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Radiative electroweak corrections to polarized top quark decays

Acknowledgements and Scope

This presentation is based on work done in collaboration with H. S. Do (Australia), M. Fischer (Bensheim) and S. Groote (University of Tartu).

The presentation is aimed at young students and postdocs. In my presentation I use and illustrate a number of simple tricks of the trade which allow one to understand some of the present papers on the subject.

My talk is mostly about the imaginary parts of the electroweak one-loop contributions. If time allows I will also briefly comment on the real parts of the electroweak radiative corrections.

Singly Produced Top Quarks: Rates, Polarization and Luminosity

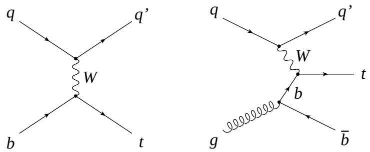


Figure: t-channel production of single top quarks.

- t-channel production of single top quarks through parity-violating weak interactions. Necessary condition for non-vanishing polarization of top quarks.
- Polarization of singly produced top quarks is calculated to be $P_t \sim 90\,\%$ (polarization along spectator quark). Experimentally confirmed by the ATLAS and CMS Collaborations.

Singly Produced Top Quarks: Rates, Polarization and Luminosity cont'd

- **SM rates for single top production:**

$$\sigma^{8\,\mathrm{TeV}} = 55 \cdot 10^3 \; fb$$
$$\sigma^{13\,\mathrm{TeV}} = 136 \cdot 10^3 \; fb$$

- **Data samples**
  Large samples of singly produced top quarks exist at present ($\sim 10^7$ events)

  The projected overall luminosity at the HL-LHC is $3ab^{-1}$ which corresponds to 400 $\times 10^7$ events. HL-LHC is projected to start in 2023.

- **Planned $(e^+ - e^-)$–colliders (ILC, FCC-ee, CEPC)**
  With a little bit of fine-tuning of the beam polarization one can achieve top quark polarizations close to 100 %.

- **Polarization Retention**
  Since the top quark decays before it can hadronize the top quark keeps its polarization at birth when it decays

Dynamical Degrees of Freedom

Before getting into the details of the problem we want to count the number of independent dynamical functions that describe the decay of a polarized top quark.
Let me draw an anology to the corresponding counting for the symmetry groups SU(2) and SU(3).

- $\pi + N \to \pi + N$

  $1 \otimes 1/2 = 1/2 \oplus 3/2$

  2 reduced matrix elements

- $M_8 + B_8 \to M_8 + B_8$

  $8 \otimes 8 = 1 \oplus 8_s \oplus 8_{as} \oplus 10 \oplus \bar{10} \oplus 27$
  6 reduced matrix elements

Number of Lorentz Invariants

The decay $t \to b + \ell^+ + \nu_\ell$ is described by the contraction $\mathcal{H}^{\mu\nu}\mathcal{L}_{\mu\nu}$.
Task: To build $\mathcal{H}^{\mu\nu}$ write down second rank tensors $t^{\mu\nu}$ built from p_t^μ, p_t^ν, $g^{\mu\nu}$ and Levi-Civita tensor. Do not consider q^μ and q^ν since $q^\mu L_{\mu\nu} = 0$

Unpolarized case :
Three covariants $t^{\mu\nu}$ and thereby three invariants $\mathcal{H}_i (i = 1, 2, 3)$ (called strucure functions)

$$\mathcal{H}^{\mu\nu} = -g^{\mu\nu}\,\mathcal{H}_1 + p_t^\mu p_t^\nu\,\mathcal{H}_2 - i\epsilon(\mu, \nu, p_t, q)\,\mathcal{H}_3$$

Polarized case:
Add spin four-vector s_t^μ as building element. Nine covariants $t^{\mu\nu}$ and thereby nine invariants $\mathcal{G}_i (i = 1, ..., 9)$ (remember $p_t \cdot s_t = 0$)

$$\begin{aligned}
\mathcal{H}^{\mu\nu}(s_t) \;=\; & (q \cdot s_t)\Big(- g^{\mu\nu}\,G_1 + p_t^\mu p_t^\nu\,G_2 - i\epsilon(\mu\nu p_t\, q)\,G_3 \Big) \\
& \Big(s_t^\mu p_t^\nu + \mu \leftrightarrow \nu\Big)\,G_4 + \Big(s_t^\mu p_t^\nu - \mu \leftrightarrow \nu\Big)\,G_5 \\
& i\epsilon(\mu\nu p_t\, s_t)\,G_6 + i\epsilon(\mu\nu q s_t)\,G_7 \\
& + \Big(i\, p_t^\mu \epsilon(\nu\, q p_t\, s_t) + \mu \leftrightarrow \nu\Big) G_8 \\
& + (i p_t^\mu \epsilon(\nu\; q\; p_t\; s_t) - \mu \leftrightarrow \nu) G_9
\end{aligned}$$

The Levy-Civita Tensor

We have used a notation such that

$$i\,\varepsilon^{\nu\,\alpha\,\beta\,\gamma} q_\alpha p_{t\beta} q_\gamma = i\,\epsilon(\nu\; q\; p_t\; s_t)$$

Can be dangerous, since $i\,\epsilon(\nu\; q\; p_t\; s_t)$

$$i\,\epsilon(\nu\; q\; p_t\; s_t) = i\,\varepsilon^{\nu\,\alpha\,\beta\,\gamma} q_\alpha p_{t\beta} q_\gamma$$

or

$$i\,\epsilon(\nu\; q\; p_t\; s_t) == \; i\varepsilon_{\nu\,\alpha\,\beta\,\gamma} q^\alpha p^{t\beta} q^\gamma$$

Must be decided on in the context

Number of Invariants cont'd

A problem has occurred. We have overcounted the number of covariants and thereby the number of invariant structure functions by two. This can cause severe problems in calculations such as

$$\begin{vmatrix} a & 3a \\ b & 3b \end{vmatrix} = 3ab - 3ab = 0$$

How do we know that we have overcounted?

The count is best done by considering the independent double spin density elements $\mathcal{H}^{\lambda_t\,\lambda'_t}_{\lambda_W\,\lambda'_W}$ **of the** W **which form a hermitian** (6×6) **matrix**

$$\left(\mathcal{H}^{\lambda_t\,\lambda'_t}_{\lambda_W\,\lambda'_W}\right)^{\dagger} = \left(\mathcal{H}^{\lambda_t\,\lambda'_t}_{\lambda_W\,\lambda'_W}\right)$$

There are only ten independent helicity structure functions (3+5 T-even and 2 T-odd).
Here they are:

$$H^{++}_{++} \quad H^{--}_{++} \quad H^{++}_{--} \quad H^{--}_{--} \quad H^{++}_{00} \quad H^{--}_{00}$$
$$Re\,H^{+-}_{-0} \quad Im\,H^{+-}_{-0} \quad Re\,H^{-+}_{+0} \quad Im\,H^{-+}_{+0}$$

Dynamical degrees of freedom: 3 unpolarized T-odd, 5 polarized T-even, 2 polarized T-odd

Schouten Identity

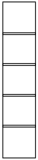
Schouten identity:  $= 0$

Figure: **Young Tableaux with 5 vertical boxes is zero in 4 dimensions**

Schouten identity true in four dimensions:

$$g^{\mu\alpha_1}\epsilon(\alpha_2\,\alpha_3\,\alpha_4\,\alpha_5) + \text{cycl.}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = 0$$

Not true in $d = 4 + \epsilon$–dimensions (dimensional regularization). This is at the origin of the so-called γ_5–problem in dimensional regularization (remember Tr$\gamma_5 \not{a} \not{b} \not{c} \not{d} = 4i\varepsilon(abcd)$))

Two Schouten Identities for Covariants

There are two nontrivial identities between the 7 polarized T-even covariants that can be derived using the Schouten identity

- **contract** $q_\alpha p_{t,\beta} s_{t,\gamma} \left(q_\alpha \varepsilon_{\mu\nu\beta\gamma} + cycl. \right)$

$$q \cdot s_t \, \epsilon(\mu\nu \, p_t \, q) - q^2 \epsilon(\mu\nu p_t s_t) + q \, p_t \, \epsilon(\mu\nu q s_t) = 0$$

- **contract** $q_\alpha p_{t,\beta} s_{t,\gamma} \left(p_{t\,\mu} \varepsilon_{\nu\alpha\beta\gamma} + cycl. \right)$

$$(p_t^\mu \epsilon(\nu \, q \, p_t \, s_t) - \mu \leftrightarrow \nu) - m_t^2 \epsilon(\mu\nu \, q \, s_t) + p_t q \, \epsilon(\mu\nu \, p_t \, s_t) = 0$$

Beware! There are other processes in which one overcounts the number of dynamical degrees of freedom by counting the number of covariants. Examples are,

- $e^+ e^- \to q\bar{q}g$ **and crossed processes thereof in DIS and DY Körner, Sieben 1991 (too early for [hep-ph])**
- $\gamma^* - \gamma^* \to \pi^+ \pi^-$ **Hoferichter 2019 [hep-ph] 1905.13198**

The Born Term Tensor as an Example

Born term amplitude (omit coupling factors):

$$M^{\mu} = \bar{u}_b \gamma^{\mu}(1-\gamma_5)u_t.$$

Square the amplitude and sum over the spin of the *b* quark

$$B^{\mu\nu} = \sum(b\text{-}spin) M^{\mu} M^{*\nu} = \mathbf{Tr}\,(\not p_b + m_b)\gamma^{\mu}(1-\gamma_5)\underbrace{(\not p_t + m_t)\frac{1}{2}(1+\gamma_5 \not s_t)}_{\not p_t + m_t \gamma_5 \not s_t}\gamma^{\nu}(1-\gamma_5).$$

A common trick:
Since only even-numbered γ-matrix strings survive between the two $(1-\gamma_5)$-factors one can compactly write ($p_t = p_b + q$)

$$B^{\mu\nu} = 2(\bar{p}_t^{\mu} p_b^{\nu} + \bar{p}_t^{\nu} p_b^{\mu} - g^{\mu\nu}\,\bar{p}_t \cdot p_b + i\epsilon^{\mu\nu\alpha\beta} p_{b,\alpha}\bar{p}_{t,\beta})$$

where

$$\bar{p}_t^{\mu} = p_t^{\mu} + m_t s_t^{\mu}.$$

Number of Invariants cont'd

$$\begin{aligned} B^{\mu\nu} &= 2(2p_t^\mu p_t^\nu + m_t(s_t^\mu p_t^\nu + s_t^\nu p_t^\mu) - g^{\mu\nu}(p_t \cdot p_b + m_t q \cdot s_t) - m_t i\epsilon^{\mu\nu\alpha\beta} p_{t,\alpha} s_{t,\beta} \\ &+ i\epsilon^{\mu\nu\alpha\beta} q_\alpha p_{t,\beta}) + m_t i\epsilon^{\mu\nu\alpha\beta} q_\alpha s_{t,\beta}) \end{aligned}$$

The Born term populates the invariants $H_1, H_2, H_3, G_1, G_4, G_6, G_7$

Common trick: Replace p_t^μ **in the unpolarized calculation with** $\bar{p}_t^\mu = p_t^\mu + m_t s_t^\mu$ **to obtain the polarized result. Trick does not work in higher order calculations.**

Special Cases

Born term helicity structure functions when $m_b \neq 0$:
Remove those structure functions where $\lambda_t + \lambda_W = \lambda_b = \pm 1/2$ is not satisfied

$$\underbrace{H_{++}^{++}}_{0} \quad H_{++}^{--} \quad H_{--}^{++} \quad \underbrace{H_{--}^{--}}_{0} \qquad H_{00}^{++} \qquad H_{00}^{--}$$

$$Re\, H_{-0}^{+-} \quad Im\, H_{-0}^{+-} \quad Re\, H_{+0}^{-+} \quad Im\, H_{+0}^{-+}$$

Dynamical degrees of freedom: 6 T-even, 2 polarized T-odd

It is common practise to define unpolarized and polarized transverse-plus helicity structure functions H_{T_+} and $H_{T_+}^P$ by writing

$$H_{T_+} = \underbrace{H_{++}^{++}}_{0} + H_{++}^{--} \qquad H_{T_+}^P = \underbrace{H_{++}^{++}}_{0} - H_{++}^{--}$$

At the Born term level one has

$$H_{T_+}(Born) = -H_{T_+}^P(Born)$$

Zero Bottom Quark Mass

Born term helicity structure functions for $m_b = 0$. Relevant for top quark decays since $m_b/m_t \approx 0$.
Remove in addition those structure functions where $\lambda_t + \lambda_W = \lambda_b = -1/2$ is not satisfied

$$\underbrace{H_{++}^{--}}_{0} \quad H_{--}^{++} \qquad \underbrace{H_{00}^{++}}_{0} \qquad H_{00}^{--}$$

$$Re\,H_{-0}^{+-} \quad Im\,H_{-0}^{+-} \quad \underbrace{Re\,H_{+0}^{-+}}_{0} \quad \underbrace{Im\,H_{+0}^{-+}}_{0}$$

Dynamical degrees of freedom: 3 T-even, 1 T-odd,

Much more difficult to count the number of dynamical degrees of freedom in the covariant representation

Hermiticity

The hadronic tensor is Hermitian ($H^{\mu\nu} \sim M^{\mu} M^{*\,\nu}$)

$$\boldsymbol{H^{\mu\nu\,\dagger} = H^{\mu\nu}}$$

- **H_1 is real** $\quad (H_1 g^{\mu\nu})^{\dagger} = H_1^{*} g^{\nu\mu} = H_1 g^{\mu\nu}$
- **H_3 is real**
  $(H_3 i\varepsilon(\mu\nu p_t q))^{\dagger} = -H_3^{*} i\varepsilon(\nu\mu p_t q) = H_3^{*} i\varepsilon(\nu\mu p_t q) = H_3 i\varepsilon(\mu\nu p_t q)$
- **G_5 is imaginary**
  $(G_5(s_t^{\mu} p_t^{\nu} - \mu \leftrightarrow \nu))^{\dagger} = G_5^{*}(s_t^{\nu} p_t^{\mu} - \mu \leftrightarrow \nu))$
  $= -G_5{}^{(s_t^{\mu} p_t^{\nu} - \mu \leftrightarrow \nu) = G_5(s_t^{\mu} p_t^{\mu} - \mu \leftrightarrow \nu)}$
- **G_8 is imaginary**
  $\left(G_8(i\, p_t^{\mu} \epsilon(\nu\, q p_t\, s_t) + \mu \leftrightarrow \nu)\right)^{\dagger} = -G_8^{*}(i\, p_t^{\nu} \epsilon(\mu\, q p_t\, s_t) + \mu \leftrightarrow \nu) =$
  $-G_8^{*}(i\, p_t^{\mu} \epsilon(\nu\, q p_t\, s_t) + \mu \leftrightarrow \nu) = G_8(i\, p_t^{\mu} \epsilon(\nu\, q p_t\, s_t) + \mu \leftrightarrow \nu)$

The imaginary invariants G_5 and G_8 are called T−odd observables

T-odd Correlations

Evaluate $H^{\mu\nu} L_{\mu\nu}$ for the T-odd terms:

$$
\begin{aligned}
L^{\mu\nu} &= Tr\left[\not{p}_\ell \gamma^\mu (1-\gamma_5) \not{p}_\nu \gamma^\nu (1-\gamma_5)\right] \\
&= 8\left(p_\ell^\mu p_\nu^\nu + p_\ell^\nu p_\nu^\mu - p_\ell \cdot p_\nu \, g^{\mu\nu} + i\varepsilon^{\mu\nu\alpha\beta} p_{\ell\,\alpha} p_{\nu\,\beta}\right)
\end{aligned}
$$

$$
\begin{aligned}
G_5\,(s_t^\mu p_t^\nu - \mu \leftrightarrow \nu)\, i\,\varepsilon(\mu\nu p_e p_\nu) &= 2i\,G_5\,\varepsilon(s_t p_t p_e p_\nu) \\
&= -2i\,G_5\,\vec{s}_t \cdot (\vec{p}_\ell \times \vec{p}_\nu)
\end{aligned}
$$

$$
G_8\,(i\,p_t^\mu \epsilon(\nu\, q\, p_t\, s_t) + \mu \leftrightarrow \nu)\,(p_\ell^\mu p_\nu^\nu + p_{\ell,\nu} p_{\nu,\mu} - p_\ell \cdot p_\nu\, g_{\mu\nu}) \sim 2i\,G_8\,\vec{s}_t \cdot (\vec{p}_\ell \times \vec{p}_\nu)
$$

Under $t \to -t$ one has $\vec{p} \to -\vec{p}$ and $\vec{s} \to -\vec{s}$ ($\vec{s} \sim \vec{x} \times \vec{p}$).
The triple product $\vec{s}_t \cdot (\vec{p}_e \times \vec{p}_\nu) \to -\vec{s}_t \cdot (\vec{p}_e \times \vec{p}_\nu)$.

Two sources of imaginary contributions:

- **CP-violating imaginary parts. No contributions from the CKM matrix.**
- **Imaginary parts from loop integrals (also called rescattering corrections)**

Self-Interference Contributions are Zero

Assume

$$\begin{aligned} M^\mu &= \bar{u}_b \Big(\underbrace{\gamma^\mu (1-\gamma_5)}_{Born\ term} + \underbrace{i\, Im\, f_L \gamma^\mu (1-\gamma_5)}_{e.w.correction} \Big) u_t \\ &= M^\mu(LO) + M^\mu(NLO\ EW) \end{aligned}$$

Square the amplitude
Sum over the spin of the b**-quark**

$$\begin{aligned} \sum (b-spin) M^\mu M^{*\nu} &= \mathbf{Tr}\, \not{p}_b \gamma^\mu (1-\gamma_5)(1+i\, Im\, f_L)(\not{p}_t + m_t)\frac{1}{2}(1+\gamma_5 \not{s}_t) \\ &\qquad \gamma^\mu (1-\gamma_5)(1-i\, Im\, f_L) \\ &= (1+i\, Im\, f_L)(1-i\, Im\, f_L) \\ &\quad \mathbf{Tr}\, \not{p}_b \gamma^\mu (1-\gamma_5)(\not{p}_t + m_t)\frac{1}{2}(1+\gamma_5 \not{s}_t)\gamma^\mu (1-\gamma_5) \end{aligned}$$

$$-i\, Im\, f_L + +i\, Im\, f_L = 0$$

Non-Self-Interfering Amplitudes

Self-Interfering amplitudes do not contribute to T-odd observables

If, however, the matrix element is given by

$$M^\mu = \bar{u}_b \Big(\underbrace{\gamma^\mu (1-\gamma_5)}_{Born\ term} \quad - \quad \underbrace{i\, Im\, g_R \frac{i\sigma^{\mu\nu} q_\nu}{m_t}(1-\gamma_5)}_{e.w. correction} \Big) u_t$$

you obtain a nonzero interference contribution to the T-odd invariants

NLO electroweak One-Loop Vertex Graphs

There are 18 NLO electroweak three-point one-loop Feynman diagrams that contribute to $t \to b + W^+$. The corresponding one-loop integrals have five mass scales: m_t, m_b, m_W, m_Z, m_H.

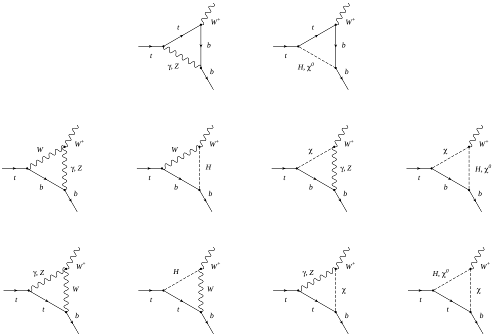


Figure: **NLO one-loop vertex Feynman diagrams contributing to $t \to b + W^+$ in the Feynman 't Hooft gauge. The χ and χ^0 are the charged and neutral Goldstone bosons. H is the Higgs boson.**

NLO electroweak One-Loop Vertex Graphs for $m_b = 0$

There are 13 NLO electroweak three-point one-loop Feynman diagrams that contribute to $t \to b + W^+$ in the limit $m_b = 0$: Omit 5 diagrams because $g_{Hbb} = g_{\chi bb} = 0$

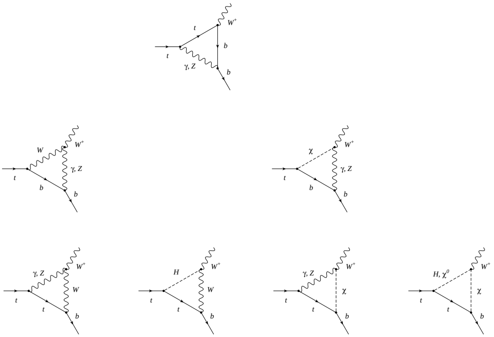


Figure: **13 NLO one-loop vertex Feynman diagrams in the limit $m_b = 0$ contributing to $t \to b + W^+$ in the Feynman 't Hooft gauge.**

NLO Electroweak Two-Point One-Loop Graphs

There are numerous two-point one-loop graphs which are also needed for the renormalization of the NLO calculation. We do not list them here.

Imaginary Parts of the One-Loop Graphs

There is a vast output of formulas for the one-loop results. How to identify the imaginary contributions? The solution is straightforward. Identify those graphs that allow the intermediate particles to be on their mass-shell.

For example: 2 self-enegy graphs

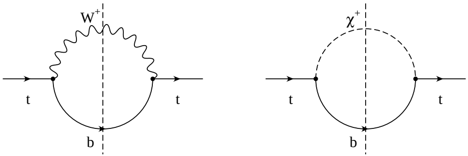


Figure: **Absorptive parts of the two two-point one-loop Feynman diagrams that contribute to the T-odd correlations in polarized top decays.**

Since the self-energy graphs are attached to the Born-term amplitudes they are self-interfering. No contribution to the T-odd observables.

Electroweak One-Loop Vertex Graphs that admit Absorptive Cuts

There are 4 NLO electroweak one-loop Feynman diagrams for $t \to b + W^+$ that admit of absorptive cuts (often also referred to as final state interactions or rescattering corrections):

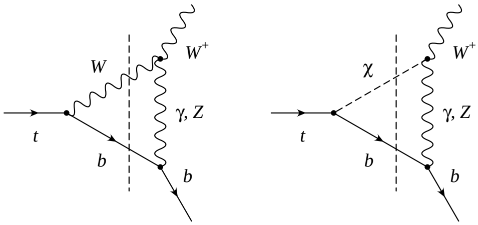


Figure: **Absorptive parts of the four Feynman diagrams that contribute to the T-odd correlations in polarized top decays.**

No Absorptive Cut in NLO QCD

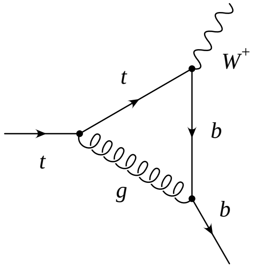


Figure: **One-loop graph in QCD**

- **The NLO one-loop graph in QCD does not admit of an absorptive cut**

Effective Current

Before giving our results for the imaginary parts we want to discuss three related items

- Definition of the amplitudes to which the imaginary parts contribute
- Positivity bounds on the imaginary contributions
- Experimental bounds on T-odd contributions

The effective current:

$$\mathcal{J}^{\mu}_{\rm eff} = -\frac{g_w}{\sqrt{2}}\bar{b}\Big\{\gamma_\mu(f_L P_L + f_R P_R) - \frac{i\sigma^{\mu\nu}q_\nu}{m_t}(g_L P_L + g_R P_R)\Big\}t$$

For the Standard Model Born term one has $f_R,\ g_L, g_R = 0$ and $f_L = V_{tb}$. Complex values of coupling factors $f_L,\ f_R,\ g_L, g_R$ may be put in by hand. Contributions of $V_L,\ V_R,\ g_L, g_R$ to the spin density matrix elements of the W^+ compete with higher order perturbative corrections (we will also refer to the spin density matrix elements as helicity structure functions).

- Imaginary parts of coupling factors can be generated by final state interactions (SM; CP-conserving) or by introducing non-SM CP-violating imaginary couplings
- Next I will present our reults on the imaginary parts resulting from the absorptive parts of the NLO electroweak one loop contributions

Imaginary Parts cont'd

The observable imaginary part ($x = m_W/m_t,\ x_Z = m_Z/m_T$):

$$\mathrm{Im}\,\delta g_R = -2e^2 Q_b\,\pi\,x^2 + \frac{e^2(1+2Q_b s_w^2)\,\pi}{s_w^2}\,\frac{1}{(1-x^2)^3}\Big\{(1-x^2)^2$$

$$\Big[x^2(1-x^2)+2x_Z^2\Big] - \Big[(1-x^2)x_Z^2 + 2x_Z^4\Big]\,\ell_Z\Big\}$$

Imaginary Parts contd

where

$$\ell_Z = \ln\left(\frac{(x_Z^2 + (1-x^2)^2)^2}{(x_Z^2 - x^2(1-x^2))(x_Z^2 + (1-x^2)(1-2x^2))}\right)$$

- **Identify γ-exchange and Z-exchange by the coupling factors Q_b and $(1 + 2Q_b s_w^2)/s_w^2$**
- **Agrees with Arhib, Jueid (2016) arXiv:1606.05270**
  Disagrees with Gonzales-Sprinberg, Martinez, Vidal by factor of two (2011,2013) arXiv:1606.05270
- **Our results are analytical, whereas their results are partly numerical**

- $\mathrm{Im} f_L = 2.26$ **not observable**
- $\mathrm{Im} g_R = -2.175 \times 10^{-3}$
- **Result on g^R is well inside the positivity and experimental bounds to be discussed in the following.**

Positivity bound and experimental bound

- **Positivity bound**
  $\mathrm{Im} g^R \in [-0.0420, +0.0420]$
- **Experimental bound** $\mathrm{Im} g^R \in [-0.18, +0.06]$

Polarized Top Decay $t(\uparrow) \to b + \ell^+ + \nu_\mu$ $(\ell = e, \mu, \tau)$

The decay $t \to b + \ell^+ + \nu_\mu$ is described by the amplitude

$$M = \bar{u}(b)\gamma^\mu(1-\gamma_5)u(t)\,\bar{u}(\nu)\gamma_\mu(1-\gamma_5)v(\ell)$$

Upon squaring and summing over spins one has

$$\begin{aligned}\sum(\text{spins})|M|^2 &= \textbf{Tr}\,\not{p}_b\gamma^\mu(1-\gamma_5)(\not{p}_t+m_t)\frac{1}{2}(1+\gamma_5\not{s}_t)\gamma^{\mu'}(1-\gamma_5)\\ &\quad \textbf{Tr}\,\not{p}_\ell\gamma_\mu(1-\gamma_5)\not{p}_\nu\gamma_{\mu'}(1-\gamma_5)\\ &= 128\,(p_b\,p_\nu)(p_t p_\ell - m_t s_t\cdot p_\mu)\end{aligned}$$

The result is very compact. Is there a reason? See next page.

$$p_t = m_t(1;0,0,0) \qquad p_\ell = E_\ell(1;0,0,1) \qquad s_t = (0,\vec{s}_t)$$

Correlation between the spin of the top quark and the momentum of the lepton given by (set $|\vec{P}_t| = 1; \vec{s}_t = \vec{P}_t$

$$\frac{d\Gamma}{d\cos\theta} \sim m_t E_\mu(1+\cos\theta_P)$$

Positivity of the rate is (barely!) guaranteed at the Born term level!

Polarized Top Decay $t(\uparrow) \to b + \ell^+ + \nu_\mu$ $(\ell = e, \mu, \tau)$

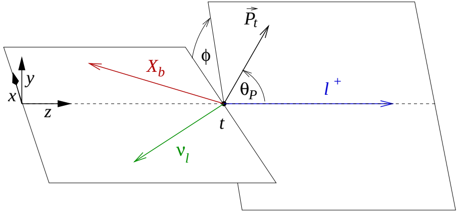


Figure: **Definition of the polar angles θ_P and the azimuthal angle ϕ in the helicity system Ib for the quasi three-body decay $t(\uparrow) \to X_b + \ell^+ + \nu_\ell$**

At NLO in QCD ($O(\alpha_s)$) this becomes

$$\frac{d\Gamma}{d\cos\theta} \sim \Big((1 - 8.54\%) + (1 - 8.71\%)\cos\theta_P\Big)$$

Again, positivity is (barely!) guaranteed at $O(\alpha_s)$!

Fierz Transformation of the Second Kind

After a Fierz transformation of the second kind one writes

$$M = 2\bar{u}(b)(1+\gamma_5)C\bar{u}^T(\nu)v^T(\ell)C^{-1}(1-\gamma_5)u(t) \qquad v = C\bar{u}^T \qquad v^T = \bar{u}C$$
$$= 2\bar{u}(b)(1+\gamma_5)v(\nu)\bar{u}(\ell)(1-\gamma_5)u(t)$$

Upon squaring and taking the spin sum

$$\begin{aligned}\sum(\text{spins})|M|^2 &= \mathbf{Tr}\, \not{p}_b(1+\gamma_5)\not{p}_\nu(1-\gamma_5) \\ &\quad \cdot \mathbf{Tr}\, \not{p}_\mu(1-\gamma_5)(\not{p}_t + m_t)\frac{1}{2}(1+\gamma_5\not{s}_t)(1+\gamma_5) \\ &= 8\mathbf{Tr}\, \not{p}_b\not{p}_\nu(1-\gamma_5)\cdot \mathbf{Tr}\, \not{p}_\ell(\not{p}_t+m_t)(1+\gamma_5\not{p}_t)(1+\gamma_5) \\ &= 128\,(p_b\, p_\ell)(p_t - m_t s_t)\cdot p_\ell)\end{aligned}$$

The same trick in the same process is used by Godbole, Peskin [hep-ph] 1809.06285

Four Structure Functions

We have learned earlier how to count the number of structure functions from Lorentz invariance.

Result:
1 unpolarized structure function
3 polarized structure functions constructed from $p_\ell s_t$, $p_\nu s_t$ and $\varepsilon(p_t\, p_\ell\, p_\nu\, s_t)$

For the spin density matrix of the top quark one has

$$\rho_{\lambda_t\,\lambda'_t} = \frac{1}{2}\mathbb{1} + P_t \cos\theta_P\, \sigma_z + P_t \sin\theta_P \cos\phi\, \sigma_x + P_t \sin\theta_P \sin\phi\, \sigma_y$$

where θ_P and ϕ describe the orientation of the polarization vector of the top quark. We expand the (2×2) decay matrix $M_{\lambda_t} M^*_{\lambda'_t}$ along the unit matrix $\mathbb{1}$ and the three σ_i matrices. One has

$$M_{\lambda_t} M^*_{\lambda'_t} = \tfrac{1}{2}\left(A\,\mathbb{1} + B\,\sigma_z + C\,\sigma_x + D\,\sigma_y\right)$$

Four Structure Functions

The angular decay distribution of the decay is obtained by folding the decay matrix $M_{\lambda_t} M^*_{\lambda'_t}$ with the spin density matrix of the top quark, i.e. by calculating the trace Tr$(\rho_{\lambda_t\,\lambda'_t} M_{\lambda_t} M^*_{\lambda'_t})$. One obtains

$$\begin{aligned}
\frac{d\Gamma}{d\cos\theta_P d\phi} &= \mathbf{Tr}\Big\{\rho_{\lambda_t\,\lambda'_t}\Big(M_{\lambda_t} M^*_{\lambda'_t}\Big)\Big\} \\
&= A + BP_t\cos\theta_P + CP_t\sin\theta_P\cos\phi + D\,P_t\sin\theta_P\sin\phi \\
&= A\left(1 + P_t\frac{B}{A}\cos\theta_P + P_t\frac{C}{A}sin\theta_P\cos\phi + P_t\frac{D}{A}\,sin\theta_P sin\phi\right)
\end{aligned}$$

Set $P_t = 1$**,** $\cos\phi = 0$ **and use** B/A

$$\frac{d\Gamma}{d\cos\theta_P d\phi} = A\left(1 + 1.0199\cos\theta + \frac{D}{A}\,sin\theta_P\right)$$

Dangerous domain: $\cos\theta_P = \pi - \delta$
Expand: $\cos\theta(\pi-\delta) = -1+\delta^2$ $\quad \sin(\pi-\delta) = \delta$ **The sin–function grows much faster than the cos–function away from** π**. Exploit this to get a positivity bound on** $\frac{D}{A}$**. The bound is defined by the zero of the rate, This gives**

$$\mathrm{Im}\,g_R \in [-0,0420,\ 0,0420]$$

Definition of Angles in Sequential Polarized Top Quark Decay

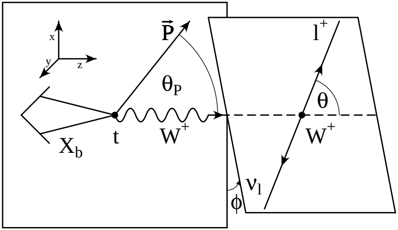


Figure: **Definition of the polar angles θ and θ_P and the azimuthal angle ϕ in the sequential decay $t(\uparrow) \rightarrow X_b + W^+(\rightarrow \ell^+ + \nu_\ell)$.**

Angular decay distribution:

$$
\begin{aligned}
W(\cos\theta,\, \cos\theta_P,\, \phi) = \\
&\Big(H_U + H_U^P\, P\cos\theta_P\Big)(1+\cos^2\theta) + \Big(H_L + H_L^P\, P\cos\theta_P\Big) 2\sin^2\theta \\
&+\Big(H_F + H_F^P P\cos\theta_P\Big)\cdot 2\cos\theta + H_I^P\, P\sin\theta_P\, 2\sqrt{2}\sin 2\theta\cos\phi \\
&+H_A^P P\sin\theta_P\, 4\sqrt{2}\sin\theta\cos\phi \\
&+H_{\mathcal{I}I}^P\, P\sin\theta_P\, 2\sqrt{2}\sin 2\theta\sin\phi + H_{\mathcal{I}A}^P P\sin\theta_P\, 4\sqrt{2}\sin\theta\sin\phi
\end{aligned}
$$

T-odd Correlations

Normalized three-vectors:

$$\begin{aligned}\hat{P}_t &= (\sin\theta_P, 0, \cos\theta_P)\\ \hat{p}_\ell &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\\ \hat{q} &= (0,0,1)\end{aligned}$$

$$\begin{aligned}\sin\theta_P\ \sin\theta\ \sin\phi &= \hat{q}\cdot(\hat{P}_t\times\hat{p}_\ell)\\ \sin\theta_P\ \sin 2\theta\ \sin\phi &= 2\,(\hat{p}_\ell\cdot\ \hat{q})\ \hat{q}\cdot(\hat{P}_t\times\hat{p}_\ell)\end{aligned}$$

Under time reversal $t\to -t$ one has $(\hat{p},\hat{P})\to(-\hat{p},-\hat{P})$.
One therefore calls the two above two correlations T-odd correlations.

Two possible sources of T-odd correlations:

1. **SM source: Imaginary parts from absorptive contributions**
2. **Non-SM source: *CP*-violating imaginary couplings**

W^+ Spin Density Matrices: Production and Decay

- **How many structure functions in $t(\uparrow) \to X_b + W^+$?**

$$\begin{array}{lll} \textit{covariant counting}: & \mathcal{H}^{\mu\nu} & 10 \ \textit{invariant structure functions} \\ \textit{helicity counting}: & \mathcal{H}_{\lambda_W \lambda'_W} & 10 \ \textit{helicity structure functions} \end{array}$$

Production spin density matrix of W^+ ($t(\uparrow) \to X_b + W^+$):

$$\mathcal{H}_{\lambda_W \lambda'_W}(\theta_P) = \begin{pmatrix} H_{++} + H^P_{++} P\cos\theta_P & H^P_{+0}\, P\sin\theta_P & 0 \\ H^P_{0+} P\sin\theta_P & H_{00} + H^P_{00} P\cos\theta_P & H^P_{0-}\, P\sin\theta_P \\ 0 & H^P_{-0}\, P\sin\theta_P & H_{--} + H^P_{--} P\cos\theta_P \end{pmatrix}$$

Decay spin density matrix of W^+ ($W^+ \to \ell^+ + \nu_\ell$; 100 % analyzing power):

$$\mathcal{L}_{\lambda_W \lambda'_W}(\theta,\phi) =$$

$$\begin{pmatrix} (1+\cos\theta)^2 & \frac{2}{\sqrt{2}}(1+\cos\theta)\sin\theta\, e^{i\phi} & \sin^2\theta\, e^{2i\phi} \\ \frac{2}{\sqrt{2}}(1+\cos\theta)\sin\theta\, e^{-i\phi} & 2\sin^2\theta & \frac{2}{\sqrt{2}}(1-\cos\theta)\sin\theta\, e^{i\phi} \\ \sin^2\theta e^{-2i\phi} & \frac{2}{\sqrt{2}}(1-\cos\theta)\sin\theta\, e^{-i\phi} & (1-\cos\theta)^2 \end{pmatrix}$$

Derivation of Angular Decay Distribution

Angular decay distribution:

$$W(\theta_P,\, \theta,\, \phi) = \sum_{\lambda_W \lambda'_W} \mathcal{H}_{\lambda_W \lambda'_W}(\theta_P)\mathcal{L}_{\lambda_W \lambda'_W}(\theta, \phi)$$

$$\boxed{W(\theta_P,\, \theta,\, \phi) = Tr\ \left\{\mathcal{H}(\theta_P) \cdot \mathcal{L}^T(\theta,\, \phi)\right\}}$$

Patterned after derivation of angular decay distribution for the sequential decay $\Xi^- \to \Lambda + \pi^-$ followed by $\Lambda \to p + \pi^-$ (any elementary particle physics text book)

Experimental Bounds

The Atlas Collaboration has performed a fit of their data on polarized top quark decays to the angular decay distribution ATLAS Collaboration, JHEP 1704 (2017) 124 arXiv:1702.08309 [hep-ex]. Their result is

$$\mathrm{Im} g^{R} \in [-0.18, +0.06]$$

Loop Calculation

There are 13 one-loop $m_b = 0$ loop diagrams. After foldig with the Born term their contributions are projected onto the 3 independent $m_b = 0$ structure functions. Their IR and mass (M) $\ln m_b$singularities are cancelled against the corresponding singularities from the tree diagrams. The UV singularities are cancelled against the UV singularities of the self-energy diagrams in the renormalization program.

NLO tree-level Feynman diagrams

There are four electroweak tree level Feynman diagrams that contribute to $t \to b + W^+ + \gamma$:

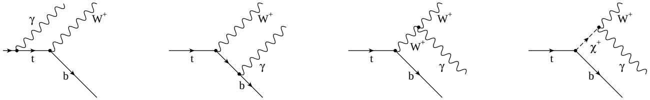


Figure: **NLO tree-level Feynman diagrams contributing to $t \to b + W^+ + \gamma$ in Feynman 't Hooft gauge. The χ^+ is the charged Goldstone boson.**

Standard procedure of calculation:

$$|M|^2(\text{hard} + \text{soft}) = \underbrace{\left\{|M|^2(\text{hard} + \text{soft}) - |M|^2(\text{soft})\right\}}_{\textit{IR and M safe}} + \underbrace{|M|^2(\text{soft})}_{\text{universal DONE before}}$$

Projection of $\left\{|M|^2(\text{hard} + \text{soft}) - |M|^2(\text{soft})\right\}$ onto 8 helicity structure functions. Phase space integrations have been done.

Summary and conclusion

- **"dislike" : Electroweak final state interaction effects in polarized top quark decays are tiny. Why bother?**
- **"like" : If T-odd effects are discovered in polarized top quark decays they must be due to non-SM CP-violating effects. No contamination from electroweak SM final state interactions.**
- **Reminder**

  When going from top quark decays $t \to b + W^+$ to anti top quark decays $\bar{t} \to \bar{b} + W^-$ one has
- **phase change $e^{i\phi} \to e^{-i\phi}$ for CP-violating phase**
- **no phase change $e^{i\phi} \to e^{i\phi}$ for CP-conserving final state interactions**