Analytic Properties of Feynman diagrams in QFT

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Feynman loop diagrams in QFT are contour integrals in the complex plane. They are analytic function in the complex plane of their external variables with the cuts related to the particles propagating in the loops, and they satisfy sertain dispersion representations.

First, the two-point function $\Sigma(p^2)$ is discussed and the dispersion representation in p^2 is derived.

Second, single and double dispersion representations for the three-point function $\Gamma(p^2, p'^2, q^2)$ are considered; the appearance of the so-called anomalous thresholds is explained.

Finally, subtleties of perturbative calculations in QCD are briefly touched.

1. Two – point function

$$\Sigma(p^2) = \frac{1}{(2\pi)^4 i} \int \frac{d^4k}{(m_1^2 - k^2 - i0)(m_2^2 - (p-k)^2 - i0)}$$

UV divergent integral, satisifes a sinlge dispersion representation with one subtraction (for $m_1 = m_2 = m$ see blackboard)

$$\Sigma(p^2) = \Sigma(0) + p^2 \int\limits_{4m^2}^{\infty} \frac{ds}{\pi\, s(s - p^2 - i0)} \sigma(s), \qquad \sigma(s) = \frac{1}{16\pi}\sqrt{1 - \frac{4m^2}{s}},$$

For $m_1 \neq m_2$ (I dont't write explicitly subtraction, but it is there)

$$\Sigma(p^2) = \int\limits_{(m_1+m_2)^2}^{\infty} \frac{ds}{\pi(s - p^2 - i0)} \frac{\sqrt{(s - (m_1 - m_2)^2)(s - (m_1 + m_2)^2)}}{s}$$

Lessons:

- **$\Sigma(p^2)$ is analytic function with cut from "unitary threshold" $(m_1 + m_2)^2$ to ∞ along real axis**
- **$\Sigma(p^2)$ satisifes dispersion representation (with subtractions)**
- **$\Sigma(p^2)$ has also another branch point on unphysical sheet of its Riemann surface that does not affect the dispersion representation**
- **The imaginary part (discontinuity) of the Feynman integral is finite and may be calculated via Cutkosky rule, i.e. by replacing $\frac{1}{m^2 - k^2 - i0} \to 2\pi i \delta(m^2 - k^2)\theta(k^0)$.**
- * **Landau equations give the location of all singularities of Feynman diagrams**

2. Three – point function

$$F(q^2, p_1^2, p_2^2|m_1, m_2, m_3)) = \frac{1}{(2\pi)^4 i}\int \frac{d^4k}{(m_3^2 - k^2 - i0)(m_2^2 - (p_1 - k)^2 - i0)(m_1^2 - (p_2 - k)^2 - i0)}$$

$p_1 = p_2 + q$.

The Feynman integral is now UV convergent. For all Euclidean external momenta

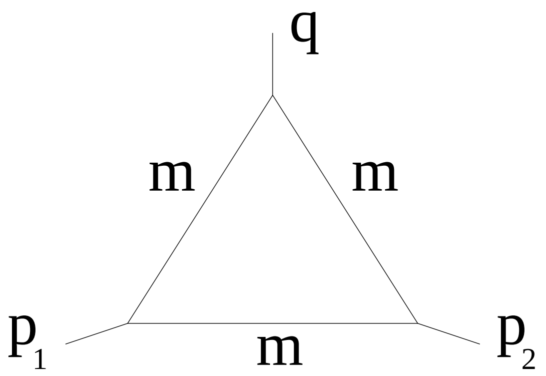
$$p_1^2 < 0, p_2^2 < 0, q^2 < 0,$$

the function $F(q^2, p_1^2, p_2^2|m_1, m_2, m_3)$ **is real and (relatively) easily calculable.**

In applications to physical phenomena (i.e. meson interations in ChPT, or hadron interactions in low-energy models, or quark-gluon interactions in QCD or in quark models) timelike values of the external momenta are needed. Dispersion representations become very useful.

One may consder *single dispersion representation* **and** *double dispersion representation*. **It turns out that at timelike external momenta, some of the singularities from the unphysical sheet move onto the physical sheet and thus lead to** *anomalous cuts and anomalous thresholds*.

2 a. Single dispersion representation in q^2



Consider the case of particles of the same mass m in the loop and $q^2 < 0$, but do not restrict the values of p_1^2 and p_2^2. A normal single dispersion representation in q^2 may be written as

$$F(q^2, p_1^2, p_2^2) = \frac{1}{\pi} \int \frac{dt}{t - q^2 - i0} \sigma(t, p_1^2, p_2^2). \tag{1}$$

For $p_1^2 < 0$ and $p_2^2 < 0$, $\sigma(t, p_1^2, p_2^2)$ may be calculated by Cutkosky rules, i.e., by placing particles attached to the q^2 vertex on the mass shell $(m^2 - k^2 - i0)^{-1} \to 2i\pi\theta(k_0)\delta(m^2 - k^2)$:

$$\sigma(t, p_1^2, p_2^2) \;=\; \frac{1}{16\pi\lambda^{1/2}(t, p_1^2, p_2^2)} \log\left(\frac{t - p_1^2 - p_2^2 + \lambda^{1/2}(t, p_1^2, p_2^2)\sqrt{1 - 4m^2/t}}{t - p_1^2 - p_2^2 - \lambda^{1/2}(t, p_1^2, p_2^2)\sqrt{1 - 4m^2/t}} \right) \theta(t - 4m^2).$$

Here $\lambda(a, b, c) = (a - b - c)^2 - 4bc$ is the triangle function.

The function $\sigma(t, p_1^2, p_2^2)$ has the branch point of the logarithm at $q^2 = t_0(p_1^2, p_2^2)$ given by the solution to the equation $(t - p_1^2 - p_2^2)^2 = \lambda(t, p_1^2, p_2^2)(1 - 4m^2/t)$, or, equivalently, to the equation

$$\frac{p_1^2 p_2^2 t}{m^2} + \lambda(p_1^2, p_2^2, t) = 0.$$

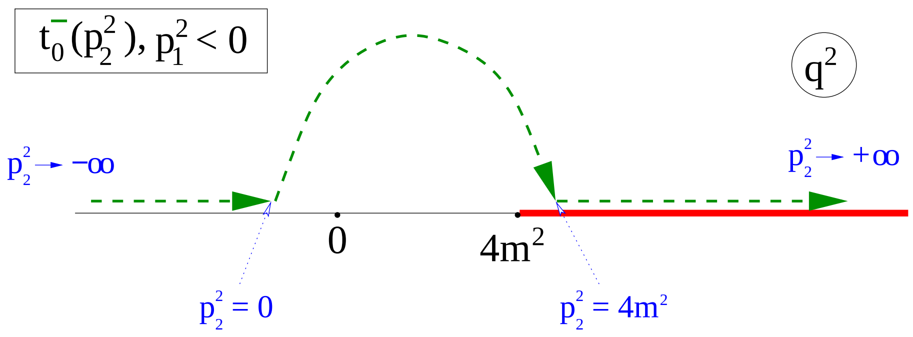
Explicitly, for $p_1^2, p_2^2 < 0$, one finds

$$t_0^\pm = p_1^2 + p_2^2 - \frac{p_1^2 p_2^2}{2m^2} \pm \frac{1}{2m^2}\sqrt{p_1^2(p_1^2 - 4m^2)p_2^2(p_2^2 - 4m^2)}.$$

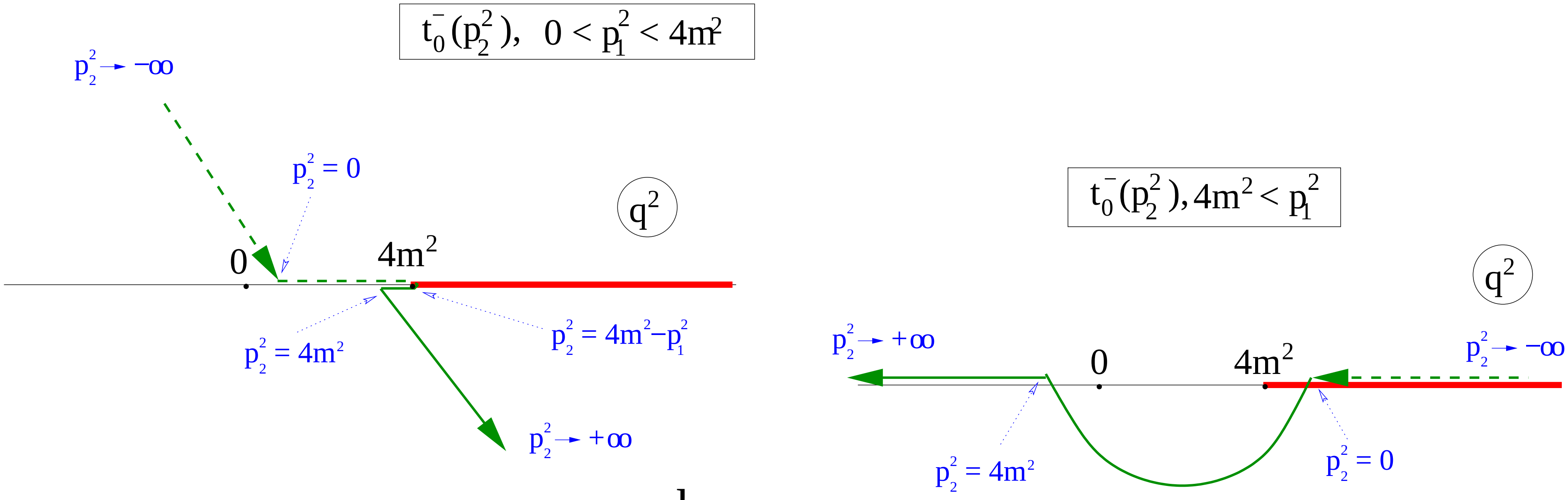
For $p_1^2 < 0$ or $p_2^2 < 0$ these branch points lie on the second (unphysical) sheet of the function σ and do not influence the q^2-dispersion representation for F.

However, in the Minkowski region of positive values of p_1^2 and p_2^2 (take care of staying on a proper branch of $\sqrt{p^2(p^2 - 4m^2)}$) the branch point t_0^- may move onto the physical sheet through the normal cut, *thus requiring the modification of the dispersion representation for F.*

The following plots show the trajectory of the branch point t_0^- vs. p_1^2 and p_2^2.



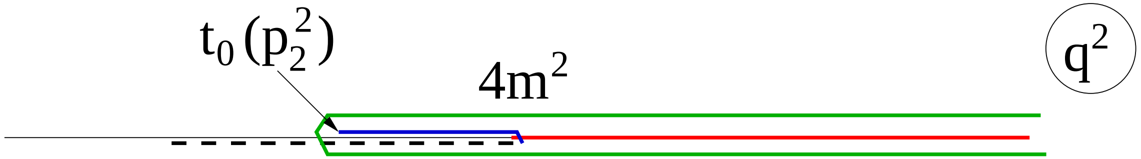
a.



b. c.

For $p_1^2 > 0$, $p_2^2 > 0$, and $p_1^2 + p_2^2 > 4m^2$, t_o^- moves onto the physical sheet and leads to anomalous cut.

Finally, for $0 < p_1^2 < 4m^2$**,** $0 < p_2^2 < 4m^2$**,** $p_1^2 + p_2^2 > 4m^2$ **the migration of** $t_0^-(p_1^2, p_2^2)$ **looks as follows**



and the single dispersion representation for F **takes the form**

$$F(q^2, p_1^2, p_2^2) \;=\; \theta(p_1^2 + p_2^2 - 4m^2) \int\limits_{t_0^-(p_1^2,p_2^2)}^{4m^2} \frac{dt}{\pi(t - q^2 - i0)} \sigma_{\text{anom}}(t, p_1^2, p_2^2)$$

$$+ \int\limits_{4m^2}^{\infty} \frac{dt}{\pi(t - q^2 - i0)} \sigma_{\text{norm}}(t, p_1^2, p_2^2).$$

For $t_0(p_1^2, p_2^2) < q^2 < 4m^2$ **(in case** $p_1^2 + p_2^2 > 4m^2$**) the imaginary part of the form factor comes from the anomalous part, while for** $q^2 > 4m^2$ **it comes from the normal part.**

Explcit expressions for $\sigma_{\text{anom}}(t, p_1^2, p_2^2)$ **and** $\sigma_{\text{norm}}(t, p_1^2, p_2^2)$ **are known**

[e.g. W. Lucha, D. M., S. Simula, PRD75, 016001 (2007); Erratum: PRD92, 019901(E) (2015) or much earlier papers which are a more difficult reading].

For a weakly bound state ($p_1^2 = p_2^2 = M^2, M = 2m - \epsilon_B, \epsilon_B \ll m$**), the anomalous threshold lies at** $t_0 = 16m\epsilon$**, and the anomalous contribution is just the one that determines the poperties of the form factor in the NR limit.**

2 b. Double dispersion representation in $p_1{}^2$ and $p_2{}^2$

For $q^2 \le 0$, the triangle diagram may be written as double dispersion representation in p_1^2 and p_2^2:

$$F(q^2, p_1^2, p_2^2) = \int \frac{ds_1}{\pi(s_1 - p_1^2 - i0)} \frac{ds_2}{\pi(s_2 - p_2^2 - i0)} \Delta(q^2, s_1, s_2). \tag{2}$$

The double spectral density $\Delta(q^2, s_1, s_2)$ may be obtained by placing all particles in the loop on the mass shell and taking the off-shell external momenta $p_1 \to \tilde{p}_1$, $p_2 \to \tilde{p}_2$, such that $\tilde{p}_1^2 = s_1$, $\tilde{p}_2^2 = s_2$, and $(\tilde{p}_1 - \tilde{p}_2)^2 = q^2$ is fixed:

$$\Delta(q^2, s_1, s_2) = \int \frac{dk_1 dk_2 dk_3}{8\pi} \delta(\tilde{p}_1 - k_2 - k_3)\delta(\tilde{p}_2 - k_3 - k_1)\theta(k_1^0)\delta(k_1^2 - m^2)\theta(k_2^0)\delta(k_2^2 - m^2)\theta(k_3^0)\delta(k_2^3 - m^2),$$

Explicitly, one finds [1]

$$\Delta(q^2, s_1, s_2) = \frac{1}{16\lambda^{1/2}(s_1, s_2, q^2)} \theta\left(s_1 - 4m^2\right)\theta\left(s_2 - 4m^2\right)\theta\left[\left(q^2(s_1 + s_2 - q^2)\right)^2 - \lambda(s_1, s_2, q^2)\lambda(q^2, m^2, m^2)\right].$$

The solution of θ-function gives the allowed intervals for integration variables s_1 and s_2:

$$4m^2 < s_2, \qquad s_1^-(s_2, q^2) < s_1 < s_1^+(s_2, q^2),$$

where

$$s_1^{\pm}(s_2, q^2) \;=\; s_2 + q^2 - \frac{s_2 q^2}{2m^2} \pm \frac{\sqrt{s_2(s_2 - 4m^2)}\sqrt{q^2(q^2 - 4m^2)}}{2m^2}.$$

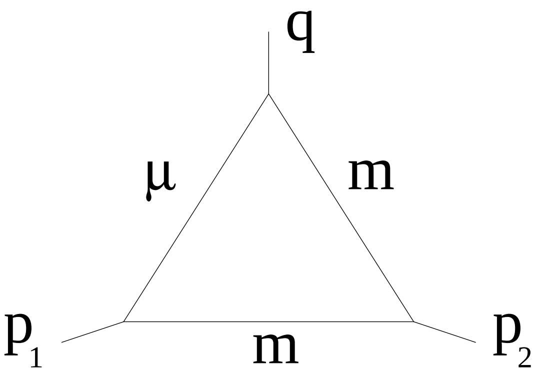
[1] The easiest way to obtain this double dispersion representation is to introduce light-cone variables in the Feynman expression, and to choose the reference frame where $q_+ = 0$ (which restricts q^2 to $q^2 < 0$). Then the k_- integral is easily done, and the remaining x and $k_\perp$ integrals may be written in the form of double spectral representation. Precisely the same way, as we did for Σ

The final double dispersion representation for the triangle diagram at $q^2 < 0$ takes the form

$$F(q^2, p_1^2, p_2^2) = \int\limits_{4m^2}^{\infty} \frac{ds_2}{\pi(s_2 - p_2^2 - i0)} \int\limits_{s_1^-(s_2,q^2)}^{s_1^+(s_2,q^2)} \frac{ds_1}{\pi(s_1 - p_1^2 - i0)} \frac{1}{16\lambda^{1/2}(s_1, s_2, q^2)}. \quad (3)$$

Notice the relation $s_1^-(s_2, q^2) > 4m^2$, which holds for all $s_2 > 4m^2$ at $q^2 < 0$: this guarantees that the integration region in s_1 always remains above the normal threshold. Clearly, the integration region does not depend on the values of p_1^2 and p_2^2. Essential for us is that no anomalous cuts emerge in the double dispersion representation in p_1^2 and p_2^2 for $q^2 < 0$. This makes the double dispersion representation particulary convenient for treating the triangle diagram for values of p_1^2 and p_2^2 above the thresholds. One should just take care about the appearance of the absorptive parts.

2 c. Double dispersion representation in $p_1{}^2$ and $p_2{}^2$ for "weak" decay knematics



For $q^2 < 0$, double dispersion representation is same as for equal masses:

$$F(q^2, p_1^2, p_2^2) = \int\limits_{4m^2}^{\infty} \frac{ds_2}{\pi(s_2 - p_2^2)} \int\limits_{s_1^-(s_2,q^2)}^{s_1^+(s_2,q^2)} \frac{ds_1}{\pi(s_1 - p_1^2)} \frac{1}{16\lambda^{1/2}(s_1, s_2, q^2)},$$

where

$$s_1^{\pm}(s_2, q^2) \;=\; \frac{s_2(m^2 + \mu^2 - q^2) + 2m^2q^2}{2m^2} \pm \frac{\lambda^{1/2}(s_2, m^2, m^2)\lambda^{1/2}(q^2, \mu^2, m^2)}{2m^2}.$$

A new feature compared to equal masses in the loop is the appearance of the region $0 < q^2 < (\mu - m)^2$, which was absent in the equal-mass case. This region corresponds to the decay of a particle of mass μ to a particle of mass m with the emission of a particle of mass $\sqrt{q^2}$.

The form factor in the region $0 < q^2 < (\mu - m)^2$ may be obtained by analytic continuation of this expression. Let us consider the structure of the singularities of the integrand in the complex s_1-plane for a fixed real value of s_2 in the interval $s_2 > 4m^2$.

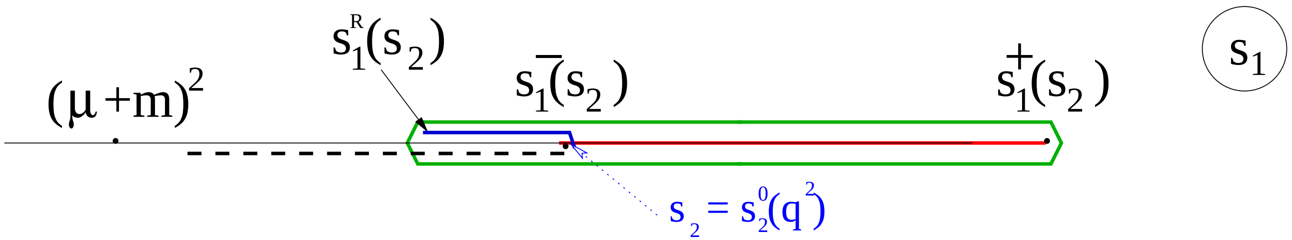
The integrand has singularities (branch points) related to the zeros of the function

$$\lambda(s_1, s_2, q^2) = (s_1 - s_1^L)(s_1 - s_1^R),$$

at $s_1^L = (\sqrt{s_2} - \sqrt{q^2})^2$ **and** $s_1^R = (\sqrt{s_2} + \sqrt{q^2})^2$**. As** $q^2 \le 0$**, these singularities lie on the unphysical sheet. However, as** q^2 **becomes positive, the point** s_1^R **may move onto the physical sheet through the cut from** s_1^- **to** s_1^+**. This happens for values of the variable** $s_2 > s_2^0$**, with** s_2^0 **obtained as the solution to the equation** $s_1^R(s_2, q^2) = s_1^-(s_2, q^2)$**. Explicitly, one finds**

$$\sqrt{s_2^0} = \frac{\mu^2 - m^2 - q^2}{\sqrt{q^2}}.$$

The trajectory of the point $s_1^R(s_2, q^2)$ **in the complex** s_1**-plane at fixed** $q^2 > 0$ **vs.** s_2 **is shown here**



As $q^2 > 0$**, for** $s_2 > s_2^0(q^2)$ **the integration contour in the complex** s_1**-plane should be deformed such that it embraces the points** s_1^R **and** s_1^+**. Respectively, the** s_1**-integration contour contains the two segments: the normal part from** s_1^- **to** s_1^+**, and the anomalous part from** s_1^R **to** s_1^-**.**

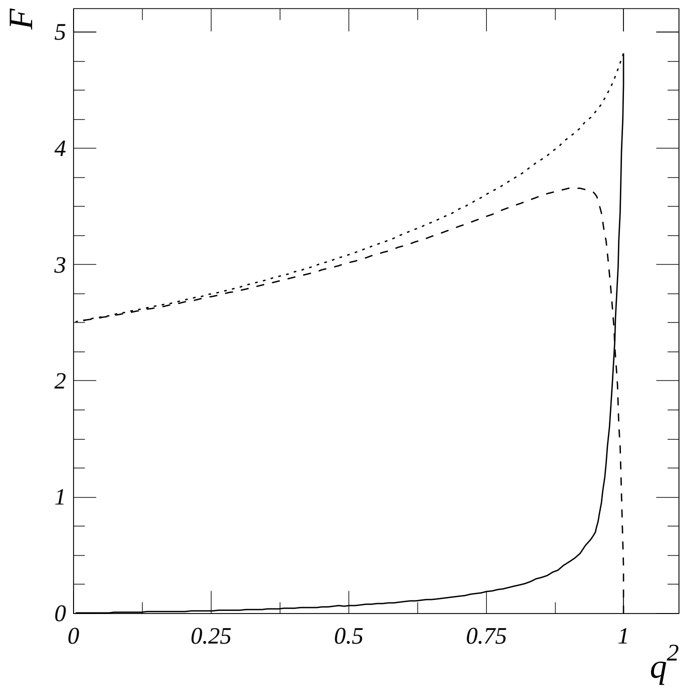
The double spectral density for the anomalous piece is just the discontinuity of the function $1/\sqrt{\lambda(s_1, s_2, q^2)}$ **that is twice the function itself** $2/\sqrt{\lambda(s_1, s_2, q^2)}$**.**

The final representation for the form factors at $0 < q^2 < (\mu - m)^2$ takes the form

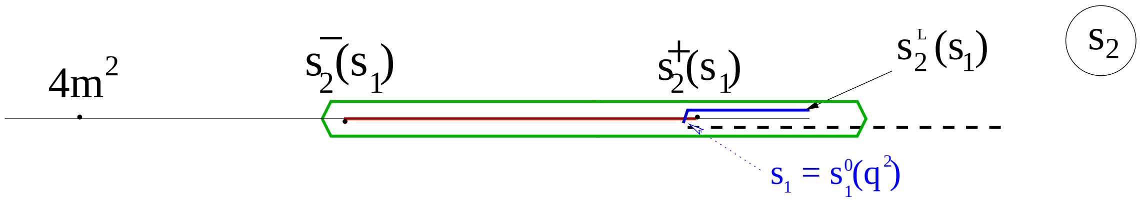
$$F(q^2, p_1^2, p_2^2) = \int\limits_{4m^2}^{\infty} \frac{ds_2}{\pi(s_2 - p_2^2 - i0)} \int\limits_{s_1^-(s_2,q^2)}^{s_1^+(s_2,q^2)} \frac{ds_1}{\pi(s_1 - p_1^2)} \frac{1}{16\lambda^{1/2}(s_1, s_2, q^2)}$$

$$+ \; 2\theta\left(0 < q^2 < (\mu - m)^2\right) \int\limits_{s_2^0(q^2)}^{\infty} \frac{ds_2}{\pi(s_2 - p_2^2 - i0)} \int\limits_{s_1^R(s_2,q^2)}^{s_1^-(s_2,q^2)} \frac{ds_1}{\pi(s_1 - p_1^2)} \frac{1}{16\lambda^{1/2}(s_1, s_2, q^2)}.$$

A typical behavior of the anomalous and the normal contributions is plotted



The picture is slightly different for the "external" s_1-integration, and the "internal" s_2-integration.



The final representation for the form factors at $0 < q^2 < (\mu - m)^2$ takes the form

$$F(q^2, p_1^2, p_2^2) = \int\limits_{(m+\mu)^2}^{\infty} \frac{ds_1}{\pi(s_1 - p_1^2 - i0)} \int\limits_{s_2^-(s_1,q^2)}^{s_2^+(s_2,q^2)} \frac{ds_2}{\pi(s_2 - p_2^2)} \frac{1}{16\lambda^{1/2}(s_1, s_2, q^2)}$$

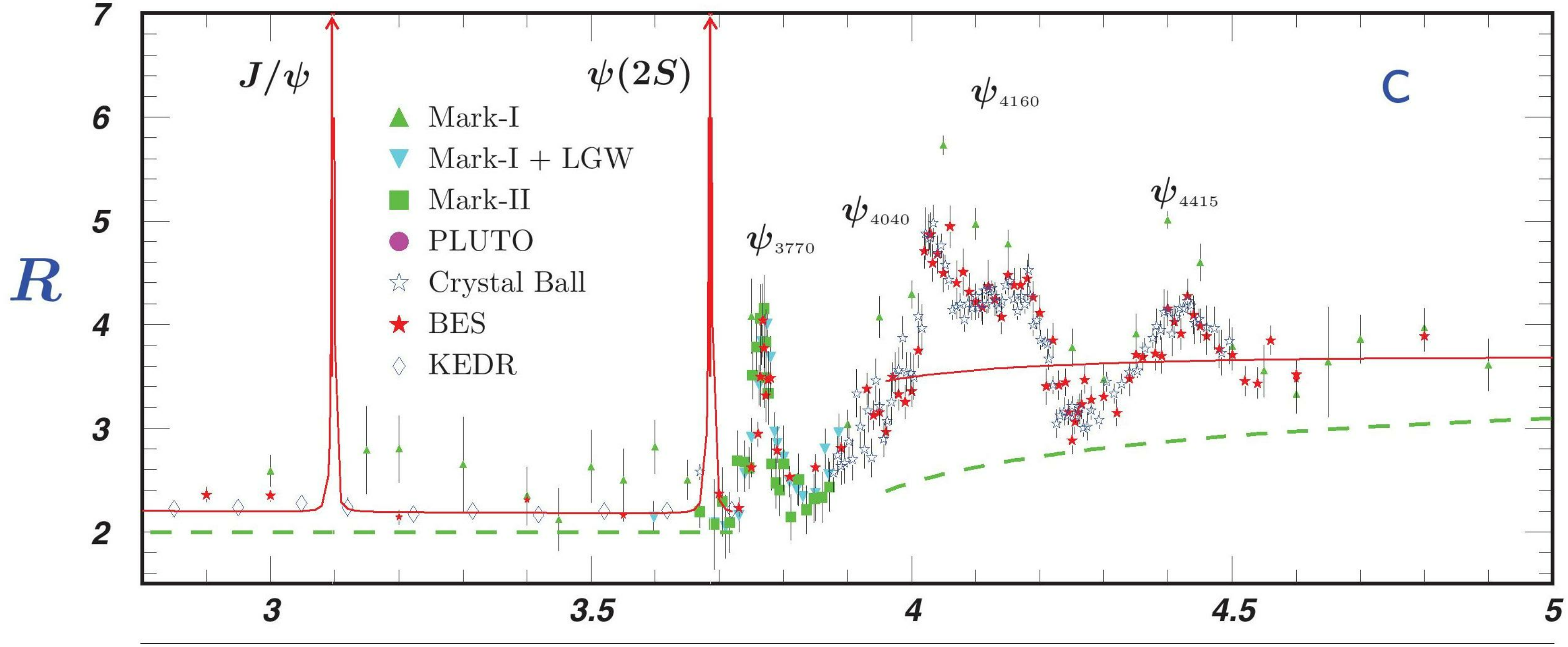
$$+ \; 2\theta\left(0 < q^2 < (\mu - m)^2\right) \int\limits_{s_1^0(q^2)}^{\infty} \frac{ds_1}{\pi(s_1 - p_1^2 - i0)} \int\limits_{s_2^+(s_2,q^2)}^{s_2^L(s_1,q^2)} \frac{ds_2}{\pi(s_2 - p_2^2)} \frac{1}{16\lambda^{1/2}(s_1, s_2, q^2)}.$$

3. Singularities of Feynman diagrams in QCD

In pQCD we work with diagrams where quarks and gluons propagate. Respectively, these diagrams contain quark and gluon singularities. However, quarks and gluons are confined and do not exist as free particles. We know that in full QCD quark/gluon singularites are replaced by hadron singularities.

(→ Blackboard)

Where quark diagrams may be used?



Summary and conclusions

• **QFT Green functions** $\Gamma(p_1^2, \ldots, p_n^2)$ **are given by contour integrals in the complex** k**-plane. As the result, the Green functions in perturbation theory are analytic functions of their variables** p_i^2 **with cuts and branch points and they satisfy dispersion representations in these variables.**

• **At spacelike values of external momenta,** $p_i^2 < 0$**, dispersion representations have only "unitary" thresholds related to the normal cuts of the Feynman diagrams. Respectively, the dispersion representations for the Green functions have only normal cuts.**
As some of the external momenta go to the timelike region, $p_i^2 > 0$**, anomalous thresholds,** *related to the motion of branch points from the unphysical sheets onto the physical sheet through the normal cuts*, **may emerge. If this happens, the anomalous cuts emerge and the corresponding anomalous contributions in the spectral representations appear.**

• **In some cases, anomalous thresholds have dramatic impact on the bound state properties: i.e. the deuteron size is determinded by the anomalous threshold at** $q^2 = 16M_N\epsilon$ **(**$\epsilon = 2.2$ **MeV) in the triangle diagram with nucleoons in the loop, and not by the normal** $\pi\pi$ **threshold at** $q^2 = 4m_\pi^2$ **in the triagle diagram with pions and the deuteron. Anomalous thresholds emerge also in other hadron decays described by triangle diagrams.**

• **Landau equations are a powerful tool to find the location of all singularities of Feynman diagrams (QFT Green functions). In particular, their solution gives also the location of anomalous thresholds.**

• **In QCD, because of confinement of quarks and gluons, diagrams of perturbation theory may be used for the description of the data far away from the quark thresholds.**