Reducible contributions (1PR) to QED in external electromagnetic fields

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Quantum Field Theory at the Limits: from Strong Fields to Heavy Quarks
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OUTLINE

- Electromagnetism and Maxwell equations
- Dirac equation
- An introduction to EHL, some physical applications
- Nonlinear QED Processes: Light-Light scattering, Pair-production
- Some of the most important laser facilities
- 1PR in constant fields
- Strong magnetic field behaviour of the 1PR
- Summary

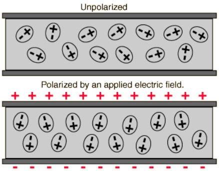
MAXWELL'S EQUATIONS

In electrodynamics Maxwell's equations are a set of four equations, that describes the behaviour of both the electric and magnetic fields as well as their interaction with matter.

Maxwell's four equations express:

- How electric charges produce electric field (Gauss's law) $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
  $\varepsilon_0 =$ vacuum electric permittivity
- The absence of magnetic monopoles $\nabla \cdot \mathbf{B} = 0$
- How currents and changing electric fields produces magnetic fields (Ampere's law) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
- How changing magnetic fields produces electric fields (Faraday's law of induction) $\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$
  $\mu_0 =$ magnetic vacuum permeability

- permittivity ($\varepsilon_0 = 8.85 \times 10^{-12} F/m$):
  More specifically, permittivity describes the amount of charge needed to generate one unit of electric flux in a particular medium. Permittivity is the measure of a material's ability to store an electric field in the polarization of the medium.



A dielectric medium showing orientation of charged particles creating polarization effects.

- permeability ($\mu_0 = 4\pi \times 10^{-7} H/m$):
  permeability is the measure of the ability of a material to support the formation of a magnetic field within itself. Hence, it is the degree of magnetization that a material obtains in response to an applied magnetic field.

HISTORICAL BACKGROUND

- 1864 Maxwell in his paper “Dynamical Theory of the Electromagnetic Field” collected all four equations

- 1884 Oliver Heaviside and Willard Gibbs gave the modern mathematical formulation using vector calculus.

DIRAC EQUATION

In particle physics, the Dirac equation is a relativistic wave equation derived by Paul Dirac in 1928. In its free form, or including electromagnetic interactions, it describes all spin-$\frac{1}{2}$ massive particles such as electrons and quarks.

$$\left\{\beta mc^2 + c\left(\sum_{n=1}^{3}\alpha_n p_n\right)\right\}\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \quad \text{(original form)}$$

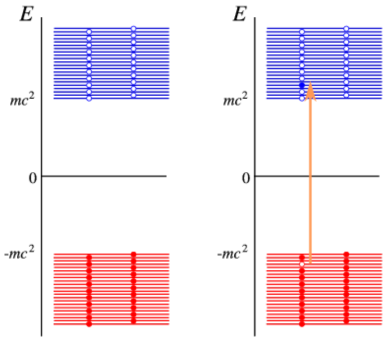
$$i\hbar\gamma_\mu\partial^\mu\psi - mc\psi = 0 \quad \text{(Covariant form)} \tag{1}$$

Main motivation: attempts to make the old quantum theory of the atom compatible with the theory of relativity based on discretizing the angular momentum had failed.

Positrons (the antimatter counterpart of the electron) predicted by the Dirac equation and experimentally conformed by Anderson (1932).

DIRAC SEA

The Dirac sea is a theoretical model of the vacuum as an infinite sea of particles with negative energy. The positron was originally conceived of as a hole in the Dirac sea, well before its experimental discovery.



THE HEISENBERG-EULER EFFECTIVE LAGRANGIAN

In 1936 Heisenberg and Euler investigated the following question:

What happens to the Dirac sea when a classical EM field is applied?

THE ENERGY LEVELS SHIFT

- An EM field produces shifting on the energy spectrum leading to important dynamical effects.

- The simplest example is a constant B-field, the Landau levels for an electron are $E_n = (2n + 1)B \pm B$

These energy shifts produce quantum corrections to the classical Maxwell Lagrangian. For instance, for spinor QED in a constant EM field Euler and Heisenberg derived an effective Lagrangian with the leading contribution.

MATHEMATICAL CHALLENGE

Euler and Heisenberg were able to calculate the energy spectrum of the Dirac-sea electrons in the presence of a constant, homogenous EM-field.

This involves solving the eigenvalue problem for the Dirac operator in the presence of a given background field.

$$\{\gamma^\mu(\partial_\mu - ieA_\mu(x)) - m\}\psi(x,t) = 0$$

But solving the eigenvalue problem is extremely hard in general!

In 1951 Schwinger reformulated the problem in terms of the fermion determinant.

Schwinger demonstrated that the effective action Γ can be written in terms of a determinant or a trace

$$\Gamma = -i \ln \det(i\not{D} - m) = -i \mathrm{Tr}\ \ln(i\not{D} - m)$$

where

$$\not{D} = \gamma_\mu D^\mu = \gamma_\mu(\partial^\mu - ieA^\mu(x))$$

- Schwinger also proved that being able to find the effective action for an arbitrary background, allows one to completely solve the theory. Thus, the effective action (or Dirac's determinant) is an essential object of the theory.

- But even in determinant representation, the calculation of the effective action is still very challenging and exact solutions are rare.

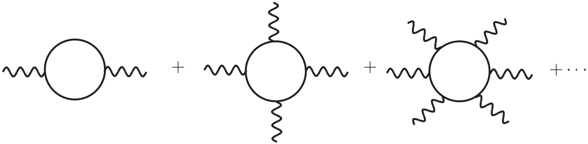
EULER-HEISENBERG LAGRANGIAN

In quantum field theory, the effective action is a modified expression for the action, which takes into account quantum-mechanical corrections:

The exact effective actions have been known only for a few configurations of electromagnetic felds. In Spinor QED, the one-loop effective action for electrons in the presence of a background electromagnetic field is :

$$S^{(1)}_{spinor} = -i \; ln \; det(\not{D} - m) = -\frac{ß}{2} \; ln \; det(\not{D}^2 + m^2) \qquad (2)$$

where A_ν is a fixed classical gauge potential with field strength tensor $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
This one-loop effective action has a natural perturbation expansion in powers of the external photon field A_μ



Euler and Heisenberg, and Weisskopf, showed that in the low energy limit for the external photon lines, in which case the background field strength $f_{\mu\nu}$ could be taken to be constant, it is possible to compute a relatively simple

$$\begin{aligned} \mathcal{L}^{(1)}_{\text{spin}} &= -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh[eaT]\tan[ebT]} - \frac{1}{3}(a^2-b^2)T^2 - 1\right] \\ \mathcal{L}^{(1)}_{\text{scal}} &= \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\sinh[eaT]\sin[ebT]} + \frac{1}{6}(a^2-b^2)T^2 - 1\right] \end{aligned} \tag{3}$$

with two invariants

$$\begin{aligned} a^2 - b^2 &= B^2 - E^2 = -\frac{1}{2}f_{\mu\nu}f^{\mu\nu} \equiv 2\mathcal{F} \\ ab &= \mathbf{E}\cdot\mathbf{B} = -\frac{1}{4}f_{\mu\nu}\tilde{f}^{\mu\nu} \equiv \mathcal{G} \end{aligned}$$

Thus

$$a = \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2} - \mathcal{F}}\ ,\quad b = \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2} + \mathcal{F}}$$

PHYSICAL APPLICATIONS

The Heisenberg and Euler result leads immediately to a number of important physical insights and applications. The EHL is nonlinear in the electromagnetic fields, the quartic and higher terms representing new nonlinear interactions, which do not occur in the tree level Maxwell action.

- Light-Light scattering:

$$\mathcal{L}^{(1)} = \xi\left[\mathcal{F}^2 + 7\mathcal{G}^2\right] = \xi\left[(\mathbf{B}^2 - \mathbf{E}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2\right] + \cdots \quad , \quad \xi = \frac{2\alpha^2\hbar^3}{45 m^4 c^5}$$

  which gives the low-energy limit f the light-light scattering. As first discussed by Euler and Kökel, these nonlinearities can be viewed as dielectric effects, with the quantum vacuum behaving as a polarizable medium.

WHAT HAS BEEN DONE FOR FOUR-PHOTON AMPLITUDE?

- In 1951 Karplus and Neuman published their results of the cross section for the one loop photon-photon scattering (Phys. Rev. **83**, 776 (1951)).
- In 1971 Costantini, De Tollis y Pistoni published the one-loop four-photon amplitude with two external photon lines on-shell and two off-shell (Nou. Cim. **2A**, 733 (1971))
- As of date there is no result for the one-loop four-photon amplitude with three off-shell and one low-energy, one on-shell, and fully off-shell cases (*N. Ahmadiniaz, M. Lopez, C. Loper-Arcos and C. Schubert*, to appear soon).

- Another remarkable thing about Heisenberg and Euler's result is that they correctly anticipated charge renormalization. The first term (on each line) on the the RHS of is the bare result, the second term is the subtraction of a field-free infinite term, and the third term is the subtraction of a logarithmically divergent term which has the same form as the classical Maxwell Lagrangian. This last subtraction corresponds precisely to what we now call charge renormalization, as was later formalized by Schwinger.

$$\mathcal{L}^{(1)}_{\text{spin}} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh[eaT]\tan[ebT]} - 1 - \frac{1}{3}(a^2-b^2)T^2\right] \tag{4}$$

- Weak-field expansions of EH: The weak field expansion expressed in terms of the Lorentz invariants a and b defined in

$$\mathcal{L}^{(1)}_{\text{spin}} \sim -\frac{m^2}{8\pi^2}\sum_{n=2}^{\infty}(2n-3)!\sum_{k=0}^{\infty}\frac{\mathcal{B}_{2k}\mathcal{B}_{2n-2k}}{(2k)!(2n-2k)!}\left(\frac{2eb}{m^2}\right)^{2n-2k}\left(\frac{2iea}{m^2}\right)^{2k}$$

where one can use the following trigonometric Taylor expansions:

$$z\coth z = \sum_{k=0}^{\infty}\frac{2^{2k}\mathcal{B}_{2k}}{(2k)!}z^{2k} \quad , \quad z\cot z = \sum_{k=0}^{\infty}\frac{(-1)^k2^{2k}\mathcal{B}_{2k}}{(2k)!}z^{2k}$$

- Pair-production from vacuum in an electric field: The presence of a background electric field accelerates and splits virtual vacuum dipole pairs, leading to e^+e^- particle production. This instability of the vacuum was realized already by Heisenberg and Euler, motivated in part by earlier work of Sauter on the Klein paradox.

  This pair production process was later formalized in the language of QED by Schwinger. Heisenberg and Euler deduced the leading pair production rate in a weak electric field to be:

$$\Gamma = 2\mathrm{Im}\mathcal{L} \sim \frac{e^2 E^2}{4\pi^3}\, \mathrm{e}^{-\frac{m^2\pi}{eE}}$$

  This rate is deduced from the imaginary part of the effective Lagrangian when the background is purely electric.

The rate is extremely small for typical electric field strengths, becoming more appreciable when the $E \sim E_c = \frac{m^2c^3}{e\hbar} \sim 10^{18} V/m$ where the work done accelerating a virtual pair apart by a Compton wavelength is of the order of the rest mass energy for the pair [1].

[1] see G. V. Dunne (arXiv:hep-th/0406216) for a comprehensive review on the EHL.

HOW STRONG ARE THE CURRENT AND FUTURE LASER FACILITIES?

LFEX(Laser for Fast Ignition Experiments):

The most powerful laser beam ever created has been recently fired at Osaka University in Japan, where the LFEX has been boosted to produce a beam with a peak power of 2,000 trillion watts -two petawatts- for an incredibly short duration, approximately a trillionth of a second ($10^{-12}s$) or one picosecond.

Values this large are difficult to grasp, but we can think of it as a billion times more powerful than a typical stadium floodlight. Imagine focusing all the sun's solar power onto a surface as wide as a human hair for the duration of a trillionth of a second: that's essentially the LFEX laser.

LFEX is only one of a series of ultra-high power lasers that are being built across the world, ranging from the gigantic 192-beam National Ignition Facility in California, to the CoReLS laser in South Korea, and the Vulcan laser at the Rutherford Appleton Laboratory outside Oxford, UK, to mention but a few.

There are other projects in design stages- of which the most ambitious is probably the **Extreme Light Infrastructure** (ELI), an international collaboration based in Eastern Europe devoted to building a laser 10 times more powerful even than the LFEX.

ELI (ELI-NP)

ELI is going to be the most advanced research facility in the world focusing on the study of photonuclear physics and its applications, comprising a very high intensity laser of two $10PW$ ultra-short pulse lasers and the most brilliant tunable gamma-ray beam.

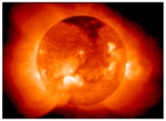
This unique experimental combination will enable ELI to tackle a wide range of research topics in fundamental physics, nuclear physics and astrophysics, and also applied research in materials science, management of nuclear materials and life sciences.

ELI facility will consist of a very high intensity laser system, with two 10 PW laser arms able to reach intensities of $10^{23}W/cm^2$ and electrical fields of $10^{15}V/m$.

lightning

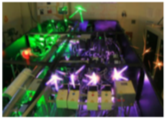
$$E_{\mathsf{lightning}} \sim 10^4 \frac{\mathsf{V}}{\mathsf{cm}}$$

the sun



- $$E_{\mathsf{sun}} \sim 10^{10} \frac{\mathsf{V}}{\mathsf{cm}}$$

lasers



$$E_{\mathsf{max}} \sim 10^{12} \frac{\mathsf{V}}{\mathsf{cm}}$$

SLAC EXPERIMENT 144 (1997)

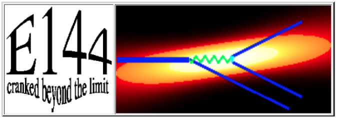
The Breit-Wheeler process or Breit-Wheeler pair production is a physical process in which a positron-electron pair is created from the collision of two photons. It is the simplest mechanism by which pure light can be potentially transformed into matter.
An electron (blue) enters the laser beam from the left, and collides with a laser photon to produce a high-energy gamma ray (wiggly yellow line). The electron is deflected downwards. The gamma ray then collides with four or more laser photons to produce an electron-positron pair. (Bula et. al PRL, 76, 3116 (1997))

Pair creation by Light (two-step process)

$$e + \omega_0 \rightarrow e' + \omega$$

then

$$\omega + n\omega_0 \rightarrow e^+ e^-$$



colliding a high-energy electrons with a counter-propagating terawatt laser pulse

VACUUM BIREFRINGENCE

PVLAS (Polarizzazione del Vuoto con LASe) is searching for vacuum polarization of laser beams crossing magnetic fields to detect effects from axion dark matter. No signal has been found and searches continue. OSQAR at CERN is also studying vacuum birefringence.

A team of astronomers from Italy, Poland, and the U.K. has reported in 2016 (**Mignani** et.al, Month. Not. Roy. Astron. Soc. 465 (2017) 492) observations of the light emitted by a neutron star (pulsar RX J1856.5?3754). The star is surrounded by a very strong magnetic field (10^{13}G), and one expects birefringence from the vacuum polarization described by the EHL. A degree of polarization of about 16% was measured and was claimed to be "large enough to support the presence of vacuum birefringence, as predicted by QED". **Fan** et al (arXiv:1705.00495) pointed that their results are uncertain due to low accuracy of star model and the direction of the neutron magnetization axis.

1PR CONTRIBUTION TO THE PROPAGATOR IN A CONSTANT BACKGROUND FIELD

The **Euler-Heisenberg Lagrangian** (EHL), one of the first serious calculations in QED describes the one-loop amplitude involving a spinor loop interacting non-perturbatively with a constant background electromagnetic field which is given by

$$\mathcal{L}^{(1)}_{\rm spin}(a,b) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2T}\,\frac{e^2ab}{\tanh(ebT)\tan(ebT)} \tag{5}$$

where a and b are related to the two invariants of the Maxwell field by

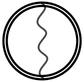
$$a^2 - b^2 = B^2 - E^2 \quad , \quad (ab)^2 = (\mathbf{E}\cdot\mathbf{B})^2 \tag{6}$$

The EHL contains the information on nonlinear QED effects such as

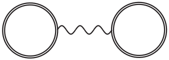
- photon-photon scattering (**Heisenberg** and **Euler**, Z. Phys. 98 (1936) 714).
- photon dispersion (**Adler**, Ann. Phys. 67 (1971) 599).
- photon splitting (**Adler**, Ann. Phys. 67 (1971) 599; **Adler** and **Schubert**, PRL 77 (1996) 1695).

The first radiative corrections to these Lagrangians (scalar and spinor), describing the effect of an additional photon exchange in the loop, were obtained in the seventies by **Ritus** (Sov. Phys. JETP 42 (1975) 774). He obtained $\mathcal{L}^{(2)}_{\text{scal,spin}}$ in terms of certain two-parameter integrals which are intracable analytically, closed-form expressions have obtained for their weak-field expansions for the purely electric or magnetic cases (**Dunne** and **Schubert**, NPB 564(2000) 59).

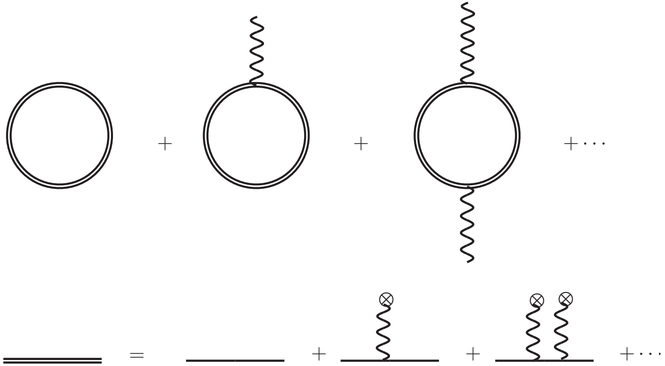
In all these calculations it was assumed that the only diagram contributing to the EHL at the two-loop level is the one particle irreducible (1PI) one:



At the same loop order, there is also the one-particle reducible (1PR) diagram



which was generally believed not to contribute!!



GIES-KARBSTEIN DISCOVERY

Recently, **Gies** and **Karbstein** (JHEP 1703 (2017) 108) made the stunning discovery that actually this diagram **does give a finite contribution**, if one takes into account the divergence of the connecting photon propagator in the zero-momentum limit which leads to the the following simple formula

$$\mathcal{L}^{(2)1\mathrm{PR}}_{\mathrm{EH}} = \frac{\partial \mathcal{L}^{(1)}_{\mathrm{EH}}}{\partial F^{\mu\nu}} \frac{\partial \mathcal{L}^{(1)}_{\mathrm{EH}}}{\partial F_{\mu\nu}} = \mathcal{F}\left[\left(\frac{\partial \mathcal{L}^{(1)}_{\mathrm{EH}}}{\partial \mathcal{F}}\right)^2 - \left(\frac{\partial \mathcal{L}^{(1)}_{\mathrm{EH}}}{\partial \mathcal{G}}\right)^2\right] + 2\mathcal{G}\frac{\partial \mathcal{L}^{(1)}_{\mathrm{EH}}}{\partial \mathcal{F}} \frac{\partial \mathcal{L}^{(1)}_{\mathrm{EH}}}{\partial \mathcal{G}} \tag{7}$$

with the following renormalized one-loop EH Lagrangian for spinor QED

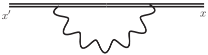
$$\mathcal{L}^{(1)}_{\mathrm{EH}} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, \mathrm{e}^{-m^2T}\left\{\frac{(eaT)(ebT)}{\tan(eaT)\tanh(ebT)} - \frac{2}{3}(eT)^2\mathcal{F} - 1\right\} \tag{8}$$

and two invariants in constant field

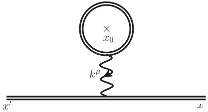
$$a = \left(\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}\right)^{1/2} \quad , \quad b = \left(\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}\right)^{1/2}$$

1PR CONTRIBUTION TO THE SCALAR PROPAGATOR

Right after this paper appeared two of my colleagues (**Edwards** and **Schubert** NBP 923 (2017) 339) pointed out that a similar 1PR addendum exists also for the QED scalar propagator already at the one-loop level:



but there is also a finite contribution from the 1PR:



By sewing the one-loop with one off-shell photon to the scalar propagator with one off-shell photon using the photon propagator in the Feynman gauge and

$$\int d^D k\, \delta^D(k) \frac{k^\mu k^\nu}{k^2} = \frac{\eta^{\mu\nu}}{D} \tag{9}$$

one obtains:

$$\begin{aligned} \mathcal{L}^{(1)1\mathrm{PR}}_{\mathrm{scal}} &= e^2 \int_0^\infty dT (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \det{}^{-\frac{1}{2}} \left[\frac{\sin \mathcal{Z}}{\mathcal{Z}}\right] \\ &\times \int_0^\infty dT' (4\pi T')^{-\frac{D}{2}} e^{-m^2 T'} \det{}^{-\frac{1}{2}} \left[\frac{\sin \mathcal{Z}'}{\mathcal{Z}'}\right] \frac{1}{D} \mathrm{tr}(\dot{\mathcal{G}}_B \cdot \dot{\mathcal{G}'}_B) \end{aligned} \tag{10}$$

where $\mathcal{Z}_{\mu\nu} = eTF_{\mu\nu}$ and

$$\mathcal{G}_B = \frac{T}{2\mathcal{Z}^2}\Big(\mathcal{Z} \cdot \cot \mathcal{Z} - 1\Big) \quad , \quad \dot{\mathcal{G}}_B = i\frac{2\mathcal{Z}}{T}\mathcal{G}_B$$

1PR CONTRIBUTION TO THE FERMION PROPAGATOR

The extension to the spinor case can be found in (N. Ahmadiniaz, F. Bastianelli, O. Corradini, J. P. Edwards and C. Schubert; NPB 924 (2017) 377). We just present the final result:

$$
\begin{aligned}
S^{(1)1PR}(p) \quad = \quad & e^2 \int_0^\infty dT\,T\, e^{-T(m^2+p\cdot\frac{\tan\mathcal{Z}}{\mathcal{Z}}\cdot p)} \int_0^\infty dT'(4\pi T')^{-\frac{D}{2}} e^{-m^2T'} \\
& \times \det{}^{-\frac{1}{2}}\left[\frac{\tan\mathcal{Z}'}{\mathcal{Z}'}\right]\Bigg\{\Big[m-\gamma\cdot(\mathbb{1}+i\tan\mathcal{Z})\cdot p\Big] \\
& \times\Big[-Tp\cdot\frac{\mathcal{Z}-\sin\mathcal{Z}\cdot\cos\mathcal{Z}}{\mathcal{Z}^2\cdot\cos^2\mathcal{Z}}\cdot\Xi'\cdot p+\Xi'_{\mu\nu}\frac{\partial}{\partial\mathcal{Z}_{\mu\nu}}\Big] \\
& -i\gamma\cdot\sec^2\mathcal{Z}\cdot\Xi'\cdot p\Bigg\}\mathrm{symb}^{-1}\Big\{e^{\frac{i}{4}\eta\cdot\tan\mathcal{Z}\cdot\eta}\Big\}, \qquad (11)
\end{aligned}
$$

where

$$
\Xi \equiv \frac{1}{\mathcal{Z}} - \frac{1}{\sin\mathcal{Z}\cdot\cos\mathcal{Z}} = \frac{d}{d\mathcal{Z}}\mathrm{tr}\,\ln\left[\frac{\tan\mathcal{Z}}{\mathcal{Z}}\right]
$$

CROSSED ELECTRIC AND MAGNETIC FIELDS

CCF are the class of constant fields with vanishing Maxwell invariants, $F_{\mu\nu}F^{\mu\nu} = 4(|\mathbf{E}|^2 - |\mathbf{B}|^2) = 0$ and $F_{\mu\nu}\widetilde{F}^{\mu\nu} = 4\mathbf{E}\cdot\mathbf{B} = 0$. Let's assume

$$\mathbf{E} = B(1,0,0) \quad , \quad \mathbf{B} = B(0,1,0) \tag{12}$$

which its field strength tensor is

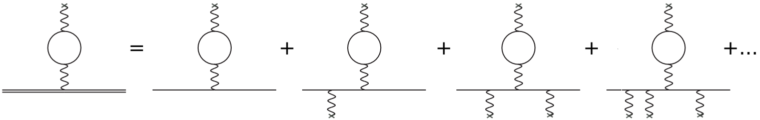
$$F_{\mu\nu} = \begin{pmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & B \\ 0 & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \end{pmatrix} \tag{13}$$

Furthermore $F^3 = 0$ for such fields and all higher powers also vanish. For the tadpole we get:

$$
\begin{aligned}
S^{(1)1PR} \;=\;& \frac{e^2}{m^2}\left(\frac{m^2}{4\pi}\right)^{\frac{D}{2}} \frac{eB}{m^2}\Gamma\left[2-\frac{D}{2}\right]\int\limits_0^\infty \mathrm{d}s\, s\; \mathrm{e}^{-is\left(p^2+m^2+\frac{z^2}{3}p^{-2}\right)} \\
& \left[\frac{1}{2}\left\{i\left(m-\not{p}\right)\,,\,\frac{4is}{9}zp^{-2}-\frac{2}{3}\gamma^{-}\gamma^{1}\right\}\right. \\
& \left. +iz\gamma^{[-}p^{1]}\left(\frac{4is}{9}zp^{-2}-\frac{2}{3}\gamma^{-}\gamma^{1}\right)\right]\left[\mathbb{1}+z\gamma^{-}\gamma^{1}\right]\,,
\end{aligned}
$$

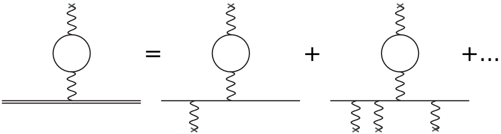
the result is linear in this coupling of the loop to the crossed field background. As such we see that this 1PR contribution can be absorbed simply by an additional (infinite, in $D = 4$) renormalisation of the photon propagator.

Spinor case:



The schematic diagrammatic expansion of the 1PR contribution to the self energy interactions with the background field for crossed fields. Being linear in the coupling of the loop, there is only one low energy photon attached thereto.

Scalar case:



The expansion of the 1PR contribution to the scalar propagator in the crossed field case. Note that in Fock-Schwinger gauge there is an odd number of free low energy photons of the background coupled to the line, and one low energy photon attached to the loop as before

CONSTANT MAGNETIC FIELD

We consider a constant **B** field in the z-direction (say), defining the field strength tensor by

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

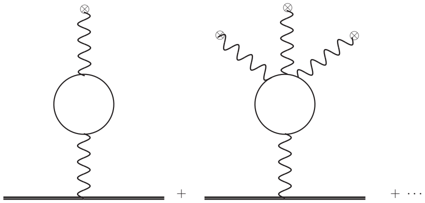
The full parameter integral form of the tadpole contribution is

$$D^{1PR}_{\rm scal}(p) = \frac{e^2}{2} \int_0^\infty dT' (4\pi i T')^{-\frac{D}{2}} \, \mathrm{e}^{-im^2T'} \, \frac{eB'T'}{\sin(eB'T')} \Big(\cot(eB'T') - \frac{1}{eB'T'} \Big)$$

$$\times \int_0^\infty dT \, T \, \frac{\mathrm{e}^{-im^2T}}{\cos(eBT)} \, \mathrm{e}^{-iT(p_\parallel^2 + \frac{\tan eBT}{eBT} p_\perp^2)} \Big\{ - \frac{T}{eBT} \Big(\frac{\tan(eBT)}{eBT} - \sec^2(eBT) \Big) p_\perp^2 + i \tan(eBT) \Big\} .$$

This time the tadpole produces terms at higher order in the background field coupling to the loop. Expanding for small magnetic field and doing the parameter integrals yields

$$\frac{e^4 B' B}{3(4\pi)^{\frac{D}{2}}} (m^2)^{\frac{D}{2}-2} \left[\Gamma\left(2 - \frac{D}{2}\right) - \frac{1}{30} \left(\frac{eB'}{m^2}\right)^2 \Gamma\left(4 - \frac{D}{2}\right) + \ldots \right] \left[\frac{2p_\perp^2}{(m^2+p^2)^4} - \frac{1}{(m^2+p^2)^3} \right] \tag{14}$$



PLANE WAVE BACKGROUND

The plane waves background of arbitrary strength and shape, which are used as models of intense laser fields. It is clear that for plane waves the (renormalised) Euler-Heisenberg effective action is zero (to any loop order, independent of whether it comes from 1PI or 1PR diagrams), because there are no Lorentz invariants which can be formed from the plane wave field strength alone.

We recently showed that the tadpole gives a nonzero contribution but that this can be renormalised away (N. Ahmadiniaz, A. Ilderton and J. P. Edwards, JHEP, 05 (2019) 038)

STRONG FIELD LIMIT OF EHL

A particularly interesting parameter regime is the regime of strong magnetic fields characterized by $eB/m^2 \gg 1$. It is in particular well known in the literature that the 1PI contribution to $\mathcal{L}_{EH}$ at l-order scales as

$$\begin{aligned}
\mathcal{L}_{EH}^{(1-loop)}(B) &= \frac{1}{2}B^2\alpha\beta_1 \log(\frac{eB}{m^2})\Big[1+\mathcal{O}(log^{-1}\frac{eB}{m^2})\Big] \\
\mathcal{L}_{EH,1PI}^{(l-loop)}(B) &= \frac{1}{2}B^2(\alpha\beta_1)^l\frac{\beta_2}{\beta_1^2(l-1)}\log^{l-1}(\frac{eB}{m^2})\Big[1+\mathcal{O}(log^{-1}\frac{eB}{m^2})\Big] \quad , \quad l = 2,3,\cdots
\end{aligned}$$

where $\beta_1 = 1/3\pi$ and $\beta_2 = 1/4\pi^2$ are renormalization scheme independent coefficients of β-function for spinor QED:

$$\beta(\alpha(\mu^2)) = \frac{1}{\alpha(\mu^2)}\mu^2\frac{d\alpha(\mu^2)}{d\mu^2} \quad \text{with} \quad \beta(\alpha) = \beta_1\alpha + \beta_2\alpha^2 + \mathcal{O}(\alpha^3)$$

governing the running of the fine structure constant.

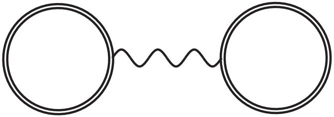
Recently **Karbstein** (arXiv: 1903.06998) the leading strong field behavior of the Heisenberg-Euler effective Lagrangian is given by

$$\mathcal{L}_{EH,1PR}^{(l-loop)}(B) = \frac{1}{2}B^2\Big[\alpha\beta_1 log(\frac{eB}{m^2})\Big]^l\Big\{1+\mathcal{O}(log^{-1}\frac{eB}{m^2})\Big\}$$

Note that this behavior is in accordance with Weisskopf's investigation showing that the logarithmic divergence at $l = 2, 3, \cdots$ loop order scales at most as the l-th power of the logarithm.

So the 1PR contribution to the EHL dominates the 1PI part in strong field limit.

1PR TO TWO-LOOP EHL



$$\mathcal{L}^{(2)1PR} = \frac{4e^2}{D}\int_0^\infty ds(4\pi is)^{-\frac{D}{2}}\,e^{-im^2s}\mathcal{J}(z)\int_0^\infty ds'(4\pi is')^{-\frac{D}{2}}\,e^{-im^2s'}\mathcal{J}(z')$$

where $\mathcal{J}(z) = (z/\tan z)(\cot z - 1/z + \tan z)$.

$$
\begin{aligned}
\mathcal{I} &= -\int_0^\infty dT\,(4\pi T)^{-\frac{D}{2}}\; e^{-m^2 T}\left[\coth z - z\,\mathrm{csch}^2 - \frac{2}{3}z\right] \\
&= -\frac{1}{eB}\left(\frac{4\pi}{eB}\right)^{-\frac{D}{2}}\int_0^\infty dz\; e^{-\frac{m^2}{eB}z} z^{-\frac{D}{2}}\left[\coth z - z\,\mathrm{csch}^2 z - \frac{2}{3}z\right] \\
&= -\frac{1}{eB}\left(\frac{4\pi}{eB}\right)^{-\frac{D}{2}}(\mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3)\,.
\end{aligned}
\tag{15}
$$

The complete integral is now finite in $D = 4$. We begin with $\mathcal{I}_1$; this is just the Laplace transform $F(\omega)$ of the function $f(z)$ where

$$
f(z) = z^{-\frac{D}{2}}\coth z \quad \text{and} \quad \omega = \frac{m^2}{eB}\,. \tag{16}
$$

The Laplace transform can be expressed in terms of the Hurwitz zeta function $\zeta[x, q]$

$$\mathcal{I}_1 = 2^{\frac{D}{2}-1}\Gamma\left[1-\frac{D}{2}\right]\left(\zeta\left[1-\frac{D}{2},\frac{m^2}{2eB}\right]+\zeta\left[1-\frac{D}{2},1+\frac{m^2}{2eB}\right]\right)$$

$$\begin{aligned}\mathcal{I}_2 = 2^{\frac{D}{2}-1}\Gamma\left[2-\frac{D}{2}\right]\Bigg\{&\zeta\left[1-\frac{D}{2},\frac{m^2}{2eB}\right]+\zeta\left[1-\frac{D}{2},1+\frac{m^2}{2eB}\right]\\ &-\frac{m^2}{2eB}\left(\zeta\left[2-\frac{D}{2},\frac{m^2}{2eB}\right]+\zeta\left[2-\frac{D}{2},1+\frac{m^2}{2eB}\right]\right)\Bigg\}\end{aligned}$$

$$\mathcal{I}_3 = \frac{2}{3}\left(\frac{eB}{m^2}\right)^{2-\frac{D}{2}}\Gamma\left[2-\frac{D}{2}\right]$$

so the leading order strong field behaviour of the reducible two-loop contribution to the EHL is

$$\mathcal{L}^{(2)1PR} \sim \frac{1}{2} B^2 \left[\alpha\beta_1 \ln\left(\frac{eB}{m^2}\right)\right]^2$$

This correctly reproduces the results presented by *Karbstein* at two-loop order.

CONCLUSION

- We briefly reviewed the EHL and its physical applications.
- We discussed the 1PR contributions recently discovered for the EHL in higher loops as well as the one-loop propagator in constant field.
- The strong field limit of the 1PR to the EHL leads to interesting observations which might change many of previous conclusions.
- One of the important step is the strong field limit of our one-loop correction to the propagator which might lead to a surprising conclusion.
- These 1PR corrections need to be investigated more!