Worldline formalism for QED amplitude calculations in vacuum and in constant fields

Naser Ahmadiniaz

n.ahmadiniaz@hzdr.de

Helmholtz-Zentrum Dresden-Rossendorf

Quantum Field Theory at the Limits: from Strong Fields to Heavy Quarks
July 22th - August 2th (2019), JINR, Dubna

PART I AND II

- History
- Worldline formalism for Scalar QED (one-loop)
- Worldline formalism for Spinor QED (one-loop)
- Constant field background (one-loop)
- Worldline formalism for Scalar propagator
- Constant field background for Scalar propagator

PART III

1PR contribution to the electron propagator in background fields

HISTORY

- In 1948, Feynman developed the path integral approach to nonrelativistic quantum mechanics (based on earlier work by Wentzel and Dirac)
- Two years later, he started his famous series of papers that laid the foundations of relativistic quantum field theory (essentially quantum electrodynamics at the time) and introduced Feynman diagrams.
- However, at the same time he also developed a representation of the QED S-matrix in terms of relativistic particle path integrals. It appears that he considered this approach less promising.

- So no essential use was made of those path integral representations for many years after, excepting the work by **Affleck** et. al (1982) where they studied pair production in external fields.

- The potential of this particle path integral or **worldline** formalism to improve on standard field theory methods, at least for certain types of computations, was recognized only in the early nineties through the work of **Bern** and **Kosower** (1992) and later **Strassler** (1992).

- Since then many amplitude calculations has been done (in tree-level as well as loop orders) in QED and QCD (recently in curved space) which we will briefly discuss them here.

FREE SCALAR PROPAGATOR

Let's start with the scalar propagator which is the Green's function for the Klein-Gordon operator equation

$$D_0^{xx'} \equiv \langle 0|T\phi(x)\phi(x')|0\rangle = \langle x|\frac{1}{-\Box + m^2}|x'\rangle \tag{1}$$

We work in Euclidean convention defined by the following changes due to the Wick rotation from Minkowski space with metric $(- + ++)$

$$E = k^0 = -k_0 \to ik_4 \quad , \quad t = x_0 = -x_0 \to ix_4 \tag{2}$$

Thus (and we set $\hbar = c = 1$)

$$\Box = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2}$$

If we use the Schwinger proper time to exponentiate the denominator of the propagator

$$D_0^{xx'} = \langle x|\int_0^\infty dT e^{-T(-\Box+m^2)}|x'\rangle = \int_0^\infty dT e^{-m^2T}\langle x|e^{T\Box}|x'\rangle \tag{3}$$

Now let's compare with the definition of the free propagator:

$$\langle x', 0|x, t\rangle = \langle x, 0|e^{-itH}|x, t\rangle = \int_{x(0)=x'}^{x(t)=x} Dx(t)e^{i\int_0^t d\tau \frac{m}{2}\dot{x}^2} \tag{4}$$

where $H = -\frac{1}{2m}\nabla^2$. Using this action and applying the following replacement

$$\nabla^2 \to \Box \quad , \quad m \to \frac{1}{2} \quad , \quad \tau \to -i\tau \quad , \quad t \to -iT$$

we get

$$D_0^{xx'} = \int_0^\infty dT e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} Dx e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2} \tag{5}$$

This is the worldline representation of the relativistic scalar propagator in euclidean space from x' to x. The parameter T for us will just be an integration variable, but it has a deeper mathematical meaning related to one-dimensional reparametrization invariance.

Now again if we change the variables to

$$x^\mu = x^\mu_{\text{cl}}(\tau) + q^\mu(\tau) = \left[x^\mu + \frac{\tau}{T}(x^\mu - x'^\mu)\right] + q^\mu(\tau)$$
$$\dot{x}^\mu = \frac{x^\mu - x'^\mu}{T} + \dot{q}^\mu(\tau) \tag{6}$$

Plugging back to the propagator we obtain

$$D_0^{xx'} = \int_0^\infty dT\, e^{-Tm^2}\, e^{-\frac{(x-x')^2}{4T}} \int_{q(0)=q(T)=0} Dq(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau \dot{q}^2} \tag{7}$$

We recall from our previous discussion that the normalization factor which in D dimensions was $\left(\frac{m}{2i\pi t}\right)^{D/2}$ with the above substitution is given now as

$$\int_{q(0)=q(T)=0} Dq(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau \dot{q}^2} = \left(4\pi T\right)^{-D/2} \tag{8}$$

therefore

$$D_0^{xx'} = \int_0^\infty dT \left(4\pi T\right)^{-D/2} e^{-Tm^2} e^{-\frac{(x-x')^2}{4T}} \tag{9}$$

which is the x-space represenaion of the free scalar propagator. If we Fourier transform it we get the familiar expression

$$\begin{aligned} D_0^{pp'} &= \int d^D x \int d^D x' \, e^{ipx} e^{ip'x'} D_0^{xx'} \\ &= \int_0^\infty dT \left(4\pi T\right)^{-D/2} e^{-Tm^2} \int d^D x \int d^D x' \, e^{ipx+ip'x'} e^{-\frac{(x-x')^2}{4T}} \end{aligned} \tag{10}$$

Changing the integration variables from x and x' to

$$x - x' = x_- \quad , \quad x + x' = 2x_+$$

finally leads to

$$\boxed{D_0^{pp'} = (2\pi)^D \delta(p+p') \frac{1}{p^2+m^2}} \tag{11}$$

One-loop correction in scalar and spinor QED in vacuum and in constant background fields

COUPLING TO THE ELECTROMAGNETIC FIELD

Now we consider coupling with an external vector field A_μ, the worldline action becomes

$$S[x, A] = \int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\dot{x} \cdot A(x) \right) \tag{12}$$

which is what we expect from Maxwell theory. The full scalar propagator that interacts with the background field $A_\mu(x)$ reads as

$$D^{xx'}[A] = \int_0^\infty dT\, e^{-Tm^2} \int_{DBC} Dx(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x)\right)} \tag{13}$$

For the one-loop case, we already introduced the effective action in the background field which takes into account the one-loop correction, namely

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int_{\mathrm{PBC}} Dx(\tau)\, e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ie\dot{x}\cdot A(x(\tau))\right)} \tag{14}$$

One-loop correction$\rightarrow$ virtual particles

The effective action contains the quantum effects caused by the presence of such particles in the vacuum for the background field. In particular, it causes electrodynamics to become a nonlinear theory at the one-loop level, where photons can interact with each other in an indirect fashion.

GAUSSIAN INTEGRALS

Techniques for efficient calculation of these path integrals developed much later than their first discovery by Feynman in 1950. Presently there are three different methods available, namely

- The analytic or **string-inspired** approach, based on the use of worldline Green's functions.
- The semi-classical approximation, based on a stationary trajectory (**worldline instanton**).
- Direct numerical computation of the path integral (Worldline Monte Carlo)

Here we follow the first approach and we introduce Green's functions.

In the **string-inspired** approach all path integrals are brought into gaussian form. They are then calculated by a formal extension of the n-dimensional gaussian integration formulas to infinite dimensions.

As a reminder in n-dimensions

$$\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x} = \frac{(4\pi)^{\frac{n}{2}}}{(\det M)^{\frac{1}{2}}}$$

$$\frac{\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x + J.x}}{\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x}} = e^{j.M^{-1}.j} \tag{15}$$

M is assumed to be $n \times n$ symmetric and positive definite. Also, by multiple differentiation of the second formula with respect to the components of the vector j one gets

$$\frac{\int d^n x\, (x_i x_j)\, e^{-\frac{1}{4}x\cdot M\cdot x + J.x}}{\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x}} = 2M^{-1}_{ij}$$

$$\frac{\int d^n x\, (x_i\, x_j\, x_k\, x_l)\, e^{-\frac{1}{4}x\cdot M\cdot x + J.x}}{\int d^n x\, e^{-\frac{1}{4}x\cdot M\cdot x}} = 4\left(M^{-1}_{ij}M^{-1}_{kl} + M^{-1}_{ik}M^{-1}_{jl} + M^{-1}_{il}M^{-1}_{jk}\right)$$

$$\vdots \qquad\qquad \vdots$$

DRESDEN concept HZDR

N-PHOTON AMPLITUDE, ONE-LOOP

Now we derive a master formula for the one-loop correction to the scalar QED. To obtain such N-photon amplitude we consider a scalar particle, while moving along the closed trajectory in spacetime, absorbs or emits a fixed number of N of quanta of the background field, that is photons of fixed momentum k and polarization ε. In field theory to do this we specialze the background field which so far was arbitrary, specialze to plane wave background

$$A^{\mu}(x) = \sum_{i=}^{N} \varepsilon_i^{\mu} e^{ik_i \cdot x(\tau)} \tag{16}$$

Then if we expand the interaction part of the amplitude to A^N order we get

$$\frac{(-ie)^N}{N!} \Big(\int_0^T d\tau \sum_{i=1}^{N} \varepsilon_i \cdot \dot{x}(\tau_i) e^{ik_i \cdot x(\tau_i)} \Big) \tag{17}$$

In total there are N^N terms from which only $N!$ are **totally mixed** terms with N different polarization and momenta, therefore the $1/N!$ cancels out and we have

$$(-ie)^N \int_0^T d\tau_1 \varepsilon_1 \cdot \dot{x}(\tau_1) \cdots \int_0^T d\tau_1 \varepsilon_N \cdot \dot{x}(\tau_N) \tag{18}$$

which can be written compactly as

$$(-ie)^N V^{\gamma}_{\text{scal}}[k_1, \varepsilon_1] \cdots V^{\gamma}_{\text{scal}}[k_N, \varepsilon_N] \tag{19}$$

where

$$V^{\gamma}_{\text{scal}}[k, \varepsilon] = \int_0^T d\tau \varepsilon \cdot \dot{x}(\tau) e^{ik\cdot x(\tau)}$$

is known as the photon vertex operator. This is the same vertex operator which is used in (open) string theory to describe the emission or absorption of a photon by a string.

Therefore the N-photon amplitude can be written as

$$\Gamma[k_1, \varepsilon_1; \cdots k_N, \varepsilon_N] = (-ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{\mathrm{PBC}} Dx(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2} V^\gamma[k_1, \varepsilon_1] \cdots V^\gamma[k_N, \varepsilon_N]$$

Note that each vertex operator represents the emission or absorption of a single photon, however the moment when this happens is arbitrary and must therefore be integrated over.

Doing it for arbitrary N the way it stands would still be difficult, though, due to the factors of $\dot{x}$'s, but let's use the following trick to rewrite the vertex operator

$$V^\gamma[k_i, \varepsilon_i] = \int_0^T d\tau_i \varepsilon \cdot \dot{x}_i\, e^{ik_i \cdot x_i} \equiv \int_0^T d\tau_i\, e^{ik_i \cdot x_i + \varepsilon_i \cdot \dot{x}_i}\Big|_{\text{line in } \varepsilon} \tag{20}$$

DRESDEN concept HZDR

We can write the kinetic term as (Exercise 1)

$$\int_0^T d\tau \dot{x}^2 = \int_0^T d\tau x \left(-\frac{d^2}{d\tau^2} \right) x \tag{21}$$

We first decompose the coordinate to the zero-mode part (x_0) and the relative coordinate q

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau) \rightarrow \int Dx(\tau) = \int d^D x_0 \int Dq(\tau) \tag{22}$$

where the zero-mode corresponds to the path integral over closed trajectories includes the constant loops $x(\tau) = \text{constant}$ where the kinetic term is zero which in the Gaussian integral it corresponds to a zero eigenvalue of the matrix M. To solve it, we define the loop centre-of-mass (or average position) by

$$x_0 = \frac{1}{T} \int_0^T d\tau x^\mu(\tau) \rightarrow \int_0^T d\tau q^\mu = 0$$

This zero-mode integral eventually after reexponentiating the vertex operators leads to

$$\int d^D x_0 \, e^{\sum_{i=1}^{N} k_i \cdot x_0} = (2\pi)^D \delta^D \left(\sum_{i=1}^{N} k_i \right) \tag{23}$$

This is just the expected global delta function for energy-momentum conservation.

Now we need to use the *D*-dimensional Gaussian integral formulas to find

$$\det M = (4T)^D$$

In the reduced Hilbert space without the zero mode, the kinetic operator is invertible, and the inverse is easily found using the eigenfunctions of the derivative operator on the circle with circumference T $\{e^{2\pi i n \frac{\tau}{T}}, n \in \mathbf{Z}/0\}$

$$G_B(\tau, \tau') = 2\langle \tau | (\frac{d}{d\tau})^{-2} | \tau' \rangle = 2T \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n \frac{\tau - \tau'}{T}}}{(2\pi i n)^2}$$

$$= |\tau - \tau'| - \frac{(\tau - \tau')^2}{T} - \frac{T}{6}$$

Note that

- $G_B(\tau, \tau')$ is a function of $\tau - \tau'$ $\rightarrow$ it is translational invariance.
- the subscript B stands for Bosonic Green's function
- in flat space calculations $-T/6$ is irrelevant and can be omitted so $G_B(\tau, \tau') \rightarrow |\tau - \tau'| - \frac{(\tau-\tau')^2}{T}$

We will later on need its first and second derivative which are

$$\dot{G}_B = \partial_\tau G_B = \text{sign}(\tau - \tau') - \frac{2(\tau - \tau')}{T}$$
$$\ddot{G}_B = \partial_\tau^2 G_B = 2\delta(\tau - \tau') - \frac{2}{T}$$

Now, to use the Gaussian integral formula we have already introduced we need to define

$$j(\tau) = \sum_{i=1}^{N} \left(i\delta(\tau - \tau')k_i - \dot{\delta}(\tau - \tau')\varepsilon_i \right) \tag{24}$$

Now we can rewrite the exponent of the vertex operator

$$e^{\sum_{i=1}^{N}(ik_i \cdot q_i + \varepsilon_i \cdot \dot{q}_i)} = e^{\int_0^T d\tau j(\tau)\cdot q(\tau)} \tag{25}$$

where we used

$$\int_0^T d\tau \dot{\delta}(\tau - \tau')q(\tau) = -\dot{q}(\tau')$$

therefore

$$\frac{Dq(\tau)e^{-\int_0^T \frac{1}{4}\dot{q}^2} e^{\sum_{i=1}^{N}(ik_i \cdot q + \varepsilon_i \cdot \dot{q}_i)}}{Dq(\tau)e^{-\int_0^T \frac{1}{4}\dot{q}^2}} = \exp\left\{ -\frac{1}{2}\int_0^T d\tau \int_0^T d\tau' G_B(\tau,\tau') j(\tau)\cdot j(\tau') \right\}$$

$$= \quad \exp\left\{ \sum_{i,j=1}^{N} \left[\frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij}\varepsilon_i \cdot k_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i \cdot \varepsilon_j \right] \right\}$$

One can already see that $-T/6$ which we have omitted has no effect since it only appears in the first term and it goes away by momentum conservation.

Finally, we need also the absolute normalization of the free path integral, which turns out to be the same as in the DBC case

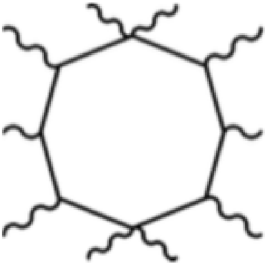
$$\int Dq(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{q}^2} = \Big(4\pi T\Big)^{-\frac{D}{2}} \tag{26}$$

Putting things together, we get the famous **Bern-Kosower master formula**

$$\Gamma_{\text{scal}}[k_1, \varepsilon_1; \cdots; k_N, \varepsilon_N] = (-ie)^N (2\pi)^D \delta^D(\sum_i k_i) \int_0^T \frac{dT}{T}\, e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times \exp\Big\{ \sum_{i,j=1}^N \Big[\frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2}\ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j\Big]\Big\}\Big|_{\text{lin } \varepsilon_1 \varepsilon_2 \cdots \varepsilon_N} \tag{27}$$

The master formula simply gives a sum over all possible diagrams in each order:

$$\Gamma[A_\mu] = \sum$$



- This formula (or rather its analogue for the QCD case) was first derived by Bern and Kosower from string theory (PRL 66 (1991) 1669, NPB 379 (1992) 45).
- It re-derived in the present approach by Strassler (NPB 385 (1992)).
- As it stands, it represents the one-loop N-photon amplitude in scalar QED.
- Bern and Kosower also derived a set of rules which allows one to construct, starting from this master formula and by purely algebraic means, parameter integral representations for the N-photon amplitudes with a fermion loop, as well as for the N-gluon amplitudes involving a scalar, spinor or gluon loop.

THE VACUUM POLARIZATION

In this part of our lecture let's use our master formula and compute the vacuum polariation diagram which is obtained by setting $N = 2$

$$\Gamma^2_{\text{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = (-ie)^2 (2\pi)^4 \delta^D(k_1 + k_2) \int_0^\infty \frac{dT}{T} (4\pi T)^{-D/2} e^{-m^2 T} \times \int_0^T d\tau_1 d\tau_2 P_2 \, e^{G_{B12} k_1 \cdot k_2} \tag{28}$$

where

$$P_2 = \dot{G}_{B12} \varepsilon_1 \cdot k_2 \, \dot{G}_{B21} \varepsilon_2 \cdot k_1 - \ddot{G}_{B12} \varepsilon_1 \cdot \varepsilon_2$$

We could perform the parameter integrals straight away, but let's add some total derivative (which vanishes by the boundary conditions) and remove the $\ddot{G}_{B12}$ term, if we add

$$P_2 + \frac{\partial}{\partial \tau_1} \left(\dot{G}_{B12} \varepsilon_1 \cdot \varepsilon_2 \, e^{G_{B12} k_1 \cdot k_2} \right) = Q_2 \tag{29}$$

in which

$$Q_2 = \dot{G}_{B12}\dot{G}_{B21}\Big(\varepsilon_1 \cdot k_2\varepsilon_2 \cdot k_1 - \varepsilon_1 \cdot \varepsilon_2 k_1 \cdot k_2\Big) = \frac{1}{2}\mathrm{tr}(f_1 f_2)e^{G_{B12}k_1\cdot k_2} \tag{30}$$

where

$$f_{\mu\nu} = k_\mu\varepsilon_\nu - k_\nu\varepsilon_\mu \to \mathrm{tr}(f_1 f_2) = f_{1\mu\nu}f_2^{\nu\mu} = 2(\varepsilon_1 \cdot k_2\varepsilon_2 \cdot k_1 - \varepsilon_1 \cdot \varepsilon_2 k_1 \cdot k_2)$$

Now if we use momentum conservation to set $k_1 = -k_2 = k$ and define

$$\Gamma_{\mathrm{scal}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = (2\pi)^D\delta^D(k_1 + k_2)\,\varepsilon_1 \cdot \Pi_{\mathrm{scal}} \cdot \varepsilon_2 \tag{31}$$

where

$$\Pi^{\mu\nu}_{\mathrm{scal}} = e^2(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T}\int_0^T d\tau_1 d\tau_2\dot{G}_{B12}\dot{G}_{B21}\,e^{-G_{B12}k^2}$$

Note that the OBP has had the effect to factor out the usual transversal projector $\delta^{\mu\nu}k^2 - k^\mu k^\nu$ already at the integrand level.

Now if $\tau_i = T u_i$ and use the translational invariance to set $u_2 = 0$ and $u_1 = u$ then we need

$$G_{B12} = T u(1-u) \quad , \quad \dot{G}_{B12} = 1 - 2u \tag{32}$$

therefore

$$\Pi^{\mu\nu}_{\text{scal}} = \frac{e^2}{(4\pi)^{D/2}} (\delta^{\mu\nu} k^2 - k^\mu k^\nu) \int_0^\infty dT T^{2-D/2} e^{-m^2 T} \int_0^1 du (1-2u)^2 \, e^{-Tu(1-u)k^2}$$

Now using

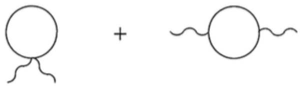
$$\int_0^\infty \frac{dx}{x} x^\lambda \, e^{-ax} = \Gamma[\lambda] a^{-\lambda} \quad , \quad a > 0$$

we obtain

$$\Pi^{\mu\nu}_{\text{scal}} = \frac{e^2}{(4\pi)^{D/2}} (\delta^{\mu\nu} k^2 - k^\mu k^\nu) \Gamma[2 - D/2] \int_0^1 du (1-2u)^2 \left[m^2 + u(1-u) k^2 \right]^{2-D/2}$$

which needs to be renormalized, but we do not proceed further.

Our result agrees, of course, with a computation of the two corresponding Feynman diagrams:



Alternatively, it should be mentioned that the same result can be obtained by using the Gaussian integrals we have mentioned before and using the following set of Wick contractions:

- The basic Wick contraction of two fields is

$$\langle q_1^\mu \, q_2^\nu \rangle = -G_{B12}\delta^{\mu\nu}$$

- For instance the contraction of four different fields

$$\langle q_1^\mu \, q_2^\nu \, q_3^\rho \, q_4^\sigma \rangle = G_{B12}G_{B34}\delta^{\mu\nu}\delta^{\rho\sigma} + 2 \text{ perm}$$

- Contract fields with exponentials according to

$$q_1^{\mu}\, e^{ik_2\cdot q_2} = i\langle q_1^{\mu} q_2^{\nu}\rangle k_{2\nu} e^{ik_2\cdot q_2}$$

- Once all elementary fields have been contracted the contraction of the remaining exponentials yields to the following universal factor

$$\begin{aligned}\langle e^{ik_1\cdot q_1}\, e^{ik_2\cdot q_2}\cdots e^{ik_N\cdot q_N}\rangle &= \exp\Big[-\frac{1}{2}\sum_{i,j=1}^{N}\langle q_i^{\mu} q_j^{\nu}\rangle k_{i\mu}k_{j\nu}\Big] \\ &= \exp\Big[\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij}k_i\cdot k_j\Big]\end{aligned}$$

Exercise: Compute the vacuum polarization diagrams by expanding the interaction term in the path integral and using the above Wick contractions.

SPINOR QED

Feynman (1950-51) presented the following generalization of the effective action for the spinor case:

$$\Gamma_{\text{spin}}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{PBC} Dx(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x)\right)} \text{Spin}[x(\tau,)A]$$

where the Spin factor

$$\text{Spin}[x(\tau), A] = \text{tr}_\gamma \mathcal{P}\text{exp}\left[i\frac{e}{4}[\gamma^\mu, \gamma^\nu]\int_0^T d\tau F_{\mu\nu}(x(\tau))\right] \tag{33}$$

trace denote the Dirac trace and $\mathcal{P}$ is the path ordering, and the minus sign implements the Fermi statistics.

A more modern way of writing the same spin factor is in terms of an additional **Grassmann** path integral which is written as

$$\mathrm{Spin}[x(\tau), A] = \int\limits_{ABC} D\psi \exp\Big[-\int\limits_0^T d\tau \Big(\frac{1}{2}\psi \cdot \dot{\psi} - ie\psi^\mu F_{\mu\nu}\psi^\nu\Big)\Big] \tag{34}$$

Here the path integration is over the space of anticommuting functions which are anti-periodic in proper-time:

$$\psi_\mu(\tau_1)\psi_\nu(\tau_2) = -\psi_\nu(\tau_2)\psi_\mu(\tau_2) \quad , \quad \psi(T) = -\psi(0)$$

The ψ^μ's effectively replace the Dirac matrices γ^μ's, but are functions of the proper-time, and thus will appear in all possible orderings after the expansion of the exponential.

Therefore we need the Grassmann Gauss integrals.

One can show thatthe Grassmann variables satisfy:

$$\int d\psi\psi = 1 \quad , \quad \psi^2 = 0 \tag{35}$$

therefore the most general function of the form

$$f(\psi) = a + b\psi \to \int d\psi(a + b\psi) = b$$

As a function of two variables this function is

$$g(\psi_1, \psi_2) = a + b\psi_1 + c\psi_2 + d\psi_1\psi_2 \to \int d\psi_1 \int d\psi_2 g(\psi_1, \psi_2) = -d$$

One can form the Gaussian integral, let's $\psi = (\psi_1, \psi_2)$ and M be a real antisymmetric matrix, then

$$e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi} = e^{-M_{12}\psi_1\psi_2} = 1 - M_{12}\psi_1\psi_2$$

therefore it follows

$$\int d\psi_1 \int d\psi_2 e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi} = M_{12}$$

But since M is antisymmetric

$$\det M = -M_{12}M_{21} = M_{12}^2 \rightarrow \int d\psi_2 \int d\psi_1 e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi} = \pm(\det M)^{\frac{1}{2}} \tag{36}$$

It is easy to generalise it to an $2n \times 2n$ antisymmetric matrix M and show

$$\int d\psi_1 \int d\psi_2 \cdots \int d\psi_{2n}\, e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi} = \pm(\det M)^{\frac{1}{2}} \tag{37}$$

Extending the Bosonic Gaussian integrals to the Grassmann fields leads to

$$\frac{\int d\psi_1 \cdots \int d\psi_n e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi + \psi \cdot J}}{\int d\psi_1 \cdots \int d\psi_n e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} = e^{\frac{1}{2} J \cdot M^{-1} \cdot J} \tag{38}$$

which by differentiation we get (note that J is also Grassmann valued variable)

$$\frac{\int d\psi_1 \cdots \int d\psi_n \psi_i\, \psi_j e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi + \psi \cdot J}}{\int d\psi_1 \cdots \int d\psi_n e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} = M_{ij}^{-1}$$

$$\frac{\int d\psi_1 \cdots \int d\psi_n\, \psi_i \psi_j \psi_k \psi_l e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi + \psi \cdot J}}{\int d\psi_1 \cdots \int d\psi_n e^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} = M_{ij}^{-1} M_{kl}^{-1} - M_{ik}^{-1} M_{jl}^{-1} + M_{il}^{-1} M_{jk}^{-1} \tag{39}$$

Now comparing to the Grassmann path integral we have, we see that M now corresponds to the first derivative $\frac{d}{d\tau}$ acting on the space of antiperiodic functions. Its inverse is quite simple:

$$G_F(\tau, \tau') = 2\langle\tau|\left(\frac{d}{d\tau}\right)^{-1}|\tau'\rangle = \text{sign}(\tau - \tau') \tag{40}$$

Then we need the Wick contraction for Grassmannian fields which are

$$\begin{aligned}
\langle\psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \frac{1}{2}G_F(\tau_1 - \tau_2)\delta^{\mu\nu} \\
\langle\psi^\mu(\tau_1)\psi^\nu(\tau_2)\psi^\alpha(\tau_3)\psi^\beta(\tau_4)\rangle &= \frac{1}{4}\Big(G_{F12}G_{F34}\delta^{\mu\nu}\delta^{\alpha\beta} - G_{F13}G_{F24}\delta^{\mu\alpha}\delta^{\nu\beta} \\
&\qquad + G_{F14}G_{F23}\delta^{\mu\beta}\delta^{\nu\alpha}\Big)
\end{aligned}$$

We also need the free path integral normalization

$$\int_{\text{ABC}} D\psi\, e^{-\frac{1}{2}\int_0^T d\tau \psi\cdot\dot{\psi}} = 2^{D/2} \tag{41}$$

where D is any even spacetime dimension, and note that $2^{D/2}$ is the number of real degrees of freedom of Dirac spinors in even dimensions.

N-PHOTON AMPLITUDE IN SPINOR QED

The processes of obtaining N-photon amplitude is completely analogous to the scalar QED case. The only modification is the pre-factor $-1/2$, a kinetic term for the Grassmann fields and a new vertex operator

$$V^{\gamma}_{\text{spin}}[k,\varepsilon] = \int_0^T d\tau \big(\varepsilon \cdot \dot{x}(\tau) + 2i\varepsilon \cdot \psi(\tau)\, k \cdot \psi(\tau)\big)\, e^{ik\cdot x(\tau)} \tag{42}$$

Therefore the N-photon amplitude is simply

$$\Gamma_{\text{spin}}[k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = -\frac{1}{2}(-ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} Dx(\tau) e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2}$$
$$\times \int_{ABC} D\psi e^{-\frac{1}{2}\int_0^T d\tau \psi\cdot\dot{\psi}} V^{\gamma}_{\text{spin}}[k_1,\varepsilon_1]\cdots V^{N}_{\text{spin}}[k_N,\varepsilon_N]$$

To generalize the **Bern-Kosower** master formula to spinor QED one needs to use the worldline supersymmetry which we just quote the master formula here (see Schubert 2001):

$$\Gamma_{\rm spin}[A] = -2(-ie)^N(2\pi)^D\delta(\sum_i k_i)\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-D/2}\prod_{i=1}^N\int_0^T d\tau_i\int d\theta_i$$

$$\times\exp\left\{\sum_{i,j=1}^N\left[\frac{1}{2}\hat{G}_{ij}k_i\cdot k_j + iD_i\,\hat{G}_{ij}\,\varepsilon_i\cdot k_j + \frac{1}{2}D_iD_j\hat{G}_{ij}\,\varepsilon_i\cdot\varepsilon_j\right]\right\}\Big|_{\text{lin }\varepsilon_1\cdots\varepsilon_N}$$

where

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau}$$

is the super-derivative acting on the super-field

$$X^\mu = x^\mu + \sqrt{2}\theta\psi^\mu$$

and the super-Green's function

$$\hat{G}(\tau_i,\theta_i;\tau_j,\theta_j) = G_{Bij} + \theta_i\theta_j G_{Fij}$$

with θ a Grassmann valued number $\int d\theta\theta = 1$.

THE VACUUM POLARIZATION (SPINOR QED)

In reality there is a more efficient way to proceed and obtain spinor loop computation. Let's first compute the vacuum diagram for spinor QED:

One can simply use two spinor vertex operator and do all possible contractions for bot x and ψ fields:

$$\Gamma_{\text{spin}}[k_1, \varepsilon_1; k_2, \varepsilon_2] = -\frac{1}{2}(-ie)^2 \int \frac{dT}{T} e^{-m^2 T} \int Dx \int D\psi \int_0^T d\tau_1 \int_0^T d\tau_2$$

$$\times \varepsilon_{1\mu} \left(\dot{x}_1^\mu + 2i\psi_1^\mu k_1 \cdot \psi_1 \right) e^{ik_1 \cdot x_1} \varepsilon_{2\nu} \left(\dot{x}_2^\nu + 2i\psi_2^\nu k_2 \cdot \psi_2 \right) e^{ik_2 \cdot x_2} \, e^{-\int_0^T d\tau (\frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi \cdot \dot{\psi})}$$

Since the Wick contraction does not mix the x and the ψ fields, the calculation of the Dx is identical to the scalar case, so one needs to do the Wick contractions for the Grassmann fields:

$$(2i)^2 \langle \psi_1^\mu k_1 \cdot \psi_1 \, \psi_2^\nu k_2 \cdot \psi_2 \rangle = G_{F12}^2 (\delta^{\mu\nu} k_1 \cdot k_2 - k_2^\mu k_1^\nu)$$

Exercise: Add these contractions to the integrand for the scalar case and multiply the free path integral for the Grassmann fields to obtain the spinor result:

$$\Pi^{\mu\nu}_{\text{spin}}[k] = -\frac{8e^2}{(4\pi)^{D/2}} (\delta^{\mu\nu} k^2 - k^\mu k^\nu) \Gamma[2-D/2] \int_0^1 du\, u(1-u)[m^2 + u(1-u)k^2]^{D/2-2}$$

Note that up to the normalization, the parameter integral for the spinor loop is obtained from the one for the scalar loop simply by replacing

$$\dot{G}_{B12} \dot{G}_{B21} \to \dot{G}_{B12} \dot{G}_{B21} - G_{F12} G_{F21}$$

INTEGRATION BY PARTS (IBP)

The substitution we mentioned is only the simplest case of a more general **replacement rule** due to Bern and Kosower (PRL, 66, 1669 (1992)). According to this replacement rule after expanding the interaction term in the scalar loop case, at the integrand level one obtains:

$$P_N(\dot{G}, \ddot{G})\, e^{\frac{1}{2}\sum_{i,j=1} G_{Bij} k_i \cdot k_j} \tag{43}$$

where P_N is a polynomial in $\dot{G}$ and $\ddot{G}$ and the kinematic invariants. It is possible to remove all the second derivatives by adding some IBP leading to a new polynomial $\sim Q_N(\dot{G})e^{\cdots}$. After doing this properly, Q_N will have some τ-cycles $(\dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_{n-1} i_n}\dot{G}_{i_n i_1})$. The integrand of the spinor case in the integrand level is simply replacing these chains with the following modified chains

$$\dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_{n-1} i_n}\dot{G}_{i_n i_1} - G_{F i_1 i_2} G_{F i_2 i_3}\cdots G_{F i_{n-1} i_n} G_{F i_n i_1} \tag{44}$$

and supplying the factor of -2 which comes from the statistics and the Grassmann free path integral (in $D = 4$).

With IBP we do not need the Wick contractions for the Grassmann fields anymore, and besides removing the $\ddot{G}$'s has following properties:

- It should maintain the permutation symmetry between the photons.
- It should lead to Q_N in which all polarization vectors (ε_i's) are absorbed into the corresponding field strength tensor

$$f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu$$

  which assures the manifest transversality at the integrand level, which is extremely helpful.

- It should be systematic enough to be computerized.

We have developed such an algorithm not many years ago, (JHEP 1301, 312 (2013)) and applied it to the form factor decomposition of the three- and four-gluon vertices (off-shell) and also off-shell four-photon amplitude.

Exercise

For $N = 3$ after expanding the master formula one obtains the following polynomial P_3

$$\begin{aligned} P_3 &= \dot{G}_{1i}\varepsilon_1 \cdot k_i \dot{G}_{2j}\varepsilon_2 \cdot k_j \dot{G}_{3k}\varepsilon_3 \cdot k_k - \ddot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2\, \dot{G}_{3i}\varepsilon_3 \cdot k_i \\ &\quad -\ddot{G}_{B13}\varepsilon_1 \cdot \varepsilon_3\, \dot{G}_{2i}\varepsilon_2 \cdot k_i - -\ddot{G}_{B23}\varepsilon_2 \cdot \varepsilon_3\, \dot{G}_{1i}\varepsilon_1 \cdot k_i \end{aligned} \tag{45}$$

Add some total derivatives to P_3 to remove all second derivatives and find the following new polynomial Q_3

$$Q_3 = Q_3^3 + Q_3^2 \tag{46}$$

in which

$$\begin{aligned} Q_3^3 &= \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}\mathrm{tr}(f_1 f_2 f_3) \\ Q_3^2 &= \frac{1}{2}\dot{G}_{B12}\dot{G}_{B21}\mathrm{tr}(f_1 f_2)\dot{G}_{B3i}\varepsilon_3 \cdot k_i + 2\ \mathrm{perm} \end{aligned} \tag{47}$$

Hint:

To remove the term involving $\ddot{G}_{B12}\dot{G}_{B31}$ you need to add the following term to P_3

$$-\partial_2\left(\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2\, \dot{G}_{B31}\varepsilon_3 \cdot k_1\, e^{\frac{1}{2}\sum_{i,j=1}^{3} G_{Bij} k_i \cdot k_j}\right) \tag{48}$$

and similar terms for other terms. After removing all the $\ddot{G}$'s and collecting all terms you should obtain Q_3^3 and Q_3^2.

EULER-HEISENBERG LAGRANGIAN

Right from beginning when we introduced the coupling of a particle with electromagnetic field (A^μ), we mentioned that this field is general and it could be either quantum field (which we considered as a sum of plane waves and later we derived the Bern-Kosower master formula in vacuum) or it could be a classical external field or a sum of two. In this part of our lecture we consider only an external constant electromagnetic field to derive the well-known Euler-Heisenberg Lagrangian which *Felix Karbstein* also derived in his lectures. For a constant field it is convenient for us to work in the **Fock-Schwinger** gauge with a fixing center-point x_c to be defined as

$$(x - x_c)_\mu \mathcal{A}^\mu(x) = 0$$

Therefore, the gauge field can be Taylor expanded as

$$\mathcal{A}_\mu(x_c + q) = \frac{1}{2}\mathcal{F}_{\nu\mu} q^\nu + \frac{1}{3}(\partial_\alpha \mathcal{F}_{\nu\mu}) q^\alpha q^\nu + \cdots$$

which for a constant field strength tensor one just have the first term in the rhs:

$$\mathcal{A}_\mu(x_c + q) = \frac{1}{2}\mathcal{F}_{\nu\mu} q^\nu \tag{49}$$

Therefore the effective action in the presence of an external field for the spinor-loop reads as

$$\Gamma_{\text{spin}}[\mathcal{A}] = -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int d^D x_0 \int_{\text{PBC}} \mathcal{D}q(\tau) e^{-\int_0^T d\tau [\frac{1}{4}\dot{q}^2 + \frac{1}{2} ieq\cdot\mathcal{F}\cdot\dot{q}]} \times \int_{\text{ABC}} \mathcal{D}\psi(\tau)\, e^{-\int_0^T d\tau [\frac{1}{2}\psi\cdot\dot{\psi} - ie\psi\cdot\mathcal{F}\cdot\psi]} \tag{50}$$

Note that the zero-mode integral $\int d^D x_0$ is empty since we have no dependence of the zero-mode in the effective action (as expected for a constant field), it gives an infinite volume factor, therefore we introduce the effective Lagrangian as

$$\Gamma_{\text{spin}}[\mathcal{A}] = \int d^D x_0\, \mathcal{L}$$

where

$$\mathcal{L}[\mathcal{F}] = -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T} \int_{\text{PBC}} Dq(\tau) e^{-\int_0^T d\tau[\frac{1}{4}\dot{q}^2+\frac{1}{2}ieq\cdot\mathcal{F}\cdot\dot{q}]} \times \int_{\text{ABC}} D\psi(\tau)\, e^{-\int_0^T d\tau[\frac{1}{2}\psi\cdot\dot{\psi}-ie\psi\cdot\mathcal{F}\cdot\psi]} \quad (51)$$

And this time we will not need any expansions to get the worldline path integrals into gaussian form they are already Gaussian. Using our formulas we introduced for the Gaussian integrals for bosonic and Grassmann fields we can write the path integrals in terms of determinant factors:

Using the formulas for the free-path integrals:

$$\begin{aligned} \int_{\text{PBC}} Dq\, e^{-\int_0^T d\tau \frac{1}{4}\dot{q}^2} &= \Big(4\pi T\Big)^{-D/2} \\ \int_{\text{ABC}} D\psi\, e^{-\int_0^T d\tau \frac{1}{2}\psi\cdot\dot{\psi}} &= \Big(2\Big)^{D/2} \end{aligned} \tag{52}$$

We set $D = 4$ (we multiply and divide the effective action with the normalization factors for the bosonic and Grassmann fields according to Eqs. (15,26;37,41)

$$\begin{aligned} \mathcal{L}[F] &= -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2 T}(4\pi T)^{-2} \frac{\int_{\text{PBC}} Dq e^{-\int_0^T d\tau[\frac{1}{4}\dot{q}^2 + \frac{1}{2} ieq\cdot F\cdot\dot{q}]}}{\int_{\text{PBC}} Dq e^{-\int_0^T d\tau \frac{1}{4}\dot{q}^2}} \\ &\times(2^2)\frac{\int_{\text{ABC}} D\psi(\tau)\, e^{-\int_0^T d\tau[\frac{1}{2}\psi\cdot\dot{\psi} - ie\psi\cdot F\cdot\psi]}}{\int_{\text{ABC}} D\psi(\tau)\, e^{-\int_0^T d\tau\frac{1}{2}\psi\cdot\dot{\psi}}} \end{aligned} \tag{53}$$

which can be written in terms of determinants (see eqs. 15 and 37)

$$\begin{aligned}\mathcal{L}[\mathcal{F}] &= -2\int_0^\infty \frac{dT}{T} e^{-m^2T}(4\pi T)^{-2}\frac{\mathrm{Det}'^{-\frac{1}{2}}_{\mathrm{PBC}}\left(-\frac{1}{4}\frac{d^2}{d\tau^2}+\frac{1}{2}ie\mathcal{F}\frac{d}{d\tau}\right)}{\mathrm{Det}'^{-\frac{1}{2}}_{\mathrm{PBC}}\left(-\frac{1}{4}\frac{d^2}{d\tau^2}\right)} \\ &\qquad\times\frac{\mathrm{Det}'^{\frac{1}{2}}_{\mathrm{ABC}}\left(\frac{d}{d\tau}-2ie\frac{1}{2}\mathcal{F}\right)}{\mathrm{Det}'^{\frac{1}{2}}_{\mathrm{ABC}}\left(\frac{d}{d\tau}\right)} \\ &= -2\int_0^\infty \frac{dT}{T} e^{-m^2T}(4\pi T)^{-2}\mathrm{Det}'^{-\frac{1}{2}}_{\mathrm{PBC}}\left(\mathbb{1}-2ie\mathcal{F}\left(\frac{d}{d\tau}\right)^{-1}\right) \\ &\qquad\times\mathrm{Det}'^{\frac{1}{2}}_{\mathrm{ABC}}\left(\mathbb{1}-2ie\frac{1}{2}\mathcal{F}\left(\frac{d}{d\tau}\right)^{-1}\right) \qquad (54)\end{aligned}$$

Here note that we have eliminated the zero-mode contributions (the zero eigenvalues) which in the determinant it is indicated by a prime.

Thus we now have to calculate the determinant of the same operator:

$$\mathcal{O}(\mathcal{F}) = \mathbb{1} - 2ie\mathcal{F}(\frac{d}{d\tau})^{-1} \tag{55}$$

acting once in the space of periodic functions and once in the space of anti-periodic functions. Note also that the determinant of $\mathcal{O}(\mathcal{F})$ must be a Lorentz scalar, and it is not possible to form such a scalar with an odd number of field strength tensors F. We can write

$$|\mathcal{O}(\mathcal{F})|^2 = |\mathbb{1} - 2ie\mathcal{F}(\frac{d}{d\tau})^{-1}|\,|\mathbb{1} + 2ie\mathcal{F}(\frac{d}{d\tau})^{-1}| = |\mathbb{1} + 4e^2\mathcal{F}^2(\frac{d}{d\tau})^{-2}|$$

In classical electrodynamics for a generic constant field there is always a Lorentz frame in which both electric and magnetic fields are pointing on the z-axis. The euclidean field strength tensor then takes the form

$$\mathcal{F} = \begin{pmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & iE \\ 0 & 0 & -iE & 0 \end{pmatrix} \tag{56}$$

and

$$\mathcal{F}^2 = \begin{pmatrix} -B^2 & 0 & 0 & 0 \\ 0 & -B^2 & 0 & 0 \\ 0 & 0 & E^2 & 0 \\ 0 & 0 & 0 & E^2 \end{pmatrix} \tag{57}$$

Plugging back to the determinant and taking the square root

$$|\mathcal{O}(\mathcal{F})| = |\mathbb{1} + 4e^2E^2(\frac{d}{d\tau})^{-2}|\,|\mathbb{1} - 4e^2B^2(\frac{d}{d\tau})^{-2}| \tag{58}$$

Thus we have managed to reduce the original matrix operator to usual (one-component) operators.

Next we determine the spectrum of the operator $\frac{d^2}{d\tau^2}$ for two boundary conditions. To do so, we need to solve the following eigenvalue equations:

$$-\frac{d^2}{d\tau^2}f(\tau) = \lambda_n f(\tau) \quad (59)$$

For the periodic case (the bosonic field), a basis of eigenfunctions is given by

$$\{\chi_n, \tilde{\chi}_n\} = \left\{\cos(\frac{2\pi n\tau}{T}), \sin(\frac{2\pi n\tau}{T})\right\} \quad , \quad n = 1, 2, \cdots \quad (60)$$

with

$$\lambda_n = \frac{(2\pi n)^2}{T^2}$$

For the antiperiodic case

$$\{\kappa_n, \tilde{\kappa}_n\} = \left\{\cos(\frac{2\pi(n+\frac{1}{2})\tau}{T}), \sin(\frac{2\pi(n+\frac{1}{2})\tau}{T})\right\}, \ n = 0, 1, 2, \cdots \quad (61)$$

with

$$\lambda_n = \frac{(2\pi(n+\frac{1}{2}))^2}{T^2}$$

After using the following Euler infinite series

$$\begin{aligned}
\frac{\sin x}{x} &= \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\\
\frac{\sinh x}{x} &= \prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2\pi^2}\right)\\
\frac{\cos x}{x} &= \prod_{n=0}^{\infty}\left(1-\frac{x^2}{(n+\frac{1}{2})^2\pi^2}\right)\\
\frac{\cosh x}{x} &= \prod_{n=0}^{\infty}\left(1+\frac{x^2}{(n+\frac{1}{2})^2\pi^2}\right)
\end{aligned} \tag{62}$$

and putting all together the EHL is obtained as

$$
\begin{aligned}
\mathcal{L}[\mathcal{F}] &= -2\int_0^\infty \frac{dT}{T}(4\pi T)^{-2}\, e^{-m^2T}\,\frac{eET}{\sin(eET)}\frac{eBT}{\sinh(eBT)}\cos(eET)\cosh(eBT) \\
&= -2\int_0^\infty \frac{dT}{T}(4\pi T)^{-2}\, e^{-m^2T}\,\frac{eET}{\tan(eET)}\frac{eBT}{\tanh(eBT)} \qquad (63)
\end{aligned}
$$

This is the famous Lagrangian found by Euler and Heisenberg in 1936 as one of the first nontrivial results in quantum electrodynamics. It tell us that even though in classical electrodynamics there is no interaction between photons but in quantum electrodynamics (QED) (after quantization) because of the presence of virtual electron-position pairs such interactions do arise and lead to very interesting nonlinear effects.

SCHWINGER PAIR-PRODUCTION

The effective Lagrangian we obtained needs to be analyzed carefully since it has an UV divergence at $T = 0$, but Felix Karbstein in his lecture showed us how to deal with this problem properly. Beside this issue the term $\frac{1}{\tan(eET)}$ has poles in

$$T_n = \frac{n\pi}{eE} \tag{64}$$

but it is a simple application of complex analysis to show for a pure electric field ($B = 0$) this effective Lagrangian has an imaginary part

$$\mathrm{Im}\mathcal{L} = \frac{(eE)^2}{8\pi^3} \sum_{n=1}^{\infty} e^{-n\pi \frac{m^2}{eE}} \tag{65}$$

Which Sauter in 1932 interpreted as the vacuum instability which leads the virtual electron-positron pairs to gain enough energy from the electric field to turn real. However, the probability for this to happen becomes significant only at about $E_c \approx 10^{18} V/cm$.

ONE-LOOP MULTIPHOTON AMPLITUDE IN THE PRESENCE OF A CONSTANT FIELD

In this part of our lecture we generalize the Bern-Kosower master formula and include an external constant field. Mathematically, the constant field is one of the very few known field configurations for which the Dirac equation can be solved exactly. In the following we use the Fock-Schwinger gauge defined in Eq. (49) centered at $x_c = x_0$. Let's first rewrite the effective action for Spinor QED in the presence of a constant field

$$L_{\text{spin}}[A, \mathcal{A}] = -\frac{1}{2} \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int_{\text{PBC}} Dx(\tau) \int_{\text{ABC}} D\psi(\tau)\, e^{-\int_0^T d\tau L_{\text{spin}}} \tag{66}$$

where $A(F)$ is the quantum gauge field (quantum field strength tensor) and $\mathcal{A}(\mathcal{F})$ is the external vector field (classical field strength tensor), the Lagrangian is given by

$$L_{\text{spin}} = \frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi \cdot \dot{\psi} + ieA \cdot \dot{x} - ie\psi \cdot F \cdot \psi + \Delta L_{\text{spin}} \tag{67}$$

where ΔL_{spin} is due to the presence of the external constant field in Fock-Schwinger gauge

$$\Delta L_{\text{spin}} = \frac{1}{2} ieq^{\mu}\mathcal{F}_{\mu\nu}\dot{q}^{\nu} - ie\psi^{\mu}\mathcal{F}_{\mu\nu}\psi^{\nu}$$

$\mathcal{F}$ is the corresponding field strength tensor for $\mathcal{A}$. In terms of superfield formalism this extra Lagrangian is written in a much compact way as

$$\Delta L_{\text{spin}} = -\frac{1}{2} ieX^{\mu}\mathcal{F}_{\mu\nu}DX^{\nu}$$

.

Note that since the extra terms are still quadratic in the worldline fields (q, ψ), we need not need to consider it as part of the interaction Lagrangian, we can absorb it into the free worldline propagator and obtain the exact propagator in the presence of the constant field. This means to replace the Green's functions we had for the vacuum case with the following modified expressions:

$$
\begin{aligned}
2\langle\tau_i|\left(\frac{d^2}{d\tau^2} - 2ie\mathcal{F}\frac{d}{d\tau}\right)^{-1}|\tau_j\rangle &\equiv \mathcal{G}_B(\tau_i, \tau_i) \equiv \mathcal{G}_{Bij} \\
2\langle\tau_i|\left(\frac{d}{d\tau} - 2ie\mathcal{F}\right)^{-1}|\tau_j\rangle &\equiv \mathcal{G}_F(\tau_i, \tau_j) \equiv \mathcal{G}_{Fij}
\end{aligned}
\tag{68}
$$

See Appendix B of Schubert 2001 for the details of how calculating these inverses which eventually lead to

$$
\begin{aligned}
\mathcal{G}_{Bij} &= \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}e^{-i\mathcal{Z}\dot{G}_{Bij}} + i\mathcal{Z}\dot{G}_{Bij} - 1\right) \\
\mathcal{G}_{Fij} &= G_{Fij}\frac{e^{-i\mathcal{Z}\dot{G}_{Bij}}}{\cos\mathcal{Z}}
\end{aligned}
\tag{69}
$$

with $\mathcal{Z} \equiv e\mathcal{F}T$. Note that these expressions are power series of the Lorentz matrix $\mathcal{Z}$ and they do not require $\mathcal{F}$ to be invertible.

Properties of these Green's functions:

- They are still translational invariance in τ (functions of $(\tau_i - \tau_j)$).

- We have avoided an explicit case distinction between $\tau_i > \tau_j$ and $\tau_j > \tau_i$ by writing them in terms of the vacuum Green's function G_B and G_F.

- $$\mathcal{G}_B(\tau_i, \tau_j) = \mathcal{G}_B^T(\tau_j, \tau_i) \quad , \quad \dot{\mathcal{G}}_B(\tau_i, \tau_j) = -\dot{\mathcal{G}}_B^T(\tau_j, \tau_i)$$

- $$\mathcal{G}_F(\tau_i, \tau_j) = -\mathcal{G}_F^T(\tau_j, \tau_i)$$

In terms of Wick contractions they correspond to the following generalization of the vacuum case:

$$\begin{aligned} \langle q^\mu(\tau_i)\, q^\nu(\tau_j) \rangle &= -\mathcal{G}_{Bij}^{\mu\nu} \\ \langle \psi^\mu(\tau_i)\, \psi^\nu(\tau_j) \rangle &= \frac{1}{2}\mathcal{G}_{Fij}^{\mu\nu} \end{aligned} \tag{70}$$

In the following we will also need the first and second derivatives of $\mathcal{G}_{Bij}$

$$\dot{\mathcal{G}}_{Bij} \equiv 2\langle\tau_i|\Big(\frac{d}{d\tau} - 2ie\mathcal{F}\Big)^{-1}|\tau_j\rangle = \frac{i}{\mathcal{Z}}\Big(\frac{\mathcal{Z}}{\sin\mathcal{Z}}e^{-i\mathcal{Z}\dot{G}_{Bij}} - 1\Big)$$
$$\ddot{\mathcal{G}}_{Bij} \equiv 2\langle\tau_i|\Big(\mathbb{1} - 2ie\mathcal{F}(\frac{d}{d\tau})^{-1}\Big)^{-1}|\tau_j\rangle = 2\delta_{ij} - \frac{2\mathcal{Z}}{T\sin\mathcal{Z}}e^{-i\mathcal{Z}\dot{G}_{Bij}} \qquad (71)$$

The above Green's functions have non-vanishing coincidence limits contrary to the vacuum case:

$$\mathcal{G}_B(\tau,\tau) = \frac{T}{2\mathcal{Z}^2}\Big(\mathcal{Z}\cot\mathcal{Z} - 1\Big)$$
$$\dot{\mathcal{G}}_B(\tau,\tau) = i\cot\mathcal{Z} - \frac{i}{\mathcal{Z}}$$
$$\mathcal{G}_F(\tau,\tau) = -i\tan\mathcal{Z} \qquad (72)$$

which can be obtained from the Taylor expansion of Eqs. (69,71) and using the following rules for the vacuum case (from symmetry and continuity)

$$\dot{G}_B(\tau,\tau) = 0 \quad , \quad \dot{G}_B^2(\tau,\tau) = 1$$

Since we have already mentioned the super-symmetric Green's function and the Wick contractions of the super-fields for the vacuum case it is also possible to assemble $\mathcal{G}_B$ and $\mathcal{G}_F$ into a super propagator:

$$\begin{aligned}\langle Y^\mu(\tau_i,\theta_i)\,Y^\nu(\tau_j,\theta_j) &= -\hat{\mathcal{G}}^{\mu\nu}(\tau_i,\theta_i;\tau_j,\theta_j)\\ \hat{\mathcal{G}}(\tau_i,\theta_i;\tau_j,\theta_j) &\equiv \mathcal{G}_{Bij}+\theta_i\theta_j\mathcal{G}_{Fij}\end{aligned} \tag{73}$$

The only further required information one needs to write down the Bern-Kosower type master formula for multi-photon amplitude (one-loop) in a constant field background is the change in the free path integral determinants due to the presence of the background field:

$$\begin{aligned}\left(4\pi T\right)^{-\frac{D}{2}} &\rightarrow \left(4\pi T\right)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\left[\frac{\sin\mathcal{Z}}{\mathcal{Z}}\right] \quad \text{(for scalar loop)}\\ \left(4\pi T\right)^{-\frac{D}{2}} &\rightarrow \left(4\pi T\right)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\left[\frac{\tan\mathcal{Z}}{\mathcal{Z}}\right] \quad \text{(for spinor loop)}\end{aligned} \tag{74}$$

THE N-PHOTON AMPLITUDE IN A CONSTANT FIELD

Following the steps we have already mentioned to obtain the master equation for N-photon amplitude for the scalar loop in vacuum and applying the above changes it is straightforward to write down the following N-photon amplitude in the presence of a constant field (in the Fock-Schwinger gauge)

$$\Gamma_{\rm scal}[k_1,\varepsilon_1;\cdots,k_N,\varepsilon_N] = (-ie)^N (2\pi)^D \delta\Big(\sum_{i=1}^N k_i\Big)$$
$$\times \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \det{}^{-\frac{1}{2}}\left[\frac{\sin \mathcal{Z}}{\mathcal{Z}}\right] \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times \exp\Big\{ \sum_{i,j=1}^N \Big[\frac{1}{2} k_i \cdot \mathcal{G}_{Bij} \cdot k_j - i\varepsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j + \frac{1}{2}\varepsilon_i \cdot \ddot{\mathcal{G}}_{Bij} \cdot \varepsilon_j\Big]\Big\}\Big|_{\text{linear in } \varepsilon_1\varepsilon_2\cdots\varepsilon_N}$$

To avoid using the super-field master formula for the spinor case, in the presence of a constant background field one can still use the replacement rules in the vacuum case with minor modifications. The spinor QED integrand for a given number of photons is obtained from the scalar QED integrand by the following generalization of the Bern-Kosower replacement rules:

- IBP: after expanding out the exponential in Eq. (75) and taking linear terms in all polarization vectors, remove all $\ddot{\mathcal{G}}_{Bij}$ by suitable IBP as the vacuum case.
- Apply the replacement rules mentioned in Eq. (44) to the new integrand with $\dot{G}_B$ and G_F substituted by $\dot{\mathcal{G}}_B$ and $\mathcal{G}_F$ respectively. Note that since the later Green's functions are nontrivial matrices of the Lorentz indices the cycle property is defined solely in terms of the τ-indices, irrespectively of the structure of the Lorentz indices. For instance:

$$
\begin{aligned}
&\varepsilon_1 \cdot \dot{\mathcal{G}}_{B12} \cdot k_2\, \varepsilon_2 \cdot \dot{\mathcal{G}}_{B23} \cdot k_3\, \varepsilon_3 \cdot \dot{\mathcal{G}}_{B31} \cdot k_1 \rightarrow \\
&(\varepsilon_1 \cdot \dot{\mathcal{G}}_{B12} \cdot k_2\, \varepsilon_2 \cdot \dot{\mathcal{G}}_{B23} \cdot k_3\, \varepsilon_3 \cdot \dot{\mathcal{G}}_{B31} \cdot k_1 \\
&\qquad - \varepsilon_1 \cdot \mathcal{G}_{F12} \cdot k_2\, \varepsilon_2 \cdot \mathcal{G}_{F23} \cdot k_3\, \varepsilon_3 \cdot \mathcal{G}_{F31} \cdot k_1)
\end{aligned}
$$

- The only difference with respect to the replacement rule in the vacuum case is the non-vanishing coincidence limits of the Green's functions which leads to the extension of the replacement rule to include *one-cycle* as

$$\dot{\mathcal{G}}_B(\tau_i,\tau_i) \to \dot{\mathcal{G}}_B(\tau_i,\tau_i) - \mathcal{G}_F(\tau_i,\tau_i)$$

- Remember also that the free-path integral is different for spinor case therefore we also need to replace

$$\det{}^{-\frac{1}{2}}\left[\frac{\sin\mathcal{Z}}{\mathcal{Z}}\right] \to \det{}^{-\frac{1}{2}}\left[\frac{\tan\mathcal{Z}}{\mathcal{Z}}\right]$$

- Finally, one also need to multiply the usual factor of -2 for statistics and degrees of freedom of fermions as we discussed in the vacuum case.

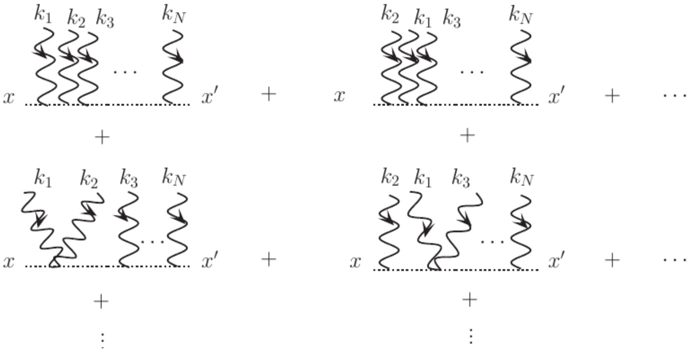
Scalar propagator in vacuum and in constant background fields

SCALAR PROPAGATOR COUPLED TO A^μ

At the beginning of our lecture we derived the wordline representation of the free scalar propagator which is

$$D_0^{xx'} = \int_0^\infty dT e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} Dx e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2} \tag{75}$$

Now we couple this scalar propagator to a gauge field A^μ to derive a **Bern-Kosower master formula** for the propagator, which along its propagation it absorbs and emits an arbitrary number of photons. This master formula (in configuration space) will represent the following set of Feynman diagrams:



k_1 k_2 k_3 k_N
x x' +
k_2 k_1 k_3 k_N
x x' + ...
+
k_1 k_2 k_3 k_N
x x' +
k_2 k_1 k_3 k_N
x x' + ...
+
⋮

According to the above diagrams, the worldline representation of this amplitude is obtained by inserting N number of scalar vertex operator defined in Eq. (20)

$$\Gamma_{\text{scal}}[x, x'; k_1, \varepsilon_1; \cdots; k_N, \varepsilon_N] = (-ie)^N \int_0^\infty dT\, e^{-m^2 T} \int_{x(0)=x'}^{x=x} Dx(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2} \times \int_0^T \prod_{i=1}^N d\tau_i\, V^\gamma[k_1, \varepsilon_1] \cdots V^\gamma[k_N, \varepsilon_N] \tag{76}$$

which after substituting the vertex operator back to the propagator, and applying the split in Eq. (6) one gets

$$\Gamma_{\text{scal}}[x, x'; k_1, \varepsilon_1; \cdots; k_N, \varepsilon_N] = (-ie)^N \int_0^\infty dT\, e^{-m^2 T}\, e^{\frac{(x-x')^2}{4T}} \int_{q(0)=q(T)=0} Dq(\tau) \times e^{-\frac{1}{4}\int_0^T d\tau \dot{q}^2} \int_0^T \prod_{i=1}^N d\tau_i\, e^{\sum_{i=1}^N [\varepsilon_i \cdot \frac{x-x'}{T} + \varepsilon_i \cdot \dot{q}_i + i k_i \cdot (x-x') \frac{\tau_i}{T} + i k_i \cdot x' + i k_i \cdot q_i]} \Big|_{\text{line } \varepsilon_1 \varepsilon_2 \cdots \varepsilon_N}$$

Now if we Fourier transform the scalar legs of the master formula to momentum space (note that all external particles are ingoing)

$$\Gamma_{\rm scal}[p,p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = \int d^Dx \int d^Dx'\, e^{ip\cdot x + ip'\cdot x'}\Gamma_{\rm scal}[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N]$$

By changing the integration variables x and x' to

$$x - x' = x_- \quad , \quad x + x' = 2x_+$$

The x_+ integral just produces the usual energy-momentum conservation factor:

$$\Gamma_{\rm scal}[p,p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N(2\pi)^D\delta^D\Big(p+p'+\sum_i k_i\Big)$$

$$\times \int_0^\infty dT\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}} \int d^Dx_-\, e^{-\frac{x_-^2}{4T}}$$

$$\times \int_0^T \prod_{i=1}^N d\tau_i\, e^{ix_-\cdot(p+\sum_i \frac{k_i\tau_i}{T})}\, e^{\sum_i \frac{\varepsilon_i\cdot x_-}{T}}\, e^{\sum_{i,j}[\Delta_{ij}k_i\cdot k_j - 2i\,{}^\bullet\!\Delta_{ij}\varepsilon_i\cdot k_j - {}^{\bullet}\!\Delta^{\bullet}_{ij}\varepsilon_i\cdot\varepsilon_j]}\Big|_{\text{line } \varepsilon_1\varepsilon_2\cdots\varepsilon_N}$$

where we have introduced the open-line Green's function (which has to satisfy the DBC)

$$\langle q^\mu(\tau) q^\nu(\tau)\rangle = -2\delta^{\mu\nu}\Delta(\tau_1, \tau_2) \tag{77}$$

which is related to the loop Green's function through

$$\Delta(\tau, \tau') = \frac{1}{2}\Big(G_B(\tau, \tau') - G_B(\tau, 0) - G_B(0, \tau') + G_B(0, 0)\Big)$$

The open line Green' function has non-trivial coincidence limit

$$\Delta(\tau, \tau) = \frac{\tau^2}{T} - \tau \tag{78}$$

and we also introduced its first and second derivatives

$$\begin{aligned} {}^\bullet\Delta(\tau_i, \tau_j) &= \frac{\tau_j}{T} + \frac{1}{2}\text{sign}(\tau_i - \tau_j) - \frac{1}{2} \\ \Delta^\bullet(\tau_i, \tau_j) &= \frac{\tau_i}{T} - \frac{1}{2}\text{sign}(\tau_i - \tau_j) - \frac{1}{2} \\ {}^\bullet\Delta^\bullet(\tau_i, \tau_j) &= \frac{1}{T} - \delta(\tau_i - \tau_j) \end{aligned} \tag{79}$$

After doing the x_- integral in our master formula we get some cancellations between terms and eventually one gets the momentum space master formula for the propagator:

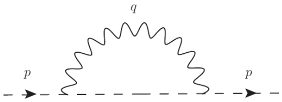
$$\Gamma_{\text{scal}}[p,p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N(2\pi)^D\delta^D\Big(p+p'+\sum_i k_i\Big)\int_0^\infty dT\, e^{-T(m^2+p^2)}$$

$$\times\int_0^T\prod_{i=1}^N d\tau_i\, e^{\sum_{i,j=0}^N\left[\frac{1}{2}|\tau_i-\tau_j|K_i\cdot K_j - i\,\text{sign}(\tau_i-\tau_j)\varepsilon_i\cdot K_j+\delta(\tau_i-\tau_j)\varepsilon_i\cdot\varepsilon_j\right]}\Big|_{\text{line } \varepsilon_1\varepsilon_2\cdots\varepsilon_N}$$

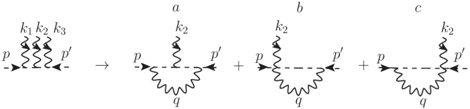
considering

$$K_0 \equiv P$$
$$K_i \equiv k_i \quad , \quad i=1,2,\cdots,N$$
$$K_{N+1}\equiv p' \quad , \quad \tau_0 = T \quad , \quad \tau_{N+1}=0 \quad , \quad \varepsilon_0=\varepsilon_{N+1}=0$$

We applied the master formula to reconstruct the one-loop corrections to the scalar propagator and its vertex in a much straightforward way than standard computations (see PRD **93** (2016) 045023)



one-loop correction to the scalar propagator using $N = 2$ and sewing the photons.



One-loop corrections to the scalar vertex using $N = 3$ and sewing two external photons.

THE PROPAGATOR IN A CONSTANT FIELD

The propagator of a scalar particle in the Maxwell background:

$$D^{xx'}[A] \quad = \quad \int_0^\infty dT \; \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} Dx \; \mathrm{e}^{-\int_0^T d\tau \left[\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x)\right]} \, . \tag{2}$$

where

$$A = A_{\mathrm{ext}} + A_{\mathrm{phot}} \tag{3}$$

Choosing Fock-Schwinger gauge, the gauge potential for a constant field can be written as

$$A^\mu(y) = -\frac{1}{2} F^{\mu\nu}(y - x')^\nu \, , \tag{4}$$

We decompose the arbitrary trajectory $x(\tau)$ into a straight-line part and a fluctuation part $q(\tau)$ obeying Dirichlet boundary conditions, $q(0) = q(T) = 0$:

$$x(\tau) = x' + \frac{\tau}{T}(x - x') + q(\tau). \tag{5}$$

After plugging this potential to the scalar propagator we get ($Q^\mu \equiv \int_0^T d\tau q^\mu(\tau)$)

$$D^{xx'}(F) \quad = \quad \int_0^\infty dT \, \mathrm{e}^{-m^2 T} \, \mathrm{e}^{-\frac{(x-x')^2}{4T}} \int Dq(\tau) \, \mathrm{e}^{-\int_0^T d\tau \frac{1}{4} q\left(-\frac{d^2}{d\tau^2} + 2ieF\frac{d}{d\tau}\right) q + \frac{ie}{T}(x-x')FQ} \, .$$

The worldline Green's function does change, but still relates to the one for string-inspired boundary conditions in the same way as in the vacuum case:

$$\underset{\smile}{\Delta}(\tau,\tau') \equiv \langle \tau \mid \Big(\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}\Big)^{-1} \mid \tau' \rangle_{\rm DBC} = \frac{1}{2}\Big(\mathcal{G}_B(\tau,\tau') - \mathcal{G}_B(\tau,0) - \mathcal{G}_B(0,\tau') + \mathcal{G}_B(0,0)\Big)\,. \tag{7}$$

Finally after using this new Green's function, we obtain the well-known proper-time representation of the constant-field propagator (E.S. Fradkin, D.M. Gitman, S.M. Shvartsman, Quantum Electrodynamics with Unstable Vacuum, Springer 199)

in configuration space:

$$D^{xx'}(F) \quad = \quad \int_0^\infty dT\, {\rm e}^{-m^2 T}(4\pi T)^{-\frac{D}{2}} \det{}^{\frac{1}{2}}\Big[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\Big] \exp\Big\{-\frac{1}{4T}(x-x')\mathcal{Z}\cot\mathcal{Z}(x-x')\Big\}\,. \tag{8}$$

and in momentum space:

$$D^{pp'}(F) \quad = \quad (2\pi)^D\delta(p+p')\int_0^\infty dT\; {\rm e}^{-m^2 T}\;\frac{{\rm e}^{-Tp(\frac{\tan\mathcal{Z}}{\mathcal{Z}})p}}{\det^{\frac{1}{2}}[\cos\mathcal{Z}]}\,.$$

THE DRESSED SCALAR PROPAGATOR IN A CONSTANT FIELD

We now wish to dress the propagator with N photons in addition to the constant field. As before, we start in configuration space. For this purpose, the potential in (2) has to be chosen as

$$A = A_{\mathrm{ext}} + A_{\mathrm{phot}} \, , \tag{10}$$

where A_{ext} is the same as in (4), and A_{phot} represents a sum of plane waves:

$$A^{\mu}_{\mathrm{phot}}(x) = \sum_{i=1}^{N} \varepsilon_i^{\mu} \ \mathrm{e}^{ik_i \cdot x} \, . \tag{11}$$

Each photon then effectively gets represented by a vertex operator

$$V^A[k,\varepsilon] = \int_0^T d\tau \, \varepsilon . \dot{x}(\tau) \, \mathrm{e}^{ik\cdot x(\tau)} \, , \tag{12}$$

integrated along the scalar line. This leads to the following path integral representation of the constant-field propagator dressed with N photons:

$$\begin{aligned} D^{xx'}(F|k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) \quad &= \quad (-ie)^N \int_0^{\infty} dT \ \mathrm{e}^{-m^2 T} \int_P Dx \ \mathrm{e}^{-\int_0^T d\tau [\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A_{\mathrm{ext}}(x)]} \\ &\quad \times V[k_1,\varepsilon_1] V[k_2,\varepsilon_2] \cdots V[k_N,\varepsilon_N] \, . \end{aligned}$$

DRESDEN concept HZDR

Applying the path decomposition and some nontrivial calculations we arrive to the following x-space representation of the dressed scalar propagator in a constant background field:

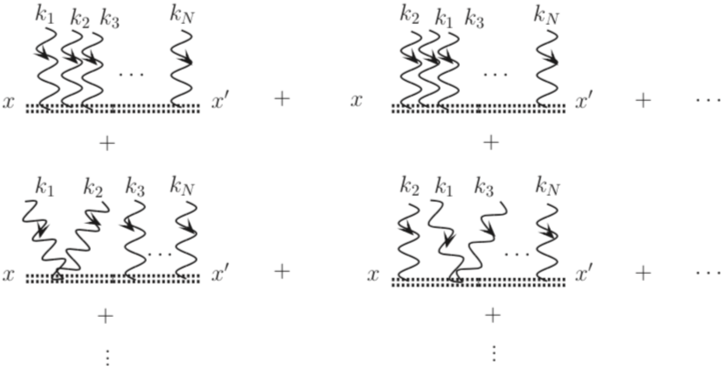
in x-space:

$$D^{xx'}(F \mid k_1, \varepsilon_1; \cdots ; k_N, \varepsilon_N) = (-ie)^N \int_0^\infty dT\, \mathrm{e}^{-m^2 T} (4\pi T)^{-\frac{D}{2}} \det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin \mathcal{Z}}\right] \mathrm{e}^{-\frac{1}{4T} x_- \mathcal{Z} \cot \mathcal{Z} x_-}$$
$$\times \int_0^T d\tau_1 \cdots \int_0^T d\tau_N\, \mathrm{e}^{\sum_{i=1}^N \left(\varepsilon_i \cdot \frac{x_-}{T} + ik_i \cdot \frac{x_- \tau_i}{T} + ik_i \cdot x'\right)}$$
$$\times \exp\left[\sum_{i,j=1}^N \left(k_i\, \underset{\smile}{\Delta}_{ij}\, k_j - 2i\varepsilon_i\, {}^{\bullet}\underset{\smile}{\Delta}_{ij}\, k_j - \varepsilon_i\, {}^{\bullet}\underset{\smile}{\Delta}^{\bullet}{}_{ij}\, \varepsilon_j\right) + \frac{2e}{T} x_- \sum_{i=1}^N \left(F\, {}^{\circ}\underset{\smile}{\Delta}_i\, k_i - iF\, {}^{\circ}\underset{\smile}{\Delta}^{\bullet}{}_i\, \varepsilon_i\right)\right]\Big|_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_N} . \tag{14}$$

left (right) 'open circle' on $\underset{\smile}{\Delta}(\tau, \tau')$ denotes an integral $\int_0^T d\tau$ ($\int_0^T d\tau'$).

For the special case of a purely magnetic field, this x - space master formula was obtained already in 1994 by McKeon and Sherry (Mod. Phys. Lett. A9 (1994) 2167).

which describes the following Feynman diagrams:



and in p-space

$$D^{pp'}(F|k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) = (-ie)^N(2\pi)^D\delta\Big(p+p'+\sum_{i=1}^N k_i\Big)\int_0^\infty dT\, \mathrm{e}^{-m^2T}\frac{1}{\det^{\frac{1}{2}}\big[\cos\mathcal{Z}\big]}$$
$$\times\int_0^T d\tau_1\cdots\int_0^T d\tau_N\, \mathrm{e}^{\sum_{i,j=1}^N\left(k_i\,\underset{\smile}{\Delta}_{ij}\,k_j-2i\varepsilon_i\,{}^{\bullet}\underset{\smile}{\Delta}_{ij}\,k_j-\varepsilon_i\,{}^{\bullet}\underset{\smile}{\Delta}^{\bullet}{}_{ij}\,\varepsilon_j\right)}\,\mathrm{e}^{-Tb\left(\frac{\tan\mathcal{Z}}{\mathcal{Z}}\right)b}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}. \quad (15)$$

Here we have defined

$$\mathcal{Z}_{\mu\nu} = eTF_{\mu\nu}$$
$$b \equiv p+\frac{1}{T}\sum_{i=1}^N\Big[\Big(\tau_i-2ieF\;{}^{\circ}\underset{\smile}{\Delta}_i\Big)k_i - i\Big(1-2ieF\;{}^{\circ}\underset{\smile}{\Delta}^{\bullet}{}_i\Big)\varepsilon_i\Big]. \quad (16)$$

A. Ahmad, **N. A**, O. Corradini, S. P. Kim and C. Schubert, NPB, **919** (2017) 9.

REFERENCES

In these lectures we have been closely following the following references:

- C. Schubert, *Perturbative quantum field theory in the string inspired formalism*, Phys. Rept. **355** (2001) 73 [arXiv: 0101036]
- O. Corradini and C. Schubert, *Spinning Particles in Quantum Mechanics and Quantum Field Theory*, arXiv: 1512.08694.
- R. Rattazzi, *Lecture notes on the path integral approach to Quantum Mechanics*
- N. Ahmadiniaz, A. Bashir ad C. Schubert, *Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED*, Phys. Rev. **D 93** (2016) 045023 [arXiv:1511.05087].
- N. Ahmadiniaz, F. Bastianelli and O. Corradini, *Dressed scalar propagator in a non- Abelian background from the worldline formalism*, Phys. Rev. **D 93** (2016) 025035 [arXiv:1508.05144].
- A. Ahmad, N. Ahmadiniaz, O. Corradini, S. P. Kim and C. Schubert, *Master formulas for the dressed scalar propagator in a constant field*, Nucl. Phys. **B 919** (2017) 9 [arXiv:1612.02944].