S = \frac{m^2 c^3}{e\hbar}$$ — critical field (F. Sauter 1931)

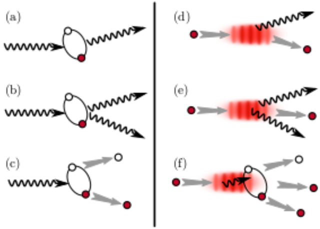
$$E_S = 1.32 \times 10^{16} \text{ V/cm} = 4.4 \times 10^{13} \text{ G}$$

$$I_S = \frac{c}{4\pi} E_S^2 \sim 10^{29} \text{ W/cm}^2$$

Examples of SFQED processes

An external field can:

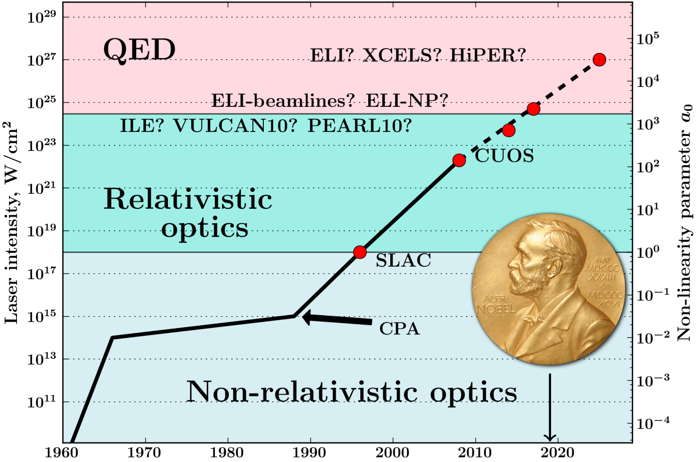
- polarize system (i.e., vacuum);
- modify (assist or suppress) a process;
- **induce a process**.

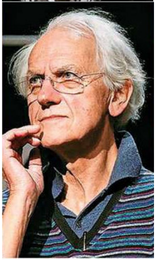


Pic. by D. Seipt

- (a): vacuum birefringence;
- (b): photon splitting;
- (c): pair production;
- (d): photon emission (nonlinear Compton scattering);
- (e): two-photon emission (plural or double Compton scattering);
- (f): trident pair production.

Timeline of the progress in laser technology





A half of the Nobel Prize in Physics 2018 was awarded jointly to Gérard Mourou and Donna Strickland “for their method of generating high-intensity, ultra-short optical pulses.”

Ordinary QED

From now on $\hbar = c = 1$

- Fundamental fields: $\psi(x)$, $A^\mu(x)$
- The Lagrangian: $\mathcal{L} = \mathcal{L}_{e^-e^+} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}}$,

$$\begin{aligned}
\mathcal{L}_{e^-e^+} &= \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi, \\
\mathcal{L}_{\text{Maxwell}} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \\
\mathcal{L}_{\text{int}} &= -J^\mu A_\mu = -e\bar{\psi}\gamma^\mu A_\mu \psi
\end{aligned}$$

- Initial state $|\Psi_i\rangle$ evolves into a final state $|\Psi_f\rangle = S|\Psi_i\rangle$, where

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots \int d^4x_n \mathrm{T}[\mathcal{L}_{\text{int}}(x_1) \ldots \mathcal{L}_{\text{int}}(x_n)]$$

- $S_{i\to f} \equiv \langle \Psi_f | S | \Psi_i \rangle$ is the amplitude giving the probability $W_{i\to f} = |S_{i\to f}|^2$ of the process
- $\alpha = e^2/4\pi = 1/137.035999074(44)$ is small $\Longrightarrow$ perturbation theory is valid (higher-order terms can be neglected)

SFQED Lagrangian

- $$A^\mu(x) = \underbrace{A^\mu_{\text{ext}}(x)}_{\text{classical field, non-perturbative}} + \underbrace{A^\mu_{\text{rad}}(x)}_{\text{quantized radiation, perturbative}}$$
- $$\mathcal{L} = \underbrace{\mathcal{L}_{e^-e^+} + \mathcal{L}^{\text{rad}}_{\text{Maxwell}} + \mathcal{L}^{\text{ext}}_{\text{int}}}_{\mathcal{L}_0} + \mathcal{L}^{\text{rad}}_{\text{int}} + \underbrace{\cancel{\mathcal{L}^{\text{ext}}_{\text{Maxwell}}}}_{\partial^\mu F^{\text{ext}}_{\mu\nu}=0} + \underbrace{\cancel{\mathcal{L}^{\text{ext+rad}}_{\text{Maxwell}}}}_{\partial^\mu F^{\text{rad}}_{\mu\nu}=0},$$

$$\mathcal{L}_{e^-e^+} + \mathcal{L}^{\text{ext}}_{\text{int}} = \bar\psi\left(i\gamma^\mu\partial_\mu - e\gamma^\mu A^{\text{ext}}_\mu - m\right)\psi,$$

$$\mathcal{L}^{(\bullet)}_{\text{Maxwell}} = -\frac{1}{4}F^{(\bullet)}_{\mu\nu}F^{\mu\nu}_{(\bullet)}, \quad \mathcal{L}^{\text{rad}}_{\text{int}} = -J^\mu A^{\text{rad}}_\mu = -e\bar\psi\gamma^\mu A^{\text{rad}}_\mu\psi$$

- Hamiltonian: $H(t) = H_{e^-e^+}(t) + H^{\text{rad}}_{\text{Maxwell}}(t) + H^{\text{ext}}_{\text{int}}(t) + H^{\text{rad}}_{\text{int}}(t)$

$$H_{e^-e^+}(t) = \int d^3x\,\bar\psi(x)\,(i\gamma\nabla - m)\,\psi(x),$$

$$H^{(\bullet)}_{\text{Maxwell}}(t) = \int d^3x\,\left[\frac{1}{2}\left(\mathcal{E}^2_{(\bullet)} + \mathcal{B}^2_{(\bullet)}\right) + \nabla A^0\mathcal{E}_{(\bullet)}\right]$$

$$H^{(\bullet)}_{\text{int}}(t) = \int d^3x\,J^\mu(x)A^{(\bullet)}_\mu(x) = e\int d^3x\,\bar\psi\gamma^\mu A^{(\bullet)}_\mu\psi$$

The Furry picture

- Let's separate interaction with the radiation field

$$\hat{H}(t) = \underbrace{\hat{H}_{e^-e^+} + \hat{H}^{\text{rad}}_{\text{Maxwell}} + \hat{H}^{\text{ext}}_{\text{int}}(t)}_{\hat{H}_0(t)\text{ — time dependent!}} + \underbrace{\hat{H}^{\text{rad}}_{\text{int}}(t)}_{\hat{H}_{\text{int}}(t)}$$

- The Schrodinger picture: $|\Psi(t)\rangle = U(t,t_0)|\Psi(t_0)\rangle, \quad \frac{d}{dt}\hat{\mathcal{O}} = 0$

$$\hat{U}(t,t_0) = T\exp\left[-i\int\limits_{t_0}^{t}\hat{H}(t)dt\right]$$

- Changing to the Furry picture:

$$|\Psi(t)\rangle^F = \hat{U}_0^{F\,\dagger}(t,t_0)|\Psi(t)\rangle, \quad \mathcal{O}^F = \hat{U}_0^{F\,\dagger}(t,t_0)\mathcal{O}^S\hat{U}_0^F(t,t_0),$$

$$\hat{U}_0^F(t,t_0) = T\exp\left[-i\int\limits_{t_0}^{t}\hat{H}_0(t)dt\right]$$

The Furry picture

- In the Furry picture:

$$|\Psi(t)\rangle^F = \hat{U}^F(t,t_0)|\Psi(t_0)\rangle^F,\ i\frac{d}{dt}\mathcal{O}^F(t) = \left[\mathcal{O}^F(t),\, \hat{H}_0^F(t)\right],$$

$$\hat{U}^F(t,t_0) = T\exp\left[-i\int\limits_{t_0}^{t}\hat{H}^F_{\text{int}}(t)dt\right],$$

- ψ and A^μ_{rad} are quantized as in QED in the Schrodinger picture

$$\left\{\hat{\psi}^F_\sigma(\mathbf{x}),\, \hat{\psi}^{F\dagger}_\lambda(\mathbf{y})\right\} = \delta_{\sigma\lambda}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

$$\left[\hat{A}^{\text{rad}}_\mu(\mathbf{x}),\, \hat{\pi}^{\text{rad}}_{A\,\nu}(\mathbf{y})\right] = ig_{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\ \text{other relations are zero}$$

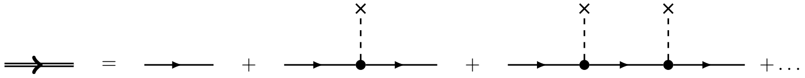
- Substitute $\hat{\psi}^F$ to Eq. for $\mathcal{O}^F$ and we get $\boxed{\left(\not{p} - e\not{A}^{\text{ext}} - m\right)\hat{\psi}^F = 0}$,

$$\hat{\psi}^F(x) = \int d^3p\left[\hat{a}_p\psi_p(x) + \hat{b}_p^\dagger\psi_{-p}(x)\right],$$

$$\hat{\overline{\psi}}^F(x) = \int d^3p\left[\hat{a}_p^\dagger\overline{\psi}_p(x) + \hat{b}_p\overline{\psi}_{-p}(x)\right]$$

K. Felix Mackenroth “Quantum Radiation in Ultra-Intense Laser Pulses” (2014)

Physical derivation: perturbation theory breakdown w.r.t. external field



$$\psi_p = \left\{ 1 + \frac{i}{\not{p} - m}(-ie\not{A}) + \frac{i}{\not{p} - m}(-ie\not{A})\frac{i}{\not{p} - m}(-ie\not{A}) + \dots \right\} \times e^{-ipx} u_p$$

Effective vertex weight:

$$\sqrt{\alpha} \mapsto \sqrt{\alpha} \times \sqrt{\bar{N}_\gamma} \simeq \frac{e}{\sqrt{\hbar c}} \times \sqrt{\left(\frac{\hbar}{mc}\right)^2 \times \frac{2\pi c}{\omega} \times \frac{E^2}{4\pi\hbar\omega}} \simeq \frac{eE}{m\omega c} \simeq \xi$$

If $\xi = \frac{e}{m}\sqrt{-\langle A^2 \rangle} \gtrsim 1$ then all the terms are comparable and must be retained – *summation needed*:

$$\psi_p = e^{-ipx} u_p + \frac{i}{\not{p} - m}(-ie\not{A})\psi_p$$

Dirac equation in Furry picture: $\boxed{\left(\not{p} - e\not{A} - m\right)\psi_p = 0}$

Volkov equation:

$$\left(\not{p} - e\not{A} - m\right)\psi_p = 0$$

Exercise 1

Solve this equation if $A_\mu = A_\mu(\varphi)$, $\varphi = kx$, $k^2 = kA = 0$.

Look up for hints in V. B. Berestetskii, E. M. Lifshitz, L P. Pitaevskii Quantum Electrodynamics (Course of Theoretical Physics, 4), paragraph 40.

Volkov solution, $\left(\hat{\not{p}} - e\not{A} - m\right)\psi_p = 0$

- Plane wave: $A^\mu = A^\mu(\varphi), \quad \varphi = kx, \quad k^2 = kA = 0$
- Solution: $\boxed{\psi_{p,\sigma}(x) = E_p(x) u_{p,\sigma}}$, $(\not{p} - m)u_{p,\sigma} = 0$

$$E_p(x) = e^{iS_p}\Sigma_p \quad \text{– Ritus } E_p\text{-function}$$

$$\Sigma_p = 1 + \frac{e}{2(kp)}\not{k}\not{A} \quad \text{– spin factor}$$

$$S_p = -px + \delta S_p(\varphi) \quad \text{– classical action}$$

$$\delta S_p(\varphi) = -\frac{e}{kp}\int_0^\varphi \left(pA(\varphi) - \frac{e}{2}A^2(\varphi)\right) d\varphi$$

- Orthonormality and completeness:

$$\int d^3x\, \overline{\psi}_{p,\sigma}(\mathbf{x},t)\gamma^0\psi_{q,\lambda}(\mathbf{x},t) = (2\pi)^3\delta_{\sigma\lambda}\delta(\mathbf{p}-\mathbf{q}),$$

$$\sum_\sigma \int \frac{d^3p}{(2\pi)^3}\psi_{p,\sigma}(\mathbf{x},t)\psi^\dagger_{p,\sigma}(\mathbf{y},t) = \delta(\mathbf{x}-\mathbf{y})$$

Volkov solution, $\left(\hat{\not{p}} - e\not{A} - m\right)\psi_p = 0$

- Plane wave: $A^\mu = A^\mu(\varphi), \quad \varphi = kx, \quad k^2 = kA = 0$
- Solution: $\boxed{\psi_{p,\sigma}(x) = E_p(x) u_{p,\sigma}}$, $(\not{p} - m) u_{p,\sigma} = 0$

$$E_p(x) = e^{iS_p}\Sigma_p \quad \text{– Ritus } E_p\text{-function}$$

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$$\delta S_p(\varphi) = -\frac{e}{kp}\int_0^\varphi \left(pA(\varphi) - \frac{e}{2}A^2(\varphi)\right) d\varphi$$

- Properties of E_p-functions:

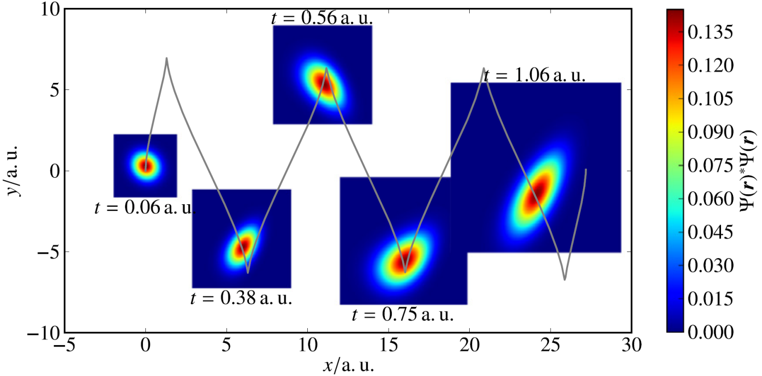
$$\int d^4x \overline{E}_p(x) E_q(x) = (2\pi)^4 \delta^{(4)}(p - q),$$

$$\int \frac{d^4p}{(2\pi)^4} E_p(x)\overline{E}_p(y) = \delta^{(4)}(x - y), \quad \overline{E}_p = \gamma^0 E_p^\dagger \gamma^0$$

Exercise 2

Prove the following property: $\hat{\not{\mathcal{P}}} E_p = E_p \not{p}, \quad \hat{\not{\mathcal{P}}} = \hat{\not{p}} - e\not{A}$

Volkov solution in plane wave



Free wave packet evolution in a plane wave field. The solid gray line indicates the center of mass trajectory, coinciding essentially with the classical trajectory, and the laser pulse travels from left to right.

Bauke and Keitel, Computer Physics Communications 182, 12 (2011); A. Di Piazza et al. Rev. Mod. Phys. 84 (2012)

S-matrix

$$|\Psi(t)\rangle^F = \hat{U}^F(t,t_0)|\Psi(t_0)\rangle^F, \quad \hat{U}^F(t,t_0) = T\exp\left[-i\int_{t_0}^{t}\hat{H}^F_{\text{int}}(t)dt\right],$$

- Scattering (no pair creation)

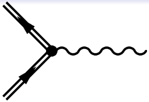
$$|\Psi_i\rangle^F \longrightarrow \sum_f |\Psi_f\rangle^F \underbrace{{}^F\langle\Psi_f|\hat{S}^F|\Psi_i\rangle^F}_{\hat{S}_{fi}},$$

$$\hat{S}^F = \hat{U}^F(t_f\to+\infty,\, t_i\to-\infty) = T\exp\left[-ie\int_{-\infty}^{+\infty}\hat{\bar{\psi}}^F\hat{\not{A}}^F_{\text{rad}}\hat{\psi}^F dt\right]$$

- Now we can construct *perturbation theory* w.r.t. interactions with A_{rad} in presence of A_{ext} (which is considered *non-perturbatively*)

Feynman rules in SFQED

Table by S. Meuren (from PhD thesis)

Vertex	$-ie\gamma^{\mu}$	
Photon propagator	$-iD_{\mu\nu}(x-y)$	$x,\ \mu$  $y,\ \nu$
Dirac propagator	$iG(x,y)$	x  y
Incoming fermion	$E_p(x)\,\frac{u_{p,\sigma}}{\sqrt{2\epsilon}}$	x  p
Outgoing fermion	$\frac{\bar{u}_{p,\sigma}}{\sqrt{2\epsilon}}\,\bar{E}_p(x)$	 x p
Incoming anti-fermion	$\frac{\bar{u}_{-p,-\sigma}}{\sqrt{2\epsilon}}\,\bar{E}_{-p}(x)$	x  $\leftarrow p$
Outgoing anti-fermion	$E_{-p}(x)\,\frac{u_{-p,-\sigma}}{\sqrt{2\epsilon}}$	 x $\leftarrow p$
Incoming photon	$\sqrt{4\pi}\,e^{\mu(\alpha)}\,e^{-ikx}$	x  $\leftarrow k$
Outgoing photon	$\sqrt{4\pi}\,e^{\mu(\alpha)^*}\,e^{ikx}$	 x $\leftarrow k$

Fermion propagator

- Green's function Eq.: $\left(\hat{\not{p}} - e\not{A} - m\right) iG(x,y) = i\delta(x-y)$
- We can use E_p functions instead of Fourier transform:

$$G(x,y) = \int \frac{d^4p'}{(2\pi)^4}\frac{d^4q'}{(2\pi)^4} E_{p'}(x) G(p',q') \overline{E}_{q'}(y)$$

- Transforming the whole equation and using the properties of E_p:

$$\underbrace{\int d^4x\, d^4y \overline{E}_p(x)\left[\left(\hat{\not{\mathcal{P}}} - m\right) G(x,y)\right] E_q(y)}_{(\not{p}-m)G(p,q)} = (2\pi)^4 \delta(p-q)$$

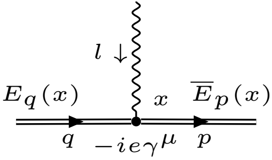
- $G(p,q) = (2\pi)^4\delta(p-q)G(p)$, then $G(p) = \dfrac{1}{\hat{\not{p}} - m} \to \dfrac{\hat{\not{p}} + m}{p^2 - m^2 + i0}$

$$\boxed{iG(x,y) = \int \frac{d^4p}{(2\pi)^4} E_p(x) \frac{i(\hat{\not{p}} + m)}{p^2 - m^2 + i0} \overline{E}_p(y)}$$

$G(x,y)$ can be derived from $iG(x,y) = \langle 0 | T \left\{ \hat{\overline{\psi}}^F(x),\, \hat{\psi}^F(y) \right\} | 0 \rangle$

E_p-representation

- Example, γ-emission: $S_{i\to f} = -ie\int d^4x\,\bar{\psi}_{p'}(x)\not{\varepsilon}_l^* e^{ilx}\psi_p(x)$
- Some operator: $O(p',q') = \int d^4x\,d^4y\overline{E}_{p'}(x)O(x,y)E_{q'}(y)$



- In various diagrams every vertex we will face a combination:

$$\Gamma^\mu(l;p,q) = \int d^4x e^{-ilx}\overline{E}_p(x)\gamma^\mu E_q(x)$$

- In E_p-representation interaction with A_{ext} can be transferred to vertices, while fermion propagator and endings are written as in free QED
- γ-emission: $S_{i\to f} = -ie\bar{u}_{p'}\varepsilon^*_{l\mu}\Gamma^\mu(-l;p',p)u_p$

Dressed vertex in CCF i

Constant crossed field: $\boxed{A_\mu = -a_\mu \varphi}$, $\varphi = kx,\ k^2 = ka = 0,$
e.g. $a^\mu = \{0,\ E/\omega,\ 0,\ 0\}$, $k_\mu = \{\omega, 0,\ 0,\ \omega\}$

- $E_q(x) = \Sigma_q \exp[-iqx + i\delta S_q(\varphi)]$, let's perform FT:

$$\Gamma^\mu(l;p,q) = \int d^4x \int\limits_{-\infty}^{\infty} ds e^{is\varphi} \times$$

$$\int\limits_{-\infty}^{\infty} \frac{d\varphi}{2\pi} e^{-is\varphi} e^{i(p-q-l)x} \left[\overline{\Sigma}_p \gamma^\mu \Sigma_q e^{-i\delta S_p + i\delta S_q}\right](\varphi)$$

$$= \int\limits_{-\infty}^{\infty} ds\, (2\pi)^4 \delta(sk + p - q - l) \tilde{\Gamma}^\mu(s|p,q)$$

- Conservation laws:

$$kp - kq - kl = 0 \Rightarrow \chi_p = \chi_q + \chi_l, \quad \chi_p = \frac{e\sqrt{-F_{\mu\nu}p^\nu}}{m^3} = \xi \frac{kp}{m^2}$$

$$\mathbf{p}_\perp = \mathbf{q}_\perp + \mathbf{l}_\perp,$$

$$s\omega + \varepsilon_p = \varepsilon_q + \varepsilon_l$$

Dressed vertex in CCF ii

Constant crossed field: $\boxed{A_\mu = -a_\mu \varphi}$, $\varphi = kx$, $k^2 = ka = 0$,
e.g. $a^\mu = \{0,\, -E/\omega,\, 0,\, 0\}$, $k_\mu = \{\omega, 0,\, 0,\, \omega\}$

Further rearranging $\tilde{\Gamma}^\mu(s|p,q)$

- $\overline{\Sigma}_p \gamma^\mu \Sigma_q = \left(1 + \frac{e}{2(kp)} \not{A}\not{k}\right) \gamma^\mu \left(1 + \frac{e}{2(kq)} \not{k}\not{A}\right) = R_0 + R_1\varphi + R_2\varphi^2$
- We have to calculate the integrals

$$A_n(s) = \int\limits_{-\infty}^{\infty} \frac{d\varphi}{2\pi}\, \varphi^n\, \exp\left[-is\varphi + i\delta S_q(\varphi) - i\delta S_p(\varphi)\right], \quad n = 0, 1, 2$$

- Actually $A_n(s) = i^n \partial^n A_0(s)/\partial s^n$
- The exponent:

$$\delta S_p(\varphi) = -\frac{e}{kp}\int_0^\varphi \left(pA(\varphi) - \frac{e}{2}A^2(\varphi)\right) d\varphi,$$

$$\delta S_q(\varphi) - \delta S_p(\varphi) = \underbrace{\frac{e}{2}\left(\frac{aq}{kq} - \frac{ap}{kp}\right)}_{\alpha/2} \varphi^2 + \underbrace{\frac{e^2 a^2}{6}\left(\frac{1}{kq} - \frac{1}{kp}\right)}_{-4\beta/3} \varphi^3$$

School implies homework!

Exercise 3

1. Consider $\bar{\Sigma}_p \gamma^\mu \Sigma_q = \left(1 + \frac{e}{2(kp)} \not{A}\not{k}\right) \gamma^\mu \left(1 + \frac{e}{2(kq)} \not{k}\not{A}\right)$.
   Using γ-matrix relations this expression can be simplified to the form
   $$\bar{\Sigma}_p \gamma^\mu \Sigma_q = \gamma^\nu \left(h_V{}^\mu{}_\nu + h_A{}^\mu{}_\nu \gamma^5\right),$$
   where h_V and h_A are some tensors, depending only on A_μ (or $F_{\mu\nu}$) and momenta. Calculate h_V and h_A for the case of CCF.
2. The function $A_0(s) = \int\limits_{-\infty}^{\infty} \frac{d\varphi}{2\pi} \exp\left[-is\varphi + i\frac{\alpha}{2}\varphi^2 - i\frac{4\beta}{3}\varphi^3\right]$ can be rewritten in the following form:
   $$A_0(s) = f(\alpha, \beta)\mathrm{Ai}\left[y(\alpha, \beta)\right], \ \mathrm{Ai}(y) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\sigma\, e^{-i\sigma^3/3 - iy\sigma}.$$
   Calculate $f(\alpha, \beta)$ and $y(\alpha, \beta)$.

We will use these results in the following lecture on Friday!

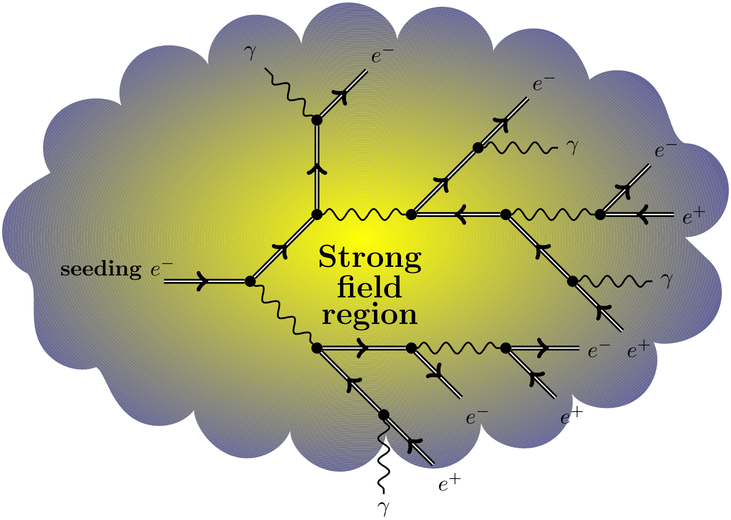
General notion of a strong field for a field-induced process

Quantum theory **allows for fluctuations restricted by the uncertainty relations**. Consider a **field-induced process** with energy lack $\Delta\mathcal{E}$ and the two characteristic times:

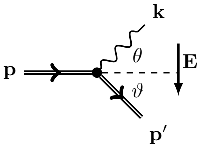
- Uncertainty time $\Delta\mathcal{E} \cdot \tau_Q \simeq \hbar \quad \Longrightarrow \quad \tau_Q \simeq \dfrac{\hbar}{\Delta\mathcal{E}}$;
- Time needed for the field to restore this energy: work produced by the field $eE \cdot v\tau_F \cdot \cos\vartheta_F \simeq \Delta\mathcal{E} \quad \Longrightarrow \quad \tau_F \simeq \dfrac{\Delta\mathcal{E}}{eEv\cos\vartheta_F}$

Then obviously the process is allowed if $\tau_F \lesssim \tau_Q$, or $\boxed{E \gtrsim \dfrac{\Delta\mathcal{E}^2}{e\hbar v\cos\vartheta_F}}$

Tree level processes and cascades



Nonlinear Compton scattering



- Consider photon emission by an ultrarelativistic ($\gamma \gg 1$) electron in a transverse field.
- An ultrarelativistic particle emits into an angle $\theta, \vartheta \simeq \gamma^{-1}$ (kinematical effect). Then $v = c$ and for an *angle* ϑ_F *with the field* $\cos\vartheta_F \simeq \gamma^{-1}$.

- Then from momentum conservation $p = p' + \frac{\hbar\omega}{c}$ and

$$\varepsilon = \sqrt{c^2p^2 + m^2c^4} \approx pc + \frac{m^2c^3}{2p}, \quad \varepsilon' = \sqrt{c^2p'^2 + m^2c^4} \approx cp' + \frac{m^2c^3}{2p'},$$

$$\Delta\mathcal{E} = \varepsilon' + \hbar\omega - \varepsilon \simeq \cancelto{0}{cp + \hbar\omega - cp'} + \mathcal{O}\left(\frac{mc^2}{\gamma}\right)$$

$$E \gtrsim \frac{\Delta\mathcal{E}^2}{e\hbar v\cos\vartheta_F} \simeq \frac{(mc^2/\gamma)^2}{e\hbar\cdot c\cdot\gamma^{-1}} = \frac{E_\mathrm{S}}{\gamma}, \quad \text{or} \quad \boxed{\chi = \frac{\gamma E}{E_S} \gtrsim 1}$$

Meaning of the dynamical quantum parameter χ

$$\chi = \frac{e\hbar}{m^3c^4}\sqrt{-(F_{\mu\nu}p^\nu)^2}$$
$$= \frac{\gamma\sqrt{\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{H}}{c}\right)^2 - \frac{(\mathbf{v}\cdot\mathbf{E})^2}{c^2}}}{E_{\mathsf{S}}} = \frac{E_{\mathrm{P}}}{E_{\mathrm{S}}} \simeq \frac{\gamma E_\perp}{E_S}$$

- Electron proper acceleration in the field measured in Compton units $mc^3/\hbar$;
- Ratio of the electric field in a *rest frame* to E_{S}
- Roughly: normalized product of energy and transverse field strength,

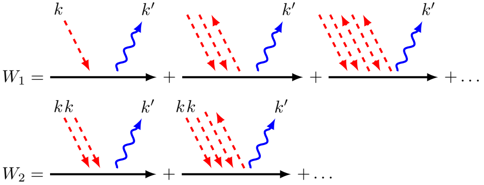
In an electron rest frame $\Delta\mathcal{E}' \simeq \hbar\omega \simeq mc^2$, $\cos\vartheta'_F \simeq 1$ and

$$E' \gtrsim \frac{\Delta\mathcal{E}'^2}{e\hbar v\cos\vartheta'_F} \simeq \frac{(mc^2)^2}{e\hbar\cdot c\cdot 1} = E_{\mathrm{S}}, \quad E'_\perp \simeq \gamma\times E_\perp,\ E'_\parallel \simeq E_\parallel$$

Unlike for vacuum processes, enough to provide $E \simeq E_{\mathrm{S}}$ in a rest frame!

NCS in circularly polarized plane wave

$$S_{i\to f} = -ie\int d^4x\,\bar{\Psi}_{p'}(x)\not{\epsilon}_l^{*}e^{ilx}\Psi_p(x) = (2\pi)^4\sum_{s>1}M^{(s)}\delta^{(4)}(q'+k'-q-sk)$$



Narozhny, Nikishov, Ritus, Sov. Phys. JETP 20, 622–629 (1965)

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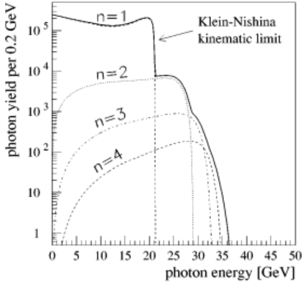
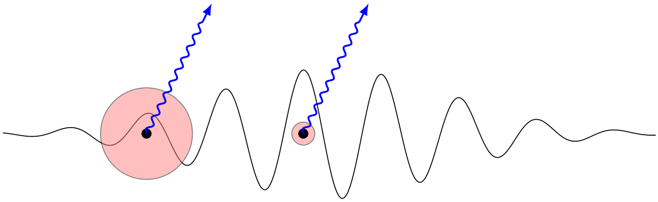


$$\boxed{\begin{aligned} W_{i\to f}^{(s)}(\xi,\chi) &= \frac{\alpha m^2}{4q_0}\int_0^{u_s}\frac{du}{(1+u)^2}\left\{-4J_s^2(z)+\xi^2\left(2+\frac{u^2}{1+u}\right)\right.\\ &\left.\times\left[J_{s+1}^2(z)+J_{s-1}^2(z)-2J_s^2(z)\right]\right\},\\ z &= \frac{\xi^2\sqrt{1+\xi^2}}{\chi}\sqrt{u\,(u_s-u)},\quad u_s=\frac{2s\chi}{\xi(1+\xi^2)}\end{aligned}}$$

Locally constant field approximation



- If $\tau_F \ll \lambda$ then one can use a locally constant field approximation (LCFA);
- Examples:
  - Atomic physics: $\tau_F = \dfrac{me}{\hbar E}$. $K = \omega\tau_F$ is called the Keldysh parameter, the field is slowly varying if $K \ll 1$.
  - SFQED: $\tau_F = \dfrac{mc^2}{eE}$ and LCFA is valid for $\xi \gg 1$ (actually if $\chi \gg 1$ then for $\xi \gg \chi^{1/3}$).
- This is not only a great simplification (allowing, e.g., to use Monte Carlo codes to track complicated backgrounds and plural events), but also a **manifestation of a truly SF regime**!

Validation (Harvey et al, PRA 2014)

$\xi \lesssim \chi^{1/3}$

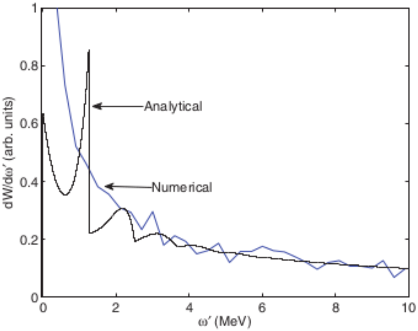


FIG. 4. (Color online) Comparison of the analytical (black) and numerical (blue or gray) frequency spectra for the case of subcritical a_0. The parameters are $a_0 = 20$ and $\gamma = 9000$.

$\xi \gg \chi^{1/3}$

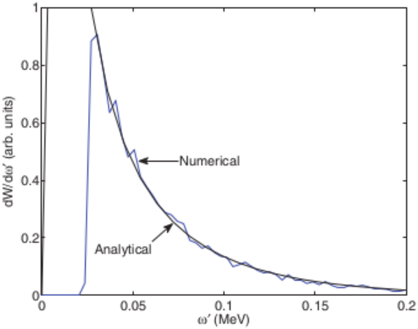


FIG. 9. (Color online) Comparison of the analytical (black) and numerical (blue or gray) angular emission rates for the case of *near*-critical a_0. The parameters are $a_0 = 30$ and $\gamma = 16$.

Invariant probabilities of processes

Probabilities of various processes W should be lorenz- and gauge invariant, thus W should be a function of

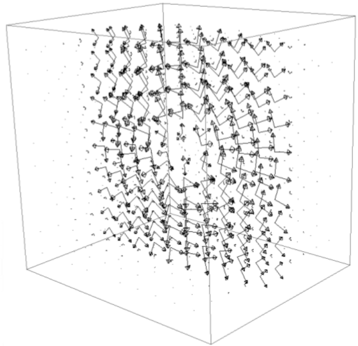
$$\xi = \frac{e\sqrt{-\langle A^\mu A_\mu \rangle}}{mc}, \quad \chi_{e,\gamma} = \frac{e\hbar\sqrt{-(F^{\mu\nu}p_\nu)^2}}{m^3c^4}$$

$$\mathcal{F} = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} = H^2 - E^2, \quad \mathcal{G} = -\frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma} = (\mathbf{E}, \mathbf{H})$$

Suppose $\xi \gg 1, \chi^{1/3} : \xi \to \infty$, formation length $l_F \sim \dfrac{\lambda}{\xi} \ll \lambda$ — **field is locally constant**

$$W = W(\chi, \mathcal{F}, \mathcal{G}) \text{ — invariant}$$

LCFA for general process



If $|\mathcal{F}|, |\mathcal{G}| \ll \min(1, \chi^2) E_S^2$:

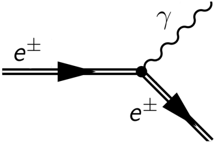
$$W \approx W(\chi, 0, 0)$$

$$\mathcal{F} = \mathcal{G} = 0$$
$$\Downarrow$$
$$E = H = const, (\mathbf{E}, \mathbf{H}) = 0$$ —
constant crossed field

In RF of an ultra-relativistic particle *any* field looks like CCF!

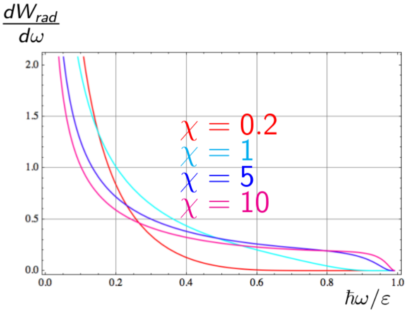
LCFA: for most physical cases it is enough to calculate $W(\chi, 0, 0)$

Nonlinear Compton Scattering in CCF



Probability rate of $e^-(e^+)$ with energy ε_e to emit γ with energy ε_γ:

$$\frac{dW_{\rm rad}(\varepsilon_\gamma,\chi_e)}{d\varepsilon_\gamma} = -\frac{\alpha m^2 c^4}{\hbar \varepsilon_e^2}\left\{\int_x^\infty \mathrm{Ai}\,(\xi)\,d\xi + \left(\frac{2}{x}+\chi_\gamma\sqrt{x}\right)\mathrm{Ai}'(x)\right\}$$



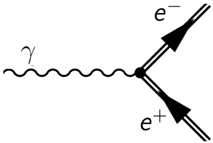
$$\chi_e = \chi_e' + \chi_\gamma,\ x = (\chi_\gamma/\chi_e\chi_e')^{2/3}$$

$$W_{\rm rad} \approx \frac{\alpha m^2 c^4}{\hbar\varepsilon_e}\times\begin{cases}1.44\chi_e, & \chi_e \ll 1\\ 1.46\chi_e^{2/3}, & \chi_e \gg 1\end{cases}$$

A.I. Nikishov, V.I. Ritus JETP, Vol. 19, 5 (1964); 19, 2, (1964);

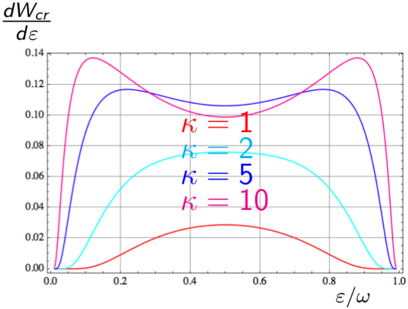
N. V. Elkina et al Phys. Rev. STAB 14, 054401 (2011)

Nonlinear Breit-Wheeler process in CCF



Probability rate of e^-e^+ production by a photon with energy ε_γ, energies of e^- and e^+ are ε_e, $\varepsilon'_e = \varepsilon_\gamma - \varepsilon_e$:

$$\frac{dW_{\mathrm{cr}}(\varepsilon_e, \chi_\gamma)}{d\varepsilon_e} = \frac{\alpha m^2 c^4}{\hbar \varepsilon_\gamma^2} \left\{ \int_x^\infty \mathrm{Ai}\,(\xi)\, d\xi + \left(\frac{2}{x} - \chi_\gamma \sqrt{x} \right) \mathrm{Ai}'(x) \right\}$$



$$\chi_\gamma = \chi_e + \chi'_e, \; x = (\chi_\gamma / \chi_e \chi'_e)^{2/3}$$

$$W_{\mathrm{cr}} \approx \frac{\alpha m^2 c^4}{\hbar \varepsilon_\gamma} \times \begin{cases} 0.23 \chi_\gamma e^{-8/3\chi_\gamma}, & \chi_\gamma \ll 1 \\ 0.38 \chi_\gamma^{2/3}, & \chi_\gamma \gg 1 \end{cases}$$

A.I. Nikishov, V.I. Ritus JETP, Vol. 19, 5 (1964); 19, 2, (1964);

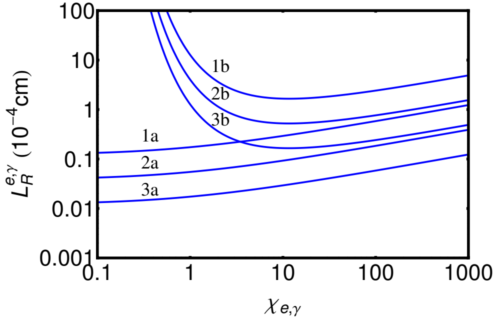
N. V. Elkina et al Phys. Rev. STAB 14, 054401 (2011)

Particle free path

$W_{\mathrm{rad,cr}}$ — total probability rates

$t_{\mathrm{free}} \sim W^{-1}_{\mathrm{rad,cr}}$ — mean free path time of $e^{\pm}$, γ

$$t_{\mathrm{free}} \propto \frac{\hbar\varepsilon}{\alpha m^2 c^4}\chi^{-2/3}, \quad \chi \gg 1$$



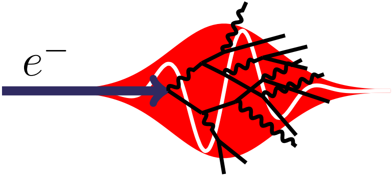
$L_R \sim ct_{e,\gamma}$ — mean free path length of a) e, b) γ propagating transversely in laser field of intensities I 10^{23} (1a, 1b), 10^{24} (2a, 2b), 10^{25} (3a, 3b) W/cm^2

*S. S. Bulanov, C. B. Schroeder et al, Phys. Rev. A, **87**, 062110 (2013).*

S-type cascades

Let's take e^- with $\varepsilon_e \sim 10 \div 100$ GeV, and "moderately" intense laser $I \sim 10^{19} \div 10^{22}$ W/cm^2.

Due to high ε_e the parameter $\chi \sim \dfrac{\varepsilon_e}{mc^2}\dfrac{E}{E_S} \gtrsim 1$



In each reaction $\chi = \chi_1' + \chi_2'$, so $\chi \searrow$ and $W_{\rm cr} \propto \exp\left(-\dfrac{8}{3\chi_\gamma}\right) \to 0$!

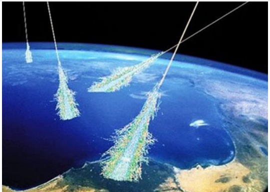
Cascade will eventually collapse, $\boxed{N_{e^-e^+} \propto \varepsilon_0}$

This cascade resembles Extensive Air Showers

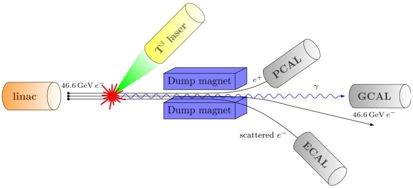
S(Shower)-type cascade

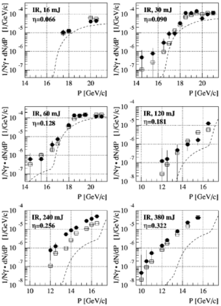
S. S. Bulanov et al Phys. Rev. A 87, 062110(2013).

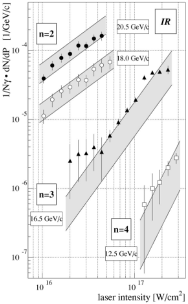
I. V. Sokolov et al PRL 105, 195005 (2010).

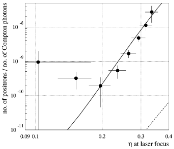


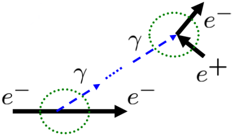
The SLAC E144 experiment (1991-1998)







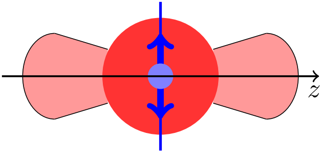




$\varepsilon_e = 46.6$ GeV, $I \approx 10^{18}$ W/cm^2, $a_0 \equiv \xi \approx 0.6$, $\chi \approx 0.13$ – slightly below the threshold of a ‘strong field domain’

Self-sustained cascades

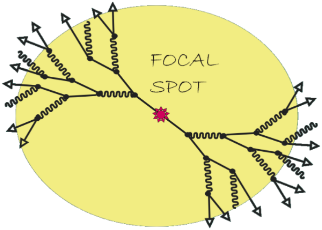
- e^- is seeded at rest in strong laser field with params λ, $E \ll E_S$
- e^- motion is quasi-classical, $d_{\text{wave packet}} \ll \lambda$



$$\frac{dp^\mu(\tau)}{d\tau} = \frac{e}{m} F^\mu{}_\nu(x(\tau)) p^\nu(\tau), \qquad \frac{dx^\mu(\tau)}{d\tau} = \frac{p^\mu(\tau)}{m},$$

$$\chi(\mathbf{r}, t) = \frac{e}{m^3} \sqrt{(p_0 \mathbf{E} + \mathbf{p} \times \mathbf{H})^2 - (\mathbf{p}\mathbf{E})^2}$$

- e^- is accelerated so that $\boxed{d\chi/dt > 0}$
- Acceleration time: $\dfrac{d\chi}{dt} t_{\text{acc}} \sim 1$
- Lifetime: $W_{\text{rad}} \cdot t_{\text{free}} \sim 1$
- Escape time: $t_{\text{esc}} \sim \lambda/c$



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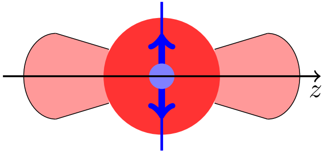
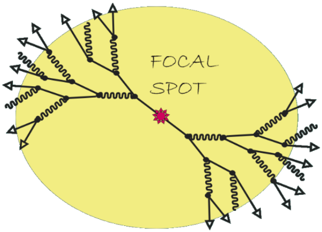


$$\frac{dp^\mu(\tau)}{d\tau} = \frac{e}{m} F^\mu{}_\nu(x(\tau)) p^\nu(\tau), \qquad \frac{dx^\mu(\tau)}{d\tau} = \frac{p^\mu(\tau)}{m},$$

$$\chi(\mathbf{r}, t) = \frac{e}{m^3} \sqrt{(p_0 \mathbf{E} + \mathbf{p} \times \mathbf{H})^2 - (\mathbf{p}\mathbf{E})^2}$$

- EM field restores ε and χ, the process is repeated, until particles leave SF area or field is depleted
- $\boxed{N_{e^-e^+}(t) \propto e^{\Gamma t}}$
- $\boxed{\text{How strong the field must be? } \Gamma\text{?}}$



Toy model: uniformly rotating electric field

Initially slow $p(0) \ll mc$ particle in a "relativistic field" $a_0 = \frac{eE_0}{mc\omega} \gg 1$

$$\chi(0) \sim E_0/E_S \ll 1$$

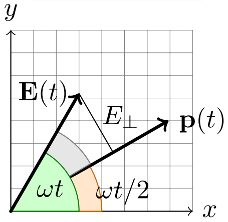
$$\boxed{\frac{mc}{eE_0} \ll t \ll \frac{1}{\omega}}$$

$$\mathbf{E}(t) = \{E_0 \cos\omega t, E_0 \sin\omega t\} \approx E_0\{1, \omega t\}$$

$$\frac{d\mathbf{p}(t)}{dt} = e\mathbf{E}(t), \quad \mathbf{p}(0) = 0$$

angle $\frac{\omega t}{2}$

$$\mathbf{p}(t) = \cancel{\mathbf{p}(0)}^{\,0} + \int_0^t e\mathbf{E}(t)\,dt = eE_0\left\{t, \frac{\omega t^2}{2}\right\}$$



$$E_\perp \sim E_0 \frac{\omega t}{2}, \quad \chi(t) \sim \frac{E_\perp \gamma}{E_S} \sim \underbrace{\frac{E_0}{E_S}}_{\text{small}} \times \underbrace{\frac{\omega t}{2}}_{\text{small}} \times \underbrace{\frac{eE_0 t}{mc}}_{\text{very large!}}$$

χ can attain unity rather quickly: $t \ll \omega^{-1}$!!! But how general is that?

General case (numerical simulations

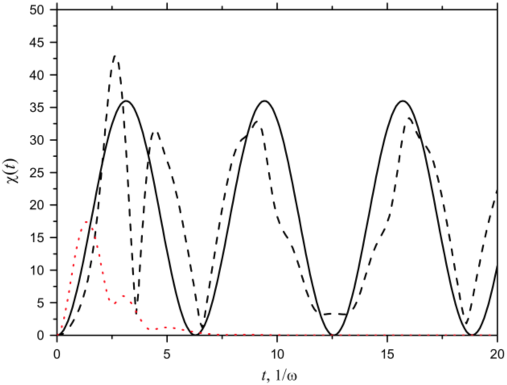


FIG. 1 (color online). Evolution of quantum dynamical parameter χ along the particle trajectory for $a_0 = 3 \times 10^3$, $\hbar\omega = 1$ eV in three cases: head-on collision of two elliptically polarized plane waves (solid line); collision at 90° of two linearly polarized plane waves with orthogonal linear polarizations (dashed line); single tightly focused e-polarized laser beam (dotted line).

Fedotov A.M. et al, PRL 105, 080402 (2010)

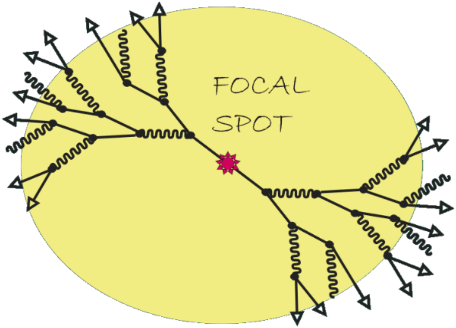
Toy model: hierarchy of times in cascade

- $\chi(t) \sim \left(\frac{E_0}{E_S}\right)^2 \frac{\hbar\omega}{mc^2}\frac{t^2}{\tau_C^2}, \quad \gamma(t) \sim \frac{E_0 t}{E_S \tau_C}, \quad \tau_C = \frac{\hbar}{mc^2}$
- $t_{\mathrm{acc}}: \ \chi(t_{\mathrm{acc}}) \simeq 1 \Longrightarrow \boxed{t_{\mathrm{acc}} \simeq \frac{\tau_C}{\alpha\mu}\sqrt{\frac{mc^2}{\hbar\omega}}}, \quad \mu = \frac{E}{\alpha E_S}$
- $t_{\mathrm{acc}} \ll t_{\mathrm{free}} \Rightarrow \gamma$ are emitted when $\chi \gg 1 \Longrightarrow W_{\mathrm{rad}} \approx \frac{\alpha}{\tau_C \gamma}\chi^{2/3}$

  $t_{\mathrm{free}}: \ W_{\mathrm{rad}}(t_{\mathrm{free}}) \simeq 1 \Longrightarrow \boxed{t_{\mathrm{free}} \simeq \frac{\tau_C}{\alpha\mu^{1/4}}\sqrt{\frac{mc^2}{\hbar\omega}}}$
- $1/\omega \lesssim t_{\mathrm{esc}} \sim \lambda$
- Hierarchy of times: $\frac{mc}{eE_0} \ll \boxed{t_{\mathrm{acc}} \lesssim t_{\mathrm{free}} \ll \frac{1}{\omega}} \lesssim t_{\mathrm{esc}}$
- Finally: $\mu \gtrsim 1$, or $\boxed{E_0 \gtrsim E_S/137}$, laser intensity $I \gtrsim 10^{25}$ W/cm^2
- $N_{e^-e^+} \sim e^{\Gamma t}$: $\Gamma = \frac{t_{\mathrm{free}}}{t_{\mathrm{acc}}} \sim \mu^{3/4}$,

  $\frac{\pi/\omega}{t_{\mathrm{free}}} \sim \pi\mu^{1/4}\sqrt{\frac{\alpha^2 mc^2}{\hbar\omega}} \quad (0.5\alpha^2 mc^2 = 13.6\text{eV} \Longrightarrow \sqrt{\ } \sim 5)$

Basic Monte-Carlo algorithm



1. Set initial particle at $\mathbf{r}_0(t_0)$
2. Determine particle lifetime $\tau \sim W^{-1}(\chi)$ with Event Generator (EG)
3. Solve Eqs of motion, $\mathbf{r}_0(t_0) \mapsto \mathbf{r}(t = \tau)$ with Particle Mover (PM)
4. Calculate out-particles with Particle Generator (PG)
5. Repeat these steps for new particles

Particles move along trajectories $(\mathbf{r}(t),\, \mathbf{p}(t))$, so

$\chi(t) = \chi(\mathbf{r}(t), \mathbf{p}(t))$, $\varepsilon(t) = \varepsilon(\mathbf{r}(t), \mathbf{p}(t))$ and $W(t) = W(\chi(t),\, \varepsilon(t))$

Steps 2 and 3 are mixed!

Details of the Monte-Carlo algorithm

Incorporate of the Particle Mover with the Event Generator
Δt — time step of numerical scheme

e^- at t_i at $\mathbf{r}(t_i)$ with ε_i, $\mathbf{p}_i$ and χ_i $\rightarrow$

EG: check if emission happens during $t_i < t < t_i + \Delta t$
random $0 < r < 1$
$r \overset{?}{<} W_{\text{rad}}(t_i)\Delta t$

False $\rightarrow$ **PM**: move e^-
$\mathbf{r}(t_i) \mapsto \mathbf{r}(t_i + \Delta t)$

True $\downarrow$ **PG**: new $e^-\gamma$
Get $\varepsilon_\gamma(t_i)$, $\mathbf{k}_\gamma(t_i)$,
then $\varepsilon'_e(t_i)$, $\mathbf{p}_\gamma(t_i)$

$$\Delta t \ll W_{\text{rad}}^{-1}(t_i)$$

Particle Generator PG

1) Random $0 < r' < 1$
2) Solve equation on ε_γ:
$$\frac{1}{W_{rad}} \int\limits_{\varepsilon_{min}}^{\varepsilon_\gamma} \frac{dW_{rad}(\varepsilon_\gamma)}{d\varepsilon_\gamma} d\varepsilon_\gamma = r'$$

1) Random $\varepsilon_{min} < \varepsilon_r < \varepsilon_e$
random $0 < r < \max(dW_{rad}(\varepsilon_\gamma)/d\varepsilon_\gamma)$
2) If $r < dW_{rad}(\varepsilon_r)/d\varepsilon_\gamma$ take ε_r

3) determine $\chi_\gamma(t_i)$, $\mathbf{k}(t_i)$
4) determine new e^- energy ε_e,
$\mathbf{p}(t_i)$ from $\chi_i = \chi_\gamma + \chi_e$

Relativistic aberration effect $\Rightarrow$ $\mathbf{p}$ of product particles lay in $\Delta\theta \sim \gamma^{-1}$ around $\mathbf{p}_0$, so all particles move in same direction

Particle Movers

$\chi \ll 1$ — classical emission
$\chi \gtrsim 1$ — quantum emission

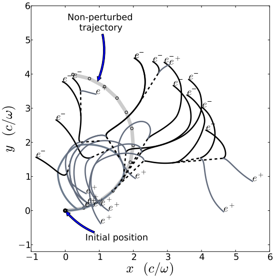
E is high, t_{acc}: $\Delta\chi(t_{acc}) \sim 1$, $\omega t_{acc} \lesssim 1$
QED cascades take place for $\chi \gtrsim 1$, so **we choose quantum description**.

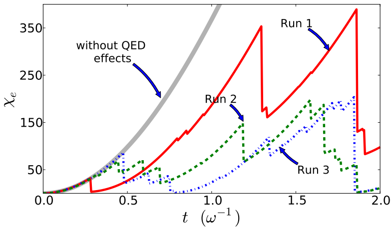
Between actions of emission $e^{\pm}$ move along classical trajectories:
$\dot{\mathbf{p}}(t) = e\left[\mathbf{E}(\mathbf{r},\mathbf{t}) + \frac{\mathbf{v}}{c} \times \mathbf{H}(\mathbf{r},\mathbf{t})\right]$
$\dot{\mathbf{r}}(t) = \mathbf{p}(t)/m$

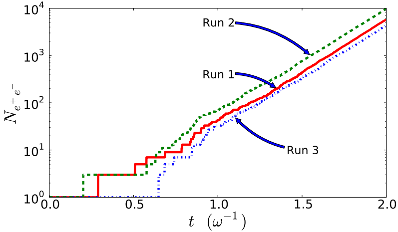
Numerical schemes

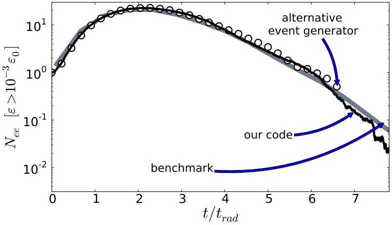
Leapfrog
Boris
Runge-Kutta
other

Simulations of cascade dynamics in rotating E field









N. V. Elkina et al Phys. Rev. STAB 14, 054401 (2011)

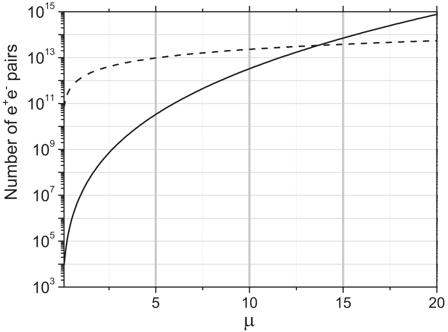
Number of e^-e^+ pairs

Maximum number of pairs (solid):

$$N \sim \exp\left(\frac{t_{esc}}{t_e}\right) \sim \exp\left(\pi\alpha\mu^{1/4}\sqrt{\frac{\hbar\omega}{mc^2}}\right)$$

Number of pairs "stored" in laser pulse (dashed):

$$N_{e,\max} \sim \frac{W}{2\varepsilon_e} \sim \alpha\mu^{5/4}\left(\frac{\hbar\omega}{mc^2}\right)^{5/2}$$



A(valanche)-type QED cascade can *deplete* driving laser field!

A. M. Fedotov, N. B. Narozhny, G. Mourou and G. Korn, PRL **105**, 080402 (2010)

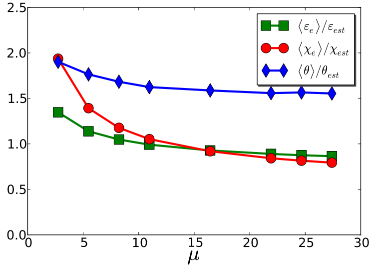
Proof of scaling

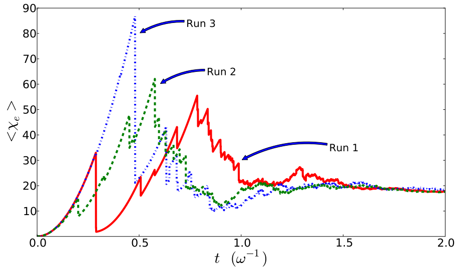
$$\chi_{est} \sim \mu^{3/2}$$

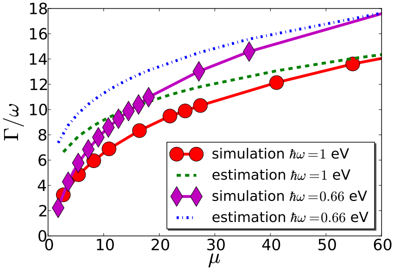
$$\varepsilon_{est} \sim mc^2 \mu^{3/4} \sqrt{\frac{mc^2}{\hbar\omega}}$$

$$\vartheta_{est} \sim \frac{1}{\alpha \mu^{1/4}} \sqrt{\frac{\hbar\omega}{mc^2}}$$

$$\Gamma \sim \alpha \mu^{1/4} \sqrt{\frac{mc^2 \omega}{\hbar}}$$







Kinetic description of generated $e^-e^+\gamma$- plasma (EPPP)

Phase space distributions of EPPP:

$$f_-(\mathbf{r},\mathbf{p},t), \quad f_+(\mathbf{r},\mathbf{p},t), \quad f_\gamma(\mathbf{r},\mathbf{p},t)$$

$$\frac{df_a}{dt} = \text{GAIN} - \text{LOSS}$$

Currently neglected:

- (Possible) degeneracy of EPPP:
  $\varepsilon_e \gg \varepsilon_F = (3\pi^2)^{1/3}\hbar c n_e^{1/3} \Longrightarrow n_e \ll (\varepsilon_e/\hbar c)^3$;
- Recombination processes $\mathcal{O}(n_a^2)$ ($e^\pm\gamma \to e^\pm$, $e^+e^- \to \gamma$):
  $n_e \ll (mc/\hbar)^2 \times (\varepsilon_e/\hbar c)$;
- “Trident” processes ($e^\pm \to e^\pm e^- e^+$, $e^\pm \to e^\pm\gamma\gamma$);
- Other $\mathcal{O}(\alpha^2)$ processes ($e^\pm\gamma \to e^\pm\gamma$, $e^+e^- \to \gamma\gamma$, $\gamma\gamma \to e^+e^-$,...);
- ...

Kinetic (cascade) equations

$$\left\{\frac{\partial}{\partial t} + \frac{\mathbf{p}}{\varepsilon}\cdot\nabla \pm e\left(\mathbf{E} + \frac{\mathbf{p}}{\varepsilon}\times\mathbf{H}\right)\cdot\frac{\partial}{\partial\mathbf{p}}\right\} f_\pm(\mathbf{p},t) =$$

$$= \underbrace{\underbrace{\int f_\pm(\mathbf{p}+\mathbf{k},t)w_{rad}(\mathbf{p}+\mathbf{k}\to\mathbf{k})d^3k}_{\text{gain } e^\pm\to e^\pm\gamma} - \underbrace{f_\pm(\mathbf{p},t)\int w_{rad}(\mathbf{p}\to\mathbf{k})d^3k}_{\text{loss } e^\pm\to e^\pm\gamma}}_{\textcolor{red}{\textbf{quantum radiation reaction (friction)}}} +$$

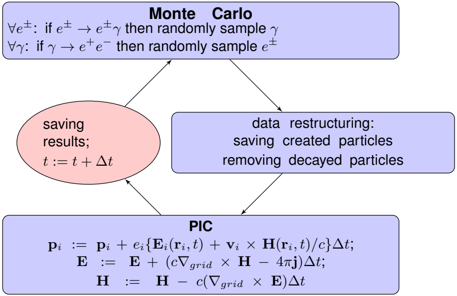
$$+ \underbrace{\int f_\gamma(\mathbf{k},t)w_{cr}(\mathbf{k}\to\mathbf{p})d^3k}_{\text{gain } \gamma\to e^- e^+}$$

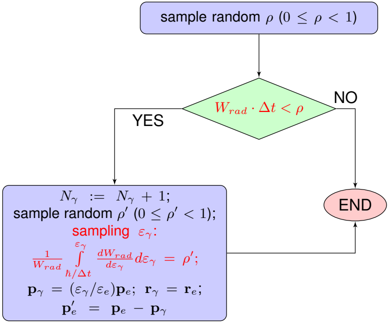
$$\left\{\frac{\partial}{\partial t} + \frac{\mathbf{k}}{\omega}\cdot\nabla\right\} f_\gamma(\mathbf{k},t) = \underbrace{\int [f_+(\mathbf{p},t) + f_-(\mathbf{p},t)]w_{rad}(\mathbf{p}\to\mathbf{k})d^3p}_{\text{gain } e^\pm\to e^\pm\gamma} -$$

$$- \underbrace{f_\gamma(\mathbf{k},t)\int w_{cr}(\mathbf{k}\to\mathbf{p})d^3p}_{\text{loss } \gamma\to e^- e^+}$$

General structure of PIC-QED code

Elkina, Fedotov et al 2011



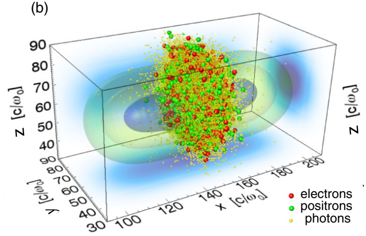


PIC-QED codes:

- SMILEI (`http://www.maisondelasimulation.fr/smilei/`)
- EPOCH (`https://gitlab.com/arm-hpc/packages/wikis/packages/EPOCH`)
- PICADOR (`http://hpc-education.unn.ru/en/research/overview/laser-plasma`
- OSIRIS (`http://epp.tecnico.ulisboa.pt/osiris/`)
- and many others

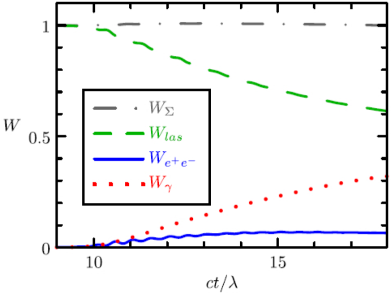
Field depletion in PIC-QED simulations

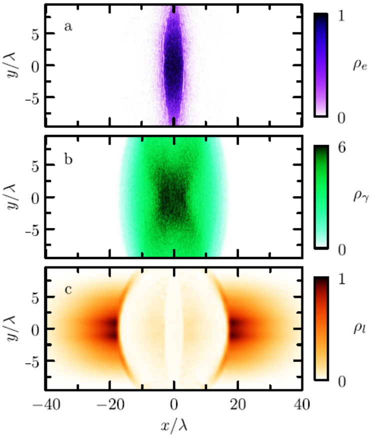
T. Grismayer et al (2017)



E.N. Nerush et al (2011)

Energy balance:





E.N. Nerush et al (2011)

Discussion i

Observation:

Typically, in more realistic simulations, self-sustained regime of QED cascades is already observed at intensities $10^{23\div 24}$W/cm^2, $1 \div 2$ orders lower than 5×10^{25}W/cm^2 $\leftrightarrow$ $\boxed{E = \alpha E_S}$.

Of ultimate importance for ELI, XCELS, etc.!

- If $R \gg 1/\omega$, then $t_{\rm esc} \simeq R \gg 1/\omega$ (if **radiative trapping** also takes place [Gonoskov et al., PRL 2014; Ji et al., PRL 2014; AF et al., PRA 2014], then even $t_{esc} \gg R$!);
- Any estimate of Γ always underestimates cascade multiplicity: $\langle e^{\Gamma t}\rangle > e^{\langle\Gamma\rangle t}$, and even $\langle e^{\Gamma t}\rangle \gg e^{\langle\Gamma\rangle t}$ for $t \gg \Gamma^{-1}$;
- Originally, we assumed $\varkappa \gtrsim 1$ as rough condition for pair production (this also approved usage of universal asymptotic for W). However, $W_{\rm cr}(\varkappa \ll 1) = \mathcal{O}(e^{-8/3\varkappa})$ remains non-negligible for even smaller values $\varkappa \gtrsim 0.1$

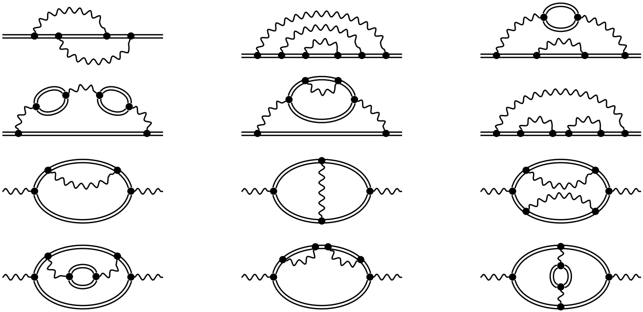
Discussion ii

- At $I \gtrsim 10^{24\div25}$W/cm^2 a **new physical regime** of laser - matter interaction should be revealed, characterized by **massive production of QED (**$e^-e^+\gamma$**) cascades** [with macroscopic multiplicity!]
  - There may be though some problems with injection of seed particles (e.g. due to radiative impenetrability of strong field region)
  - One possible solution – conversion of S-cascades to A-cascades (as hard photons may easily access focus)
- At $I \gtrsim 10^{26\div27}$W/cm^2 even focusing of laser pulses in vacuum would become unstable due to spontaneous pair creation and subsequent cascades development
- This process of fast depletion of a focused laser field in vacuum due to production of $e^-e^+\gamma$-plasma may very likely **prevent attainability of the Sauter-Schwinger critical electric field**

$$E_S = \frac{m^2c^3}{e\hbar} = 1.3 \times 10^{16}\text{V/cm}$$

  with laser fields capable for pair creation
- However, for more definite predictions **further simulations of this regime are required.**

Radiative corrections in SFQED



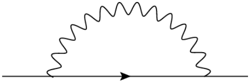
Electron self-energy

Classical Electrodynamics

- $\mathcal{E}_{\text{em}} \propto \dfrac{e^2}{2}\displaystyle\int d^3\xi \frac{\delta^{(3)}(\xi)}{|\xi|} = \frac{e^2}{2r_0} \quad r_0 \to 0$,

  $\boxed{r_0 \lesssim r_e \equiv \dfrac{e^2}{mc^2} = \alpha l_C}$ (r_e – *'classical electron radius'*)
- r_e and $E_{\text{cr}} = \dfrac{m^2c^4}{e^3} = \dfrac{E_S}{\alpha}$ limit Classical ED

Quantum Electrodynamics



- $\mathcal{E}_{\text{em}} \simeq \dfrac{e^2 m}{\pi^2}\displaystyle\int \frac{d^3\xi}{|\xi|^3} \propto e^2 m \log\left(\frac{1}{mr_0}\right), \quad r_0 \to 0$
- Pointlike charge is effectively replaced by a cloud of virtual pairs of size $\boxed{\simeq l_C = \dfrac{1}{m} \simeq 137 r_e}$ (or $\dfrac{\hbar}{mc} \simeq 4 \times 10^{-11}$cm in conventional units)

Weisskopf V. F. Growing up with Field Theory (The Development of Quantum Electrodynamics in Half a Century. Personal Recollections) (1980)

Radiation corrections in QED

$\hbar = c = 1$

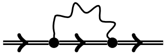
- After renormalization (which is all the same required for physical reasons, albeit $\mathcal{E}_{\text{em}} \simeq \alpha m \log\left(\frac{1}{m r_0}\right) \ll m$ for any reasonable value of r_0!), the coupling constant becomes effectively *'running'*, and its energy dependence essentially mimics the nature of divergency: $\alpha(\varepsilon) \simeq \alpha \log(\varepsilon/m)$, $\varepsilon \gg m$ (high energy 'stripping'). Note that $\alpha(\varepsilon)$ *remains small for all reasonable values of energy!*
- Review and classification of the variety of high-energy QED processes demonstrates that all the cross sections remain small $\sigma(\varepsilon) \lesssim \alpha^n r_e^2 \log^k(\varepsilon/m)$ within all the **reasonable** energy range.
  V.G. Gorshkov (1973); V.N. Baier (1981)
- Thus, **perturbation theory in ordinary QED works pretty well for all the reasonable values of parameters.**

Radiation corrections in SFQED — the first glance

- Consider e^- or γ in CCF, characterized by a single Lorentz- and gauge-invariant parameter $\chi_{e,\gamma} = \frac{e}{m^3}\sqrt{-(F_{\mu\nu}p^\nu_{e,\gamma})^2} \sim \frac{E_P}{E_S}$
-  — Volkov propagator
- Nonlinear Compton effect: $W_{\rm rad}(\chi_e) \simeq \frac{\alpha m^2}{p_0}\chi_e^{2/3}, \quad \chi_e \gg 1$

  on the other hand by optical theorem $W_{\rm rad} = \frac{2m}{p_0}\,{\rm Im}\, M^{(2)}$

$$M^{(2)}(\chi_e) =$$

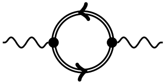


$$\simeq \boldsymbol{\alpha m \chi_e^{2/3}}, \quad \chi_e \gg 1;$$

- Nonlinear BW pair production:

$$W_{\rm cr}(\chi_\gamma) = \frac{2m}{k_0}\,{\rm Im}\,\Pi^{(2)} \simeq \frac{\alpha m^2}{k_0}\chi_\gamma^{2/3}, \quad \chi_\gamma \gg 1,$$

$$\Pi^{(2)}(\chi_\gamma) =$$



$$\simeq \boldsymbol{\alpha m \chi_\gamma^{2/3}}, \quad \chi_\gamma \gg 1;$$

IFQED radiation corrections **are growing surprisingly fast** with χ (i.e. with **both energy and field strength**)

Reminder: Volkov solution in CCF

- CCF: $A^\mu = -a^\mu \varphi, \quad \varphi = kx, \quad k^2 = ka = 0$

- $\psi_{p,\sigma}(x) = E_p(x) u_{p,\sigma}$, $(\not{p} - m) u_{p,\sigma} = 0$

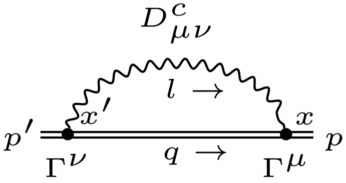
$$E_p(x) = \left[1 - \frac{e\,\not{k}\not{a}}{2(kp)}\varphi\right] \exp\left\{-ipx + i\frac{e\,ap}{2\,kp}\varphi^2 + i\frac{e^2 a^2}{6\,kp}\varphi^3\right\}$$

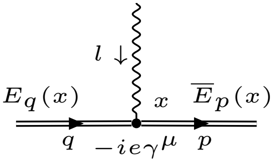
$$\overline{E}_p(x) = \left[1 - \frac{e\,\not{a}\not{k}}{2(kp)}\varphi\right] \exp\left\{ipx - i\frac{e\,ap}{2\,kp}\varphi^2 - i\frac{e^2 a^2}{6\,kp}\varphi^3\right\}$$

- e^- propagator

$$iG(x,y) = \int \frac{d^4p}{(2\pi)^4} E_p(x) \frac{i(\not{p} + m)}{p^2 - m^2 + i0} \overline{E}_p(y)$$

Electron mass operator in CCF





$$\Gamma^\mu(l;p,q) = \int d^4x e^{-ilx}\overline{E}_p(x)\gamma^\mu E_q(x)$$

$$-iM(x,x') = (-ie)^2\gamma^\mu iG^c(x,x')\gamma^\nu D^c_{\mu\nu}(x-x'),$$

$$D^c_{\mu\nu}(x-x') = \int \frac{d^4l}{(2\pi)^4}\frac{-ig_{\mu\nu}}{l^2+i0}e^{-il(x-x')}$$

In E_p representation:

$$-iM(p,p') = (-ie)^2\int d^4xd^4x'\,\bar{E}_p(x)\gamma^\mu iG^c(x,x')\gamma^\nu E_{p'}(x)D^c_{\mu\nu}(x-x')$$

$$= (-ie)^2\int \frac{d^4l}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\Gamma^\mu(l;p,q)\frac{i(\not{q}+m)}{q^2-m^2+i0}\Gamma^\nu(-l;q,p')\frac{-ig_{\mu\nu}}{l^2+i0}$$

Electron mass operator in CCF

ANNALS OF PHYSICS: **69**, 555–582 (1972)

Radiative Corrections in Quantum Electrodynamics with Intense Field and Their Analytical Properties

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Received April 10, 1971

Diagonalization and general properties of the mass and polarization operators and Greens functions of an electron and a photon in an intense crossed field are considered exactly in the external and radiation fields. Their explicit expressions are obtained in the e^2-approximation in the radiation field, and exactly in the external field. On the mass shell they determine the elastic scattering amplitudes and other physical quantities, for example, the dependence of the anomalous magnetic moment of the electron on the field and particle momentum. Due to instability of the electron and photon in the field,

Originally the diagram calculated by V.I. Ritus (1972). Unfortunately the paper lacks details and contains old notations, making it difficult to understand. Nevertheless, let's try to follow it's plan

Main goals:

- simplify as much as possible
- extract SF behaviour (dependence on χ)

$M^{(2)}$: Sketch of calculation I, conservation laws

$$-iM(p,p') = -e^2 \int \frac{d^4l\, d^4q}{(2\pi)^8} \Gamma^\mu(l;p,q) \frac{i(\not{q}+m)}{q^2-m^2+i0} \Gamma^\nu(-l;q,p') \frac{-ig_{\mu\nu}}{l^2+i0}$$

12 integrals in total (and some of them are divergent)

Let's use

- $\Gamma^\mu(l;p,q) = \int\limits_{-\infty}^{\infty} ds\,(2\pi)^4 \delta(sk+p-q-l)\tilde{\Gamma}^\mu(s|p,q)$
- $\Gamma^\nu(-l;q,p') = \int\limits_{-\infty}^{\infty} ds'\,(2\pi)^4 \delta(-s'k-p'+q+l)\tilde{\Gamma}^\nu(-s'|q,p')$
- Let's fix $\boxed{q = sk+p-l}$ and integrate over d^4q:

$$M(p,p') = -ie^2 \int d^4l\, ds\, ds'\, \delta\left[(s-s')k+p-p'\right] \times \frac{1}{(q^2-m^2+i0)(l^2+i0)} \tilde{\Gamma}^\mu(s|p,q)(\not{q}+m)\tilde{\Gamma}_\mu(-s'|q,p')$$

$M^{(2)}$: Sketch of calculation II, dressed vertices

$$M(p,p') = -ie^2 \int d^4l\, ds\, ds'\, \delta\left[(s-s')k + p - p'\right] \times \frac{1}{(q^2 - m^2 + i0)(l^2 + i0)} \tilde{\Gamma}^\mu(s|p,q)(\not{q} + m)\tilde{\Gamma}_\mu(-s'|q,p')$$

$$\tilde{\Gamma}^\mu(s|p,q) = \gamma_\lambda \left(\tilde{\Gamma}_V^{\lambda\mu}(s|p,q) + \tilde{\Gamma}_A^{\lambda\mu}(s|p,q)\gamma^5 \right),$$

$$\tilde{\Gamma}_V^{\lambda\mu}(s|p,q) = A_0\, g^{\lambda\mu} + \frac{i}{2}\left(\frac{1}{kp} + \frac{1}{kq}\right) A_1\, eF^{\lambda\mu} - \frac{1}{2(kp)(kq)} A_2\, e^2 (F^2)^{\lambda\mu},$$

$$\tilde{\Gamma}_A^{\lambda\mu}(s|p,q) = \frac{1}{2}\left(\frac{1}{kp} - \frac{1}{kq}\right) A_1\, eF^{*\lambda\mu},$$

$$F^{\lambda\mu} = k^\lambda a^\mu - k^\mu a^\lambda, \quad F^{*\lambda\mu} = \frac{1}{2}\varepsilon^{\lambda\mu\sigma\delta} F_{\sigma\delta}, \quad A_n = A_n(s|p,q)$$

- $p' = p + (s-s')k \implies \tilde{\Gamma}_\mu(-s'|q,\boxed{p'}) = \tilde{\Gamma}_\mu(-s'|q,\boxed{p})$

$M^{(2)}$: Sketch of calculation III, γ-matrix algebra

$$M(p,p') = -ie^2 \int d^4l\, ds\, ds'\, \delta\left[(s-s')k + p - p'\right]$$
$$\times \frac{1}{(q^2-m^2+i0)(l^2+i0)} \boxed{\tilde{\Gamma}^{\mu}(s|p,q)(\not{q}+m)\tilde{\Gamma}_{\mu}(-s'|q,p)}$$

- Total expression is very bulky. It much easier to expand all the γ-matrix terms with a computer algebra system like `FeynCalc`.
- Here for demonstration let's consider only terms $\propto A_0(s)$

$$\tilde{\Gamma}^{\mu}_0(s|p,q) = \gamma^{\mu} A_0(s|p,q)$$

$$M_{00}(p,p') = -ie^2 \int d^4l\, ds\, ds'\, \delta\left[(s-s')k + p - p'\right]$$
$$\times \frac{1}{(q^2-m^2+i0)(l^2+i0)} \boxed{A_0(s|p,q)A_0(-s'|q,p)\gamma^{\mu}(\not{q}+m)\gamma_{\mu}}$$

- $\gamma^{\mu}(\not{q}+m)\gamma_{\mu} = -2\not{q} + 4m$

$M^{(2)}$: Sketch of calculation IV, the A_0 function

$$M_{00}(p,p') = -ie^2 \int d^4l\, ds\, ds'\, \delta\left[(s-s')k + p - p'\right] \times \frac{-2\not{q} + 4m}{(q^2 - m^2 + i0)(l^2 + i0)} \boxed{A_0(s|p,q) A_0(-s'|q,p)}$$

- Next we need to substitute A_0-s explicitly

$$A_0(s|p,q) = \frac{1}{(4\beta)^{1/3}} \exp\left[-is\frac{\alpha}{8\beta} + i\frac{8\beta}{3}\left(\frac{\alpha}{8\beta}\right)^3\right] \mathrm{Ai}(y),$$

$$\alpha = -e\left(\frac{ap}{kp} - \frac{aq}{kq}\right), \quad \beta = \frac{e^2a^2}{8}\left(\frac{1}{kp} - \frac{1}{kq}\right)$$

$$y = (4\beta)^{2/3}\left[\frac{s}{4\beta} - \left(\frac{\alpha}{8\beta}\right)^2\right],$$

$$\mathrm{Ai}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma\, e^{-i\sigma^3/3 - iy\sigma}$$

- Don't mix up α here with $\alpha = e^2/4\pi$!

Remark on the constant crossed field I

$$F^{\mu\nu} = k^\mu a^\nu - k^\nu a^\mu, \qquad F^{*\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$

- Crossed field field strength obeys some simple relations

$$(F^*F^*)^{\mu\nu} = (F^2)^{\mu\nu} = -a^2 k^\mu k^\nu,$$
$$(F^*F)^{\mu\nu} = (FF^*)^{\mu\nu} = 0,$$
$$FFF = 0, \quad F^n = 0,\, n \geq 3$$

- If we have a 4-vector p^μ we can construct only 4 non-zero 4-vectors in combinations with F:

$$p^\mu, \quad (Fp)^\mu, \quad (F^*p)^\mu, \quad (F^2p)^\mu$$

- Their scalar combinations (other are zero):

$$p^2, \quad e^2\left[(Fp)^\mu\right]^2 = e^2\left[(F^*p)^\mu\right]^2 = -e^2 p_\mu (F^2p)^\mu = -m^6\chi_p^2$$

Remark on the constant crossed field II

- Any 4-vector can be expanded:

$$l^\mu = C_1 p^\mu + C_2 (Fp)^\mu + C_3 (F^* p)^\mu + C_4 (F^2 p)^\mu$$

- To find C_i we multiply l^μ by p_μ, $(Fp)_\mu$, $(F^* p)_\mu$, $(F^2 p)_\mu$
- Arising scalar combinations:

$$\begin{aligned} & lp \\ & lFp \longrightarrow \rho = -\frac{e\,(lFp)}{\xi m^4 \chi_l}, \\ & lF^* p \longrightarrow \tau = \frac{e\,(lF^* p)}{m^4 \chi_l} \\ & lF^2 p = \frac{m^6}{e^2} \chi_l \chi_p \end{aligned}$$

- Thus

$$C_1 = \frac{\chi_l}{\chi_p},\; C_2 = \frac{e\chi_l}{m^2 \chi_p^2} \xi \rho,\; C_2 = -\frac{e\chi_l}{m^2 \chi_p^2} \tau,\; C_4 = \frac{1}{m^6 \chi_p^2} \left(pl - \frac{\chi_l}{\chi_p} p^2 \right)$$

$$F^{\mu\nu} = k^\mu a^\nu - k^\nu a^\mu, \quad kp = m^2 \frac{\chi_p}{\xi}, \quad \xi^2 = -\frac{e^2 a^2}{m^2}$$

Remark on the constant crossed field III

Exercise 4

Using formulas from previous slides prove that

1. $l^2 = -\frac{\chi_l^2}{\chi_p^2} p^2 - \frac{\chi_l^2}{\chi_p^2}(\xi^2\rho^2 + \tau^2)m^2 + 2\frac{\chi_l}{\chi_p} pl;$

2. $C_4 = \frac{1}{2m^6\chi_p^2}\left(-\frac{\chi_l}{\chi_p}p^2 + \frac{\chi_p}{\chi_l}l^2 + \frac{\chi_l}{\chi_p}(\xi^2\rho^2 + \tau^2)m^2\right);$

3. also using the conservation law $sk + p = q + l$ show that

$$s = \frac{\xi}{2}\left[\frac{1}{\chi_q}\frac{q^2}{m^2} - \frac{1}{\chi_p}\frac{p^2}{m^2} + \frac{1}{\chi_l}\frac{l^2}{m^2} + \frac{\chi_l}{\chi_p\chi_q}(\xi^2\rho^2 + \tau^2)\right]$$

.

$M^{(2)}$: Sketch of calculation V, changing variables

$$M_{00}(p,p') = -ie^2 \int \boxed{d^4l\, ds\, ds'}\, \delta\left[(s-s')k + p - p'\right]$$
$$\times \frac{-2\not{q} + 4m}{(q^2 - m^2 + i0)(l^2 + i0)} A_0(s|p,q) A_0(-s'|q,p)$$

- Variable change (all of them $\in (-\infty, +\infty)$)

$$\left\{ \begin{array}{c} l^0 \\ \\ l^1 \\ \\ l^2 \\ \\ l^3 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} l^2 \\ \\ \chi_l = \xi \dfrac{kl}{m^2} \\ \rho = -\dfrac{e\,(lFp)}{\xi m^4 \chi_l} \\ \tau = \dfrac{e\,(lF^*p)}{m^4 \chi_l} \end{array} \right\} \qquad J^{-1} = m^2 \xi \frac{|\chi_l|}{2\chi_p^2}$$

$$s' \longrightarrow s'' = s' - s$$

$s \longrightarrow q^2$ formula from prev slide

- $$\boxed{d^4l\, ds\, ds' = \frac{\xi^2}{4\chi_p^2} \left| \frac{\chi_l}{\chi_q} \right| dq^2\, dl^2\, d\chi_l\, d\tau\, d\rho\, ds''}$$

$M^{(2)}$: Sketch of calculation VI, changing variables

- Consider $\alpha(p,q)$, $\beta(p,q)$:

$$\alpha = -e\left(\frac{ap}{kp} - \frac{aq}{kq}\right) = -\frac{e}{(kp)(kq)}\left[(kq)(ap) - (kp)(aq)\right] = -\frac{\chi_l}{\chi_p\chi_q}\xi^3\rho,$$

$$\beta = \frac{e^2a^2}{8}\left(\frac{1}{kp} - \frac{1}{kq}\right) = \frac{1}{8}\frac{\chi_l}{\chi_p\chi_q}\xi^3,$$

$$\frac{\alpha}{8\beta} = -\rho$$

- Some formulas that we use

$$sk + p = q + l, \quad kp = kq + kl,$$

$$(lFp) = -(qFp) = -(kq)(ap) + (kp)(aq) = -\frac{\xi m^4\chi_l}{e}\rho,$$

$$kp = m^2\frac{\chi_p}{\xi}, \quad \xi^2 = -\frac{e^2a^2}{m^2}$$

$M^{(2)}$: Sketch of calculation VII, combining A_0 functions

- The last change of variable $\chi_l \longrightarrow u = \chi_l/\chi_q$, $\chi_p \longrightarrow \chi$

$$\chi_q = \frac{1}{1+u}\chi, \quad \chi_l = \frac{u}{1+u}\chi, \quad d\chi_l = \chi \frac{du}{(1+u)^2},$$

$$\alpha = \frac{u}{\chi}\xi^3\rho, \quad \frac{1}{8}\frac{u}{\chi}\xi^3, \quad \frac{\alpha}{8\beta} = -\rho$$

- $$A_0(s|p,q) = \frac{1}{(4\beta)^{1/3}} \exp\left[-is\frac{\alpha}{8\beta} + i\frac{8\beta}{3}\left(\frac{\alpha}{8\beta}\right)^3\right] \mathrm{Ai}(y)$$

$$= \frac{1}{\xi}\left(\frac{2\chi}{u}\right)^{1/3} \boxed{\exp\left[i\rho s - i\frac{1}{3}\frac{u}{\chi}\xi^3\rho^3\right]} \mathrm{Ai}(y)$$

$$y = \left(\frac{u}{2\chi}\right)^{2/3}\left(\frac{1+u}{u^2}\frac{l^2}{m^2} - \frac{1}{u}\frac{p^2}{m^2} + \frac{1+u}{u}\frac{q^2}{m^2} + \tau^2\right)$$

y does not depend on ρ!

$$A_0(-s'|q,p) = \frac{1}{\xi}\left(\frac{2\chi}{u}\right)^{1/3} \boxed{\exp\left[-i\rho s' + i\frac{1}{3}\frac{u}{\chi}\xi^3\rho^3\right]} \times \mathrm{Ai}\left(y + \frac{s''}{(4\beta)^{1/3}}\right)$$

$M^{(2)}$: Sketch of calculation VIII, integrating over $d\rho\, ds''$

$$M_{00}(p,p') = -ie^2 \frac{\xi^2}{4\chi} \int \frac{dq^2\, dl^2\, du}{(1+u)^2}\, d\tau \boxed{d\rho\, ds''} |u| \delta\left(-s''k + p - p'\right)$$
$$\times \frac{\boxed{-2\not{q} + 4m}}{(q^2 - m^2 + i0)(l^2 + i0)} \boxed{A_0(s|p,q)A_0(-s'|q,p)}$$

- $(-2\not{q} + 4m)A_0(s|p,q)A_0(-s'|q,p') \propto \begin{bmatrix} 1 \\ \rho \\ \rho^2 \end{bmatrix} \times \exp\left[-is''\rho\right]$
- $\displaystyle\int d\rho\, \rho^n f(s'') \exp\left[-is''\rho\right] = (-i)^n 2\pi f^{(n)}(0) \boxed{\delta(s'')}$
- $\displaystyle\int ds'' \delta(s'') \ldots$ is now trivial
- $\delta\left(-s''k + p - p'\right) \longrightarrow \delta\left(p - p'\right)$, $M^{(2)}$ is diagonal!

  $$M^{(2)}(p,p') = (2\pi)^4 \delta\left(p - p'\right) M^{(2)}(p,F)$$

- Let's follow only $-2\not{q} + \boxed{4m}$

  $$2\pi u\, A_0(s|p,q)A_0(-s|q,p) = 2\pi \frac{1}{\xi^2} 2\chi \left(\frac{u}{2\chi}\right)^{1/3} \mathrm{Ai}(y)^2$$

$M^{(2)}$: Sketch of calculation IX, integrating over $d\tau$

$$M_{00}^{4m}(p,F) = 4m\frac{-i\pi e^2}{(2\pi)^4}\int \frac{dq^2\, dl^2\, du}{(1+u)^2}\boxed{d\tau}\frac{|u|}{u}\left(\frac{u}{2\chi}\right)^{1/3} \times \frac{1}{(q^2-m^2+i0)(l^2+i0)}\boxed{\mathrm{Ai}^2(y)}$$

$$y = \frac{t}{2^{2/3}} + \left(\frac{u}{2\chi}\right)^{2/3}\boxed{\tau^2}, \quad t = \left(\frac{u}{\chi}\right)^{2/3}\left(\frac{1+u}{u^2}\frac{l^2}{m^2} - \frac{1}{u}\frac{p^2}{m^2} + \frac{1+u}{u}\frac{q^2}{m^2}\right)$$

- Let's use the property of $\mathrm{Ai}(y)$:

$$\int_0^\infty \frac{dz}{\sqrt{z}}\mathrm{Ai}^2\left(\frac{t}{2^{2/3}}+z\right) = \frac{1}{2}\mathrm{Ai}_1(t) \equiv \frac{1}{2}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\sigma}{\sigma - i0}\, e^{-i\sigma^3/3 - it\sigma},$$

$$\int_{-\infty}^{\infty} d\tau \mathrm{Ai}^2(y) = 2\times\frac{1}{2}\left(\frac{2\chi}{u}\right)^{1/3}\int_0^\infty \frac{dz}{\sqrt{z}}\mathrm{Ai}(y) = \boxed{\frac{1}{2}\left(\frac{2\chi}{u}\right)^{1/3}\mathrm{Ai}_1(t)}$$

$M^{(2)}$: Sketch of calculation X, virtualities

$$M_{00}^{4m}(p,F) = 4m\frac{-i\pi e^2}{2(2\pi)^4}\int\limits_{-\infty}^{\infty}\frac{du}{(1+u)^2}\frac{|u|}{u}\underbrace{\boxed{\int dq^2\,dl^2\,\frac{\mathrm{Ai}_1(t)}{(q^2-m^2+i0)(l^2+i0)}}}_{I_1(u)}$$

- The last step — calculate the integrals over virtualities:

$$I_1 = \int\limits_{-\infty}^{\infty} d\mu \int\limits_{-\infty}^{\infty} d\lambda \frac{\mathrm{Ai}_1(t)}{(\mu+i0)(\lambda+i0)}$$

$$t = \left(\frac{u}{\chi}\right)^{2/3}\left(1-\frac{1}{u}\nu+\frac{1+u}{u}\mu+\frac{1+u}{u^2}\lambda\right),$$

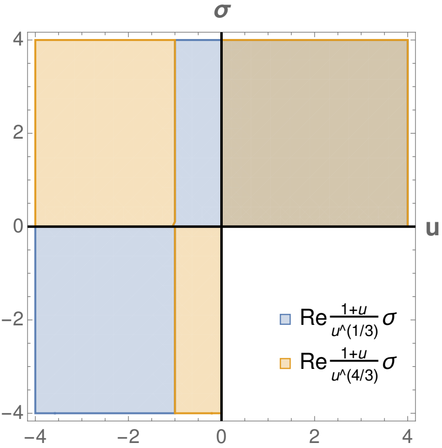
$$\mu=\frac{q^2-m^2}{m^2},\quad \lambda=\frac{l^2}{m^2},\quad \nu=\frac{p^2-m^2}{m^2}$$

- $\mathrm{Ai}_1(t) = \dfrac{1}{2\pi i}\displaystyle\int\limits_{-\infty}^{\infty}\frac{d\sigma}{\sigma-i0}\,\exp\left[-i\sigma^3/3-it\sigma\right]$

$M^{(2)}$: Sketch of calculation XI, the master integrals

- Swap the integrals:

$$I_1(u) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\sigma}{\sigma - i0}\, e^{-i\sigma^3/3 - iz\sigma}\underbrace{\int_{-\infty}^{\infty} d\mu \frac{e^{-ia\mu}}{\mu + i0}}_{-2\pi i\theta(\mathrm{Re}\, a)} \times \underbrace{\int_{-\infty}^{\infty} d\lambda \frac{e^{-ib\lambda}}{\lambda + i0}}_{-2\pi i\theta(\mathrm{Re}\, b)},$$



$$z = \left(\frac{u}{\chi}\right)^{2/3}\left(1 - \frac{1}{u}\nu\right),$$

$$a = \frac{1+u}{\chi^{2/3}u^{1/3}}\,\sigma,$$

$$b = \frac{1+u}{\chi^{2/3}u^{4/3}}\,\sigma,$$

$$\boxed{\theta(\mathrm{Re}\, a)\theta(\mathrm{Re}\, b) = \theta(u)\theta(\sigma)}$$

$M^{(2)}$: Sketch of calculation XII, renormalization I

- $I_1(u) = 2\pi i\theta(u) \int\limits_{\mathbf{0}}^{\infty} \frac{d\sigma}{\sigma - i0}\, e^{-i\sigma^3/3 - it_{(\nu)}\sigma}$ — UV divergent!

- This divergency is **purely of QED nature** and does not depend on the external field

- You may directly apply a standard regularization scheme like dimensional reg. or Pauli-Villars reg., but while we are interested in SF effects it is more efficient to do the following:

$$M^{(2)}(p,F) \to M^{(2)}_{\mathrm{ren}}(p,F) = \underbrace{\left[M^{(2)}(p,F) - M^{(2)}(p,0)\right]}_{M^{(2)}_{\mathrm{R}} \text{ – regular part}} + M^{(2)}_{\mathrm{ren}}(p,0)$$

- We need $M^{(2)}(p,0)$ (in our case $M^{4m}_{00}(p,0)$).

$M^{(2)}$: Sketch of calculation XII, renormalization II

- When we started from the dressed vertex, we got:

$$A_0(s) = \int_{-\infty}^{\infty} \frac{d\varphi}{2\pi} \exp\left[-is\varphi + i\frac{\alpha}{2}\varphi^2 - i\frac{4\beta}{3}\varphi^3\right]$$

and after substitution $\varphi \to \sigma = (4\beta)^{1/3}(\varphi - \alpha/8\beta)$ we get $\mathrm{Ai}(y)$

- $F = 0 \Longrightarrow \alpha = \beta = 0$, and the substitution is trivial $\varphi \to \sigma$:

$$A_0(s)\Big|_{F=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\sigma \, \exp\left[-is\sigma\right]$$

The rest calculation is similar, but **instead** of Ai_1 we will arrive to

$$\mathrm{Ai}_1(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\sigma}{\sigma - i0} e^{-i\sigma^3/3 - it\sigma} \longrightarrow \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\sigma}{\sigma - i0} e^{-it\sigma}$$

- The subtraction $\left[M^{(2)}(p,F) - M^{(2)}(p,0)\right]$ effectively leads to

$$\boxed{\mathrm{Ai}_1(t) \to \mathrm{Ai}_1^{\mathrm{R}}(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\sigma}{\sigma - i0} e^{-it\sigma}\left(e^{-i\sigma^3/3} - 1\right)}$$

$M^{(2)}$: Sketch of calculation, RESULT

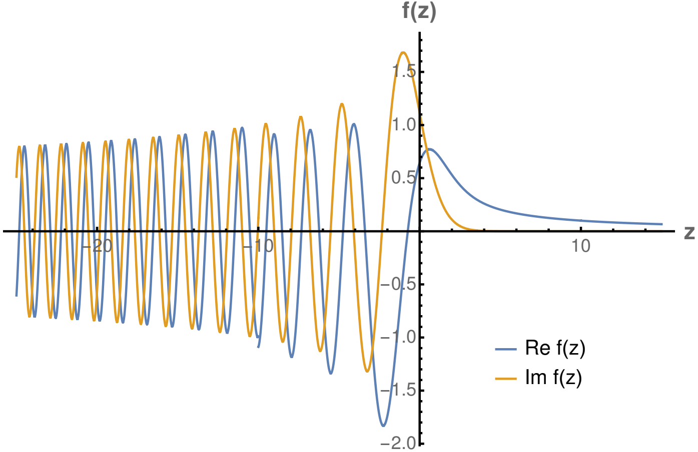
- $$I_1^{\mathrm{R}}(u) = 2\pi i\theta(u)\int\limits_0^\infty \frac{d\sigma}{\sigma - i0}\, e^{-iz\sigma}\underbrace{\left(e^{-i\sigma^3/3} - 1\right)}_{\mathcal{O}(\sigma^3),\, \sigma\to 0} = 2\pi i\theta(u) f_1(z)$$

$$\boxed{(M_{\mathrm{R}})_{00}^{4m}(p, F) = \frac{2e^2}{(4\pi)^2}\int\limits_{\mathbf{0}}^\infty \frac{du}{(1+u)^2} 2m f_1(z)}$$

- Reduced to double integral
- Considered only one term in whole γ-structure. Other terms are calculated exactly the same way.
- Regularization procedure concerns only those terms that are non-zero at $F = 0$. All other terms appear to be convergent.

The Ritus function

$$f(z) = i \int_0^\infty d\sigma \, \exp\left[-i\sigma^3/3 - iz\sigma\right]$$



$M^{(2)}$: total result

$$-iM_{\mathrm{R}}^{(2)}(p,F) = -i\frac{2e^2}{(4\pi)^2}\int\limits_0^\infty \frac{du}{(1+u)^2}$$

$$\times\left\{\left[2m - \frac{\gamma p}{1+u} - \frac{e^2(\gamma F^2 p)}{2m^4\chi^2}\left(1+\frac{u-1}{1+u}\frac{p^2}{m^2}\right)\right] f_1(z_0)\right.$$

$$-\left[\frac{e\sigma F}{m\chi} - \frac{e(\gamma F^* p)\gamma^5}{m^2\chi}\frac{2+u}{1+u}\right]\left(\frac{\chi}{u}\right)^{1/3} f(z_0)$$

$$\left.+\frac{e^2(\gamma F^2 p)}{m^4\chi^2}\frac{u^2+2u+2}{1+u}\left(\frac{\chi}{u}\right)^{2/3} f'(z_0)\right\},$$

Ritus V.I. Annals of Physics 69.2 (1972)

The on-shell matrix element

$$\mathcal{M}_{\mathrm{R}}^{(2)}(\chi) = \overline{u}_p M_{\mathrm{R}}^{(2)}(p,F) u_p$$
$$= \frac{\alpha m^2}{\pi} \int_0^\infty \frac{du}{(1+u)^2} \left\{ f_1(z) + \frac{u^2+2u+2}{1+u} \left(\frac{\chi}{u}\right)^{2/3} f'(z) - \frac{\overline{u}_p e \widehat{F^* p} \gamma^5 u_p}{2m^4(1+u)} \left(\frac{u}{\chi}\right)^{2/3} f(z) \right\},$$

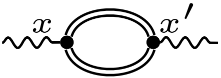
$$z = (u/\chi)^{2/3}, \quad \alpha = e^2/4\pi$$

- Imaginary part of $\mathcal{M}_{if} = \overline{u}_p M_{\mathrm{R}}^{(2)}(p,F) u_p \longrightarrow W_{\mathrm{rad}}(\chi)$ (see the 2nd lecture)
- Real part $\longrightarrow$ correction to the electron mass
- Asymptotic behaviour at $\chi \gg 1$:

$$\mathcal{M}_{\mathrm{R}}^{(2)}(\chi \gg 1) \simeq e^{-i\frac{\pi}{3}} \frac{28\sqrt[6]{3}}{27} \Gamma\left(\frac{2}{3}\right) \boxed{m^2 \alpha \chi^{2/3}}$$

Polarization operator in CCF

N.B. Narozhny Sov. Phys. JETP 28 (2) (1969); Ritus V.I. Annals of Physics 69.2 (1972)



$$i\Pi^{\mu\nu}(x,x') = e^2\,\mathrm{Tr}\left[\gamma^\mu iG(x,x')\gamma^\nu iG(x',x)\right]$$

$$\Pi^{\mu\nu}(l^2,\chi_l) = \left(Z^{-1}-1\right)g_{\mu\nu} + \pi_1(l^2,\chi_l)\varepsilon_\mu(l)\varepsilon_\nu(l) + \pi_2(l^2,\chi_l)\varepsilon^*_\mu(l)\varepsilon^*_\nu(l)$$

$$\varepsilon_\mu(l) = \frac{eF_{\mu\nu}l^\nu}{m^3\chi_l}, \qquad \varepsilon^*_\mu(l) = \frac{eF^*_{\mu\nu}l^\nu}{m^3\chi_l}$$

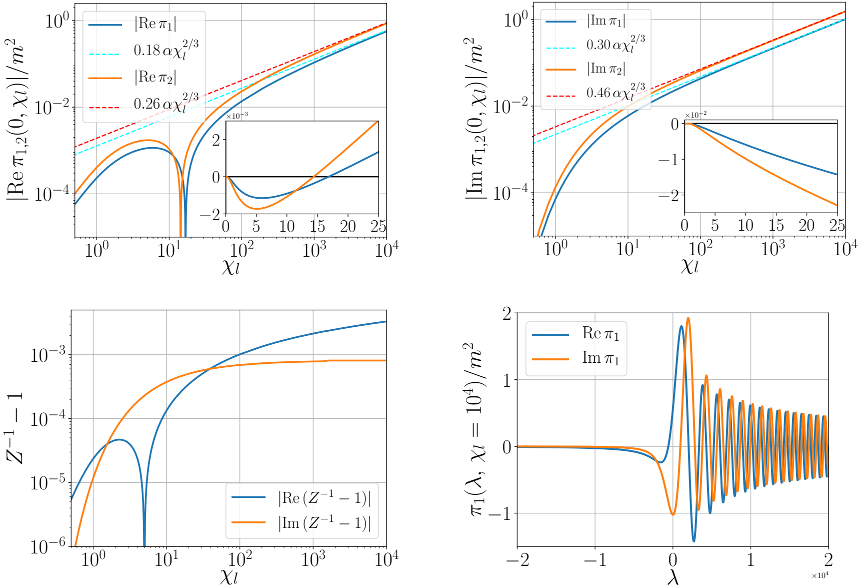
$$Z^{-1}-1 = \frac{4\alpha}{\pi}\int_4^\infty \frac{dv}{v^{5/2}\sqrt{v-4}}\left[f_1(z) - \log\left(1-\frac{1}{v}\frac{l^2}{m^2}\right)\right] \simeq -\frac{2\alpha}{9\pi}\log\chi_l \ll 1$$

$$\pi_{1,2}(l^2,\chi_l) = \frac{4\alpha\chi_l^{2/3}m^2}{3\pi}\int_4^\infty \frac{dv}{v^{13/6}}\,\frac{v\,{}^{-1}_{+2}}{\sqrt{v-4}}f'(z), \qquad z = \left(\frac{v}{\chi_l}\right)^{2/3}\left(1-\frac{l^2}{vm^2}\right)$$

$$\pi_1(0,\chi_l\gg 1) \approx e^{-i\pi/3}\frac{2}{3\sqrt[3]{6}\sqrt{\pi}}\frac{\Gamma^2(2/3)}{\Gamma(13/6)}\boxed{m^2\alpha\chi_l^{2/3}}, \qquad \pi_2(0,\chi_l\gg 1) = \frac{3}{2}\pi_1$$

Calculated either the same way as $M^{(2)}$ or via proper time representation

Graphical representation and high-χ asymptotics validity



Running coupling in CCF

- In QED polarization operator leads to renormalization of charge and:

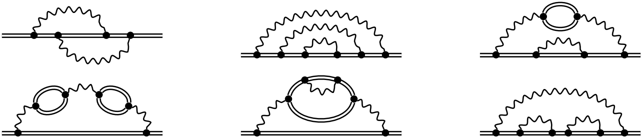
$$\alpha_{\mathrm{eff}} = \frac{\alpha}{1 - \dfrac{\alpha}{3\pi} \log \dfrac{l^2}{m^2}}$$

- In SFQED (in CCF):

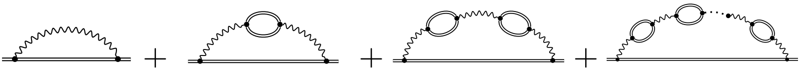
$$\alpha_{\mathrm{eff}} = \frac{\alpha}{1 - \dfrac{2\alpha}{9\pi} \log \chi_l + \ldots}$$

Ritus-Narozhny conjecture

- Expansion parameter of PT in SFQED is $\boxed{g = \alpha\chi^{2/3}}$, $\chi \sim \frac{\varepsilon}{m}\frac{E}{E_S}$
- Main contribution — loop insertions



$$\frac{\mathcal{M}}{m} =$$



$+ \cdots$

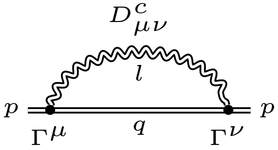
$\alpha\chi^{2/3}$	$\alpha^2\chi\log\chi$	$\alpha^3\chi^{5/3}$	$\alpha^n\chi^{(2n-1)/3}$
Ritus 1970	Ritus 1972	Narozhny 1980	conjecture

1. N.B. Narozhny, Sov. Phys. JETP 28, 371-374 (1969).
2. V.I. Ritus, Ann. Phys. 69, 555 (1972).
3. N.B. Narozhny, Phys. Rev. D 21, 1176 (1980).
4. V.I. Ritus, Journ. Russian Laser Research 6, 584 (1985).

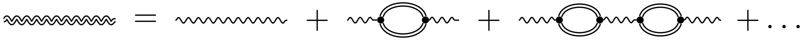
Nowadays

- A review dedicated to the memory of Nikolay Narozhny presented at LPHYS'16:
  - A.M. Fedotov, Journ. of Phys.: Conf. Series 826, 012027 (2017).
- No relation to fundamental ultraviolet behavior of QED:
  - T. Podszus and A. Di Piazza, Phys. Rev. D 99, 076004 (2019).
  - A. Ilderton, Phys. Rev. D 99, 085002 (2019).
- Experimental proposals to reach $g \gtrsim 1$:
  - C. Baumann, E.N. Nerush, A. Pukhov, I.Yu. Kostyukov, Scientific Reports 9, 9407 (2019)
  - T.G. Blackburn, A. Ilderton, M. Marklund, C. P. Ridgers, New J. Phys. 21, 053040 (2019)
  - V. Yakimenko, S. Meuren, F. Del Gaudio, C. Baumann, A. Fedotov, F. Fiuza, T. Grismayer, M.J. Hogan, A. Pukhov, L.O. Silva, G. White, Phys. Rev. Lett. 122, 190404 (2019)
- Upcoming workshop "Physics Opportunities at a Lepton Collider in the Fully Nonperturbative QED Regime", SLAC 7-9 August, 2019 `https://conf.slac.stanford.edu/npqed-2019/`.

Summation of the leading diagrams



- The exact photon propagator



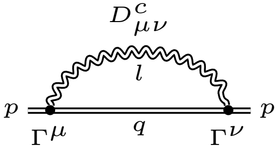
$$D^{c}_{\mu\nu}(l) = D_0(l^2, \chi_l) g_{\mu\nu} + D_1(l^2, \chi_l) \varepsilon_\mu(l) \varepsilon_\nu(l) + D_2(l^2, \chi_l) \varepsilon^*_\mu(l) \varepsilon^*_\nu(l),$$

$$D_0(l^2, \chi_l) = \frac{-i}{l^2 + i0}, \quad D_{1,2}(l^2, \chi_l) = \frac{i\pi_{1,2}}{(l^2 + i0)\,(l^2 - \pi_{1,2})}$$

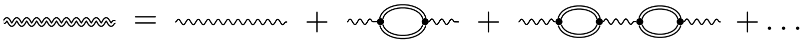
Exercise 5

Solve the Dyson equation for the exact photon propagator (derive $D^{c}_{\mu\nu}(l)$)

Summation of the leading diagrams



- The exact photon propagator



$$D^c_{\mu\nu}(l) = D_0(l^2,\chi_l)g_{\mu\nu} + D_1(l^2,\chi_l)\varepsilon_\mu(l)\varepsilon_\nu(l) + D_2(l^2,\chi_l)\varepsilon^*_\mu(l)\varepsilon^*_\nu(l),$$

$$D_0(l^2,\chi_l) = \frac{-i}{l^2+i0}, \quad D_{1,2}(l^2,\chi_l) = \frac{i\pi_{1,2}}{(l^2+i0)\,(l^2-\pi_{1,2})}$$

- We should insert $D^c_{\mu\nu}(l)$ into $M(p,p')$ and recalculate everything in the same manner. Finally, this way we arrive at:

$$\boxed{\mathcal{M}(\chi) \equiv \bar{u}_{p,s} M(p)|_{p^2=m^2} u_{p,s} = \mathcal{M}^{(2)}_{\mathrm{R}}(\chi) + \delta\mathcal{M}(\chi), \quad \delta\mathcal{M} = \sum_{i=1}^{2} \delta\mathcal{M}_i}$$

The correction $\delta\mathcal{M}_i$

$$\delta\mathcal{M}_{1,2}(\chi) = -\frac{i\alpha m^2}{(2\pi)^2}\int_{-\infty}^{+\infty}\frac{du}{(1+u)^2}\int_{-\infty}^{+\infty}\frac{d\lambda\,\pi_{1,2}/m^2}{(\lambda+i0)\,(\lambda-\pi_{1,2}/m^2)}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu+i0}$$
$$\times\left\{\left[1+\lambda\frac{u^2+2u+2}{2u^2}\right]\mathrm{Ai}_1(t)+\left(\frac{u^2+2u+2}{1+u}\pm 1\right)\left(\frac{\chi}{u}\right)^{2/3}\mathrm{Ai}'(t)\right\}$$

$$t=\left(\frac{u}{\chi}\right)^{2/3}\left(1+\frac{1+u}{u^2}\lambda+\frac{1+u}{u}\mu\right),\quad \chi_l=\frac{\chi u}{1+u},$$

$$\mathrm{Ai}_1(t)=-i\int_{-\infty}^{\infty}\frac{d\sigma}{2\pi}\,\frac{1}{(\sigma-i0)}e^{-i\sigma^3/3-it\sigma},\quad \mathrm{Ai}'(t)=-i\int_{-\infty}^{\infty}\frac{d\sigma}{2\pi}\,\sigma e^{-i\sigma^3/3-it\sigma}$$

- Main difficulty on this step — the master integral over λ

Approximating the master integral over λ

Resulting approximation:

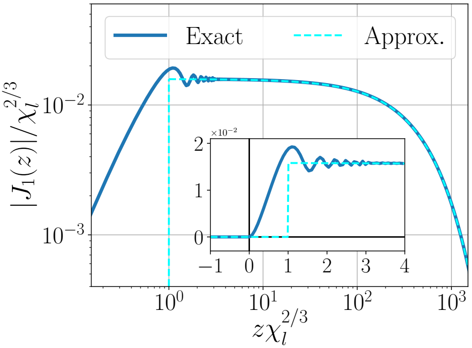
$$J_1(z) = \int_{-\infty}^{+\infty} d\lambda \, \frac{\pi(\lambda, \chi_l) e^{-i\lambda z}}{\lambda - \pi(\lambda, \chi_l)} \approx$$

$$- 2\pi i \, \theta\left(z - \chi_l^{-2/3}\right) \pi(0, \chi_l) e^{-i\pi(0,\chi_l) z}$$

Similarly:

$$J_2(z) = \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda + i0} \, \frac{\pi(\lambda, \chi_l) e^{-i\lambda z}}{\lambda - \pi(\lambda, \chi_l)} \approx$$

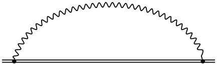
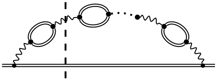
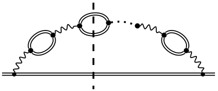
$$- 2\pi i \, \theta\left(z - \chi_l^{-2/3}\right) \left(e^{-i\pi(0,\chi_l) z} - 1\right)$$



The last step — calculating the asymptotics at $\chi \gg 1$

Calculation of $\delta\mathcal{M}$: Summary

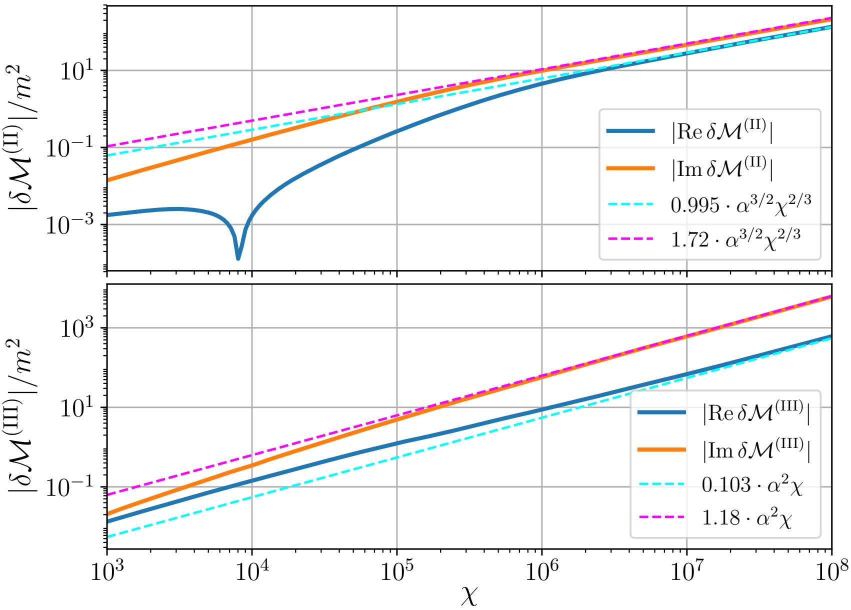
Mass radiative correction: $\mathcal{M}(\chi) = \mathcal{M}_0^{(\mathrm{ren})} + \delta\mathcal{M}, \quad \delta\mathcal{M} = \delta\mathcal{M}^{(\mathrm{II})} + \delta\mathcal{M}^{(\mathrm{III})}$

Lowest-order PQED correction $\mathcal{M}_0^{(\mathrm{ren})}$	$0.843(1 - i\sqrt{3})\alpha\chi^{2/3}m^2$	
NPQED correction due to photon emission $\delta\mathcal{M}^{(\mathrm{II})}$	$(-0.995 + 1.72i)\alpha^{3/2}\chi^{2/3}m^2$	
NPQED correction due to trident pair production* $\delta\mathcal{M}^{(\mathrm{III})}$	$-(0.103 + 1.18i)\alpha^2\chi m^2$	

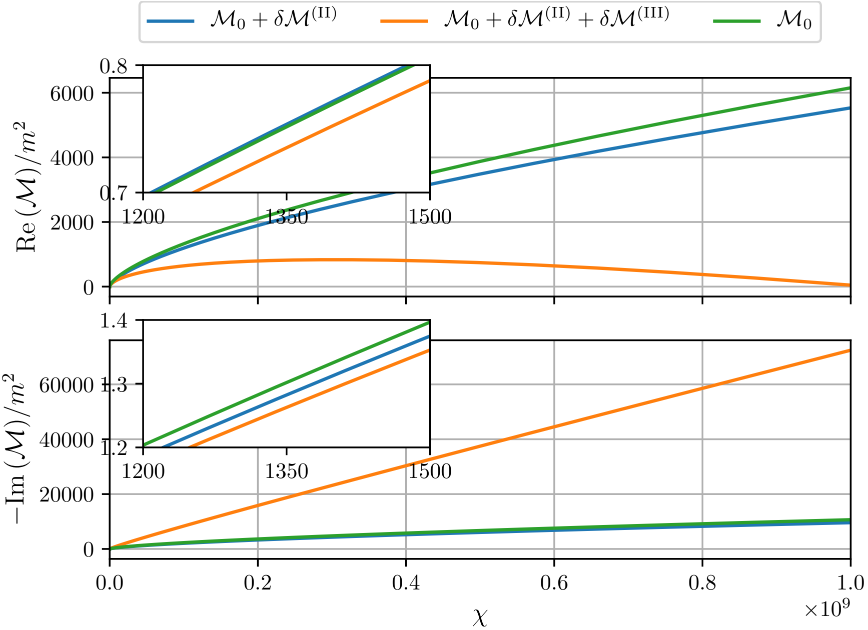
* Cf. 2-loop PQED result [Eq.(76) in Ritus 1972]:

$$\delta\mathcal{M}^{(2-\mathrm{loop})} = -[0.208 + (0.133\ln\chi - 0.725)i]\alpha^2\chi m^2$$

Justification by numerical calculation

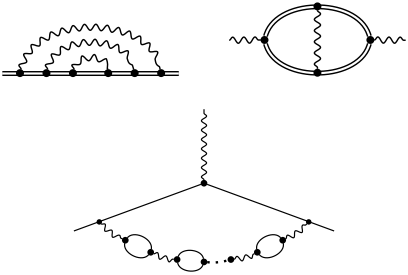


χ-dependence



Discussion

- **NOVELTY**: first truly NPQED calculation (bubble-type corrections to on-shell mass operator).
- **OBSERVATIONS**:
  - $g = \alpha\chi^{2/3} \simeq 1$: effect is small (reduction by $\lesssim 3\%$ w.r.t. 1-loop)
  - $g = \alpha\chi^{2/3} \gg 1$:
- **PROSPECTS FOR FURTHER STUDIES**:
  - diagrams with more complex virtual channels are of great potential interest!



etc. . .

Take-away message: SFQED parameters and regimes

Regime	$a_0 \ll 1$	$a_0 \gtrsim 1$
$\chi \ll 1$	classical non-relativistic (linear Thompson scattering)	classical relativistic (nonlinear Thompson =synchrotron radiation)
$\chi \gtrsim 1$	perturbative QED (Compton, Breit-Wheeler,...)	SFQED (nonlinear Compton, nonlinear Breit-Wheeler, cascades,...)

$a_0 \gg 1,\ \chi^{1/3} \Longrightarrow$ **LCFA!!!!**

$\chi \ggg 1 \Longrightarrow$ NptSFQED

I_L[W/cm^2]	PHYSICAL REGIME
5×10^{29}(?)	Sauter-Schwinger QED critical field (unstable vacuum)
2.5×10^{25}	Massive self-sustained QED cascades
10^{24}	Quantum radiation reaction, pair photoproduction ($\chi \gtrsim 1$)
5×10^{23}	Classical radiation reaction
5×10^{22}	*State-of-the-art*
3×10^{18}	Relativistic electrons ($a_0 \gtrsim 1$)
3×10^{16}	Strong field of atomic physics (immediate ionization)
10^5	External (given classical background) field concept
$< 10^5$	Week field quantum regime