Finite Density Lattice QCD for NICA energy regions

V. Bornyakov, D. Boyda,
V. Goy, A. Molochkov,
A.Nakamura, M.Wakayama
and V.I.Zakharov

Far Eastern Federal Univ.,
VladivoRCNP, Osaka Univ.,
Osaka, Japan
RIKEN, Saitama, Japan

nakamura@an-pan.org

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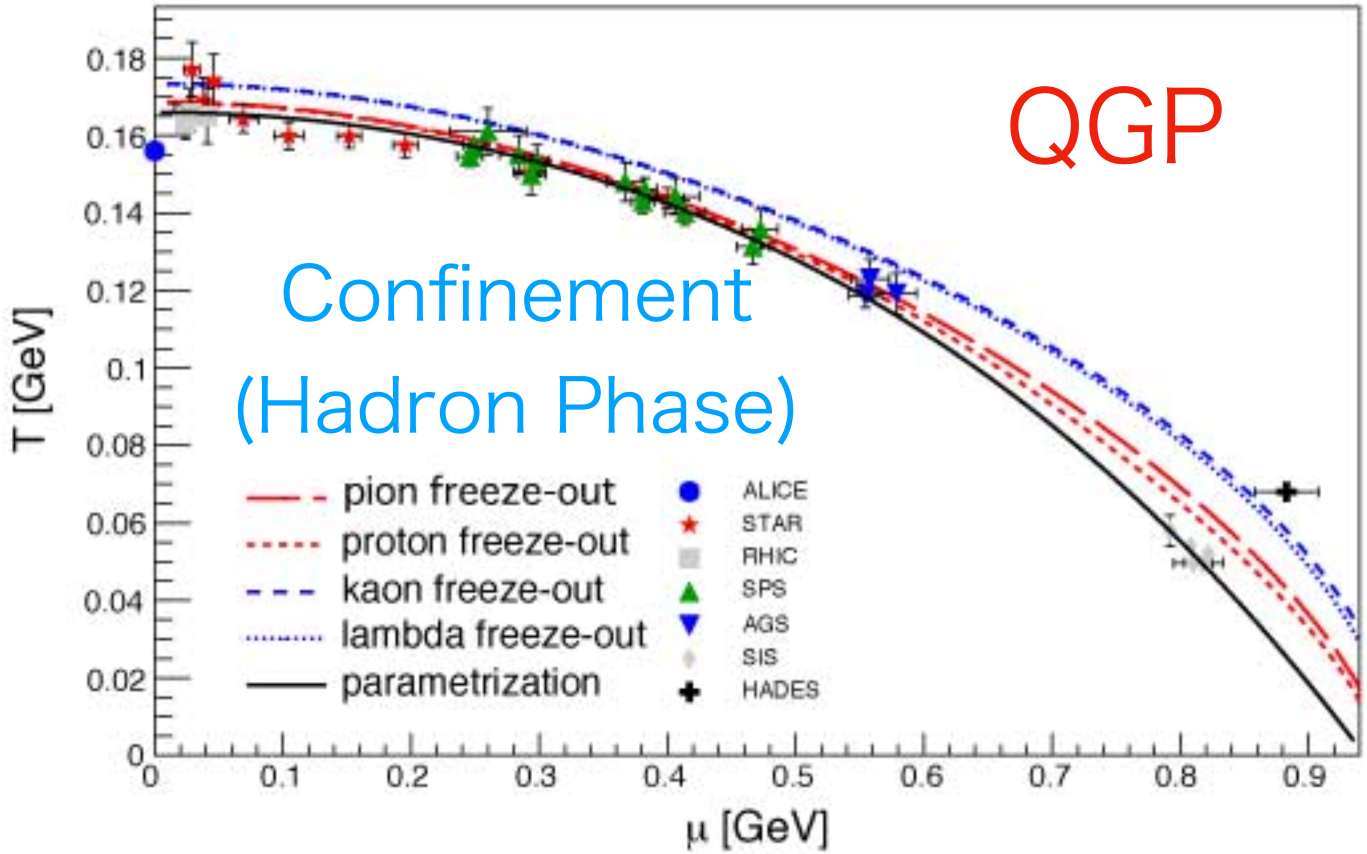
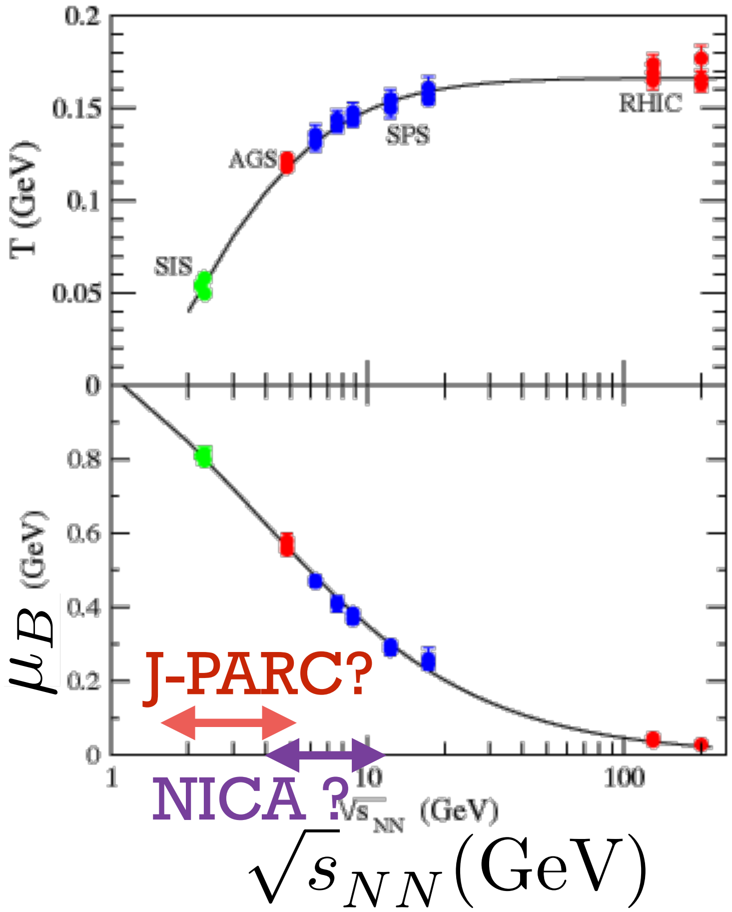
Content

1. Introduction — Finite Density Regions
2. Sign Problem
3. Canonical Approach
4. Experimental data at RHIC
5. How to find QCD phase transition line ?
6. What should we do next ?
7. Summary

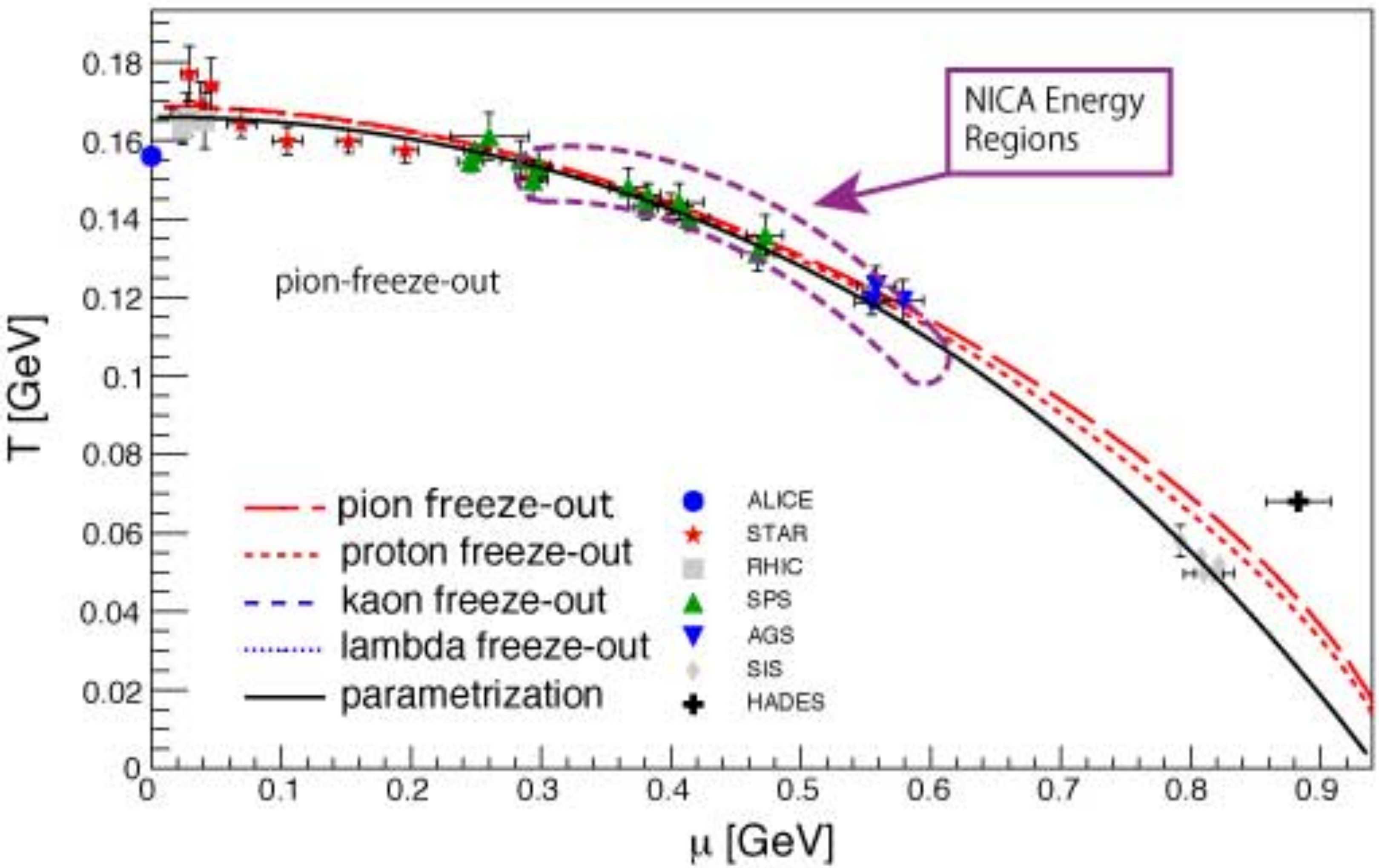
D. Blaschke, J. Jankowski, and M. Naskręt
arXiv:1705.00169



μ:Chemical Potential



J.Cleymans et al.,
Phys. Rev. C73, (2006) 034905.



D. Blaschke, J. Jankowski, and M. Naskręt
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Content

1. Introduction — Finite Density Regions

2. Sign Problem

3. Canonical Approach

4. Experimental data at RHIC — Higher Moments

5. How to find QCD phase transition line ?

6. What should we do next ?

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But

Sign Problem

Lattice QCD does not work
at finite density !

$$(\det\Delta(\mu))^* = \det\Delta(\mu)^\dagger = \det\Delta(-\mu^*)$$

For $\mu = 0$

$$(\det\Delta(0))^* = \det\Delta(0)$$

$\det\Delta$ ➡ ***Real***

For $\mu \neq 0$ (in general)

$\det\Delta$ ➡ ***Complex***

$$Z = \int \mathcal{D}U \prod_f \det\Delta(m_f, \mu_f) e^{-\beta S_G}$$

⬆

Complex ➡ Sign Problem

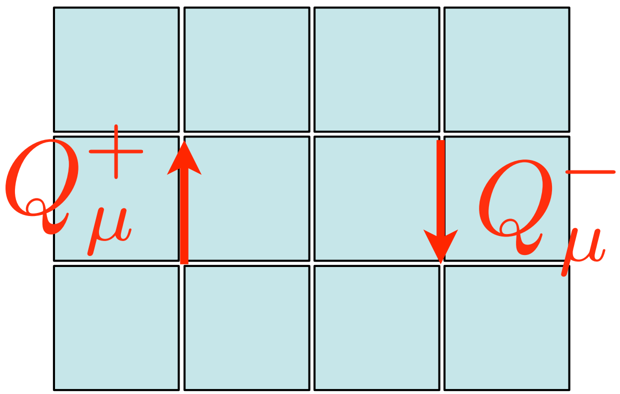
Origin of Sign Problem

Wilson Fermions $\Delta = I - \kappa Q$

KS(Staggered) Fermions $\Delta = m - Q'$

$$= m(I - \frac{1}{m}Q)$$

$$Q = \sum_{i=1}^{3} (Q_i^+ + Q_i^-) + (e^{+\mu} Q_4^+ + e^{-\mu} Q_4^-)$$



$$Q_\mu^+ = **\, U_\mu(x) \delta_{x', x+\hat{\mu}}$$

$$Q_\mu^- = **\, U_\mu^\dagger(x') \delta_{x', x-\hat{\mu}}$$

$$\det \Delta = e^{\mathrm{Tr}\log\Delta} = e^{\mathrm{Tr}\log(I-\kappa Q)}$$

$$= e^{-\sum_n \frac{1}{n}\kappa^n \mathrm{Tr} Q^n}$$

Hopping Parameter Exp.
or
Large Mass Expansion.

Closed loops do not vanish

Lowest μ depsnent terms

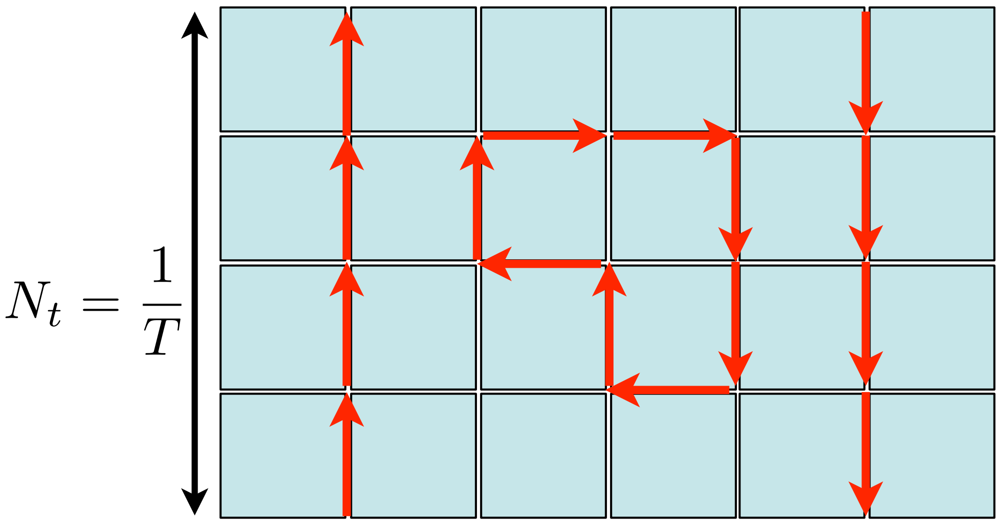
$$\kappa^{N_t} e^{\mu N_t} \mathrm{Tr}(Q^+ \cdots Q^+)$$

$$= **\kappa^{N_t} e^{\mu/T} \mathrm{Tr} L$$

$$\kappa^{N_t} e^{-\mu N_t} \mathrm{Tr}(Q^- \cdots Q^-)$$

$$= **\kappa^{N_t} e^{-\mu/T} \mathrm{Tr} L^\dagger$$

$\mathrm{Tr}L$: Polyakov Loop



Combine both

$$**\kappa^{N_t}\left(\cosh\frac{\mu}{T}\Re\mathrm{Tr}L + i\sinh\frac{\mu}{T}\Im\mathrm{Tr}L\right)$$

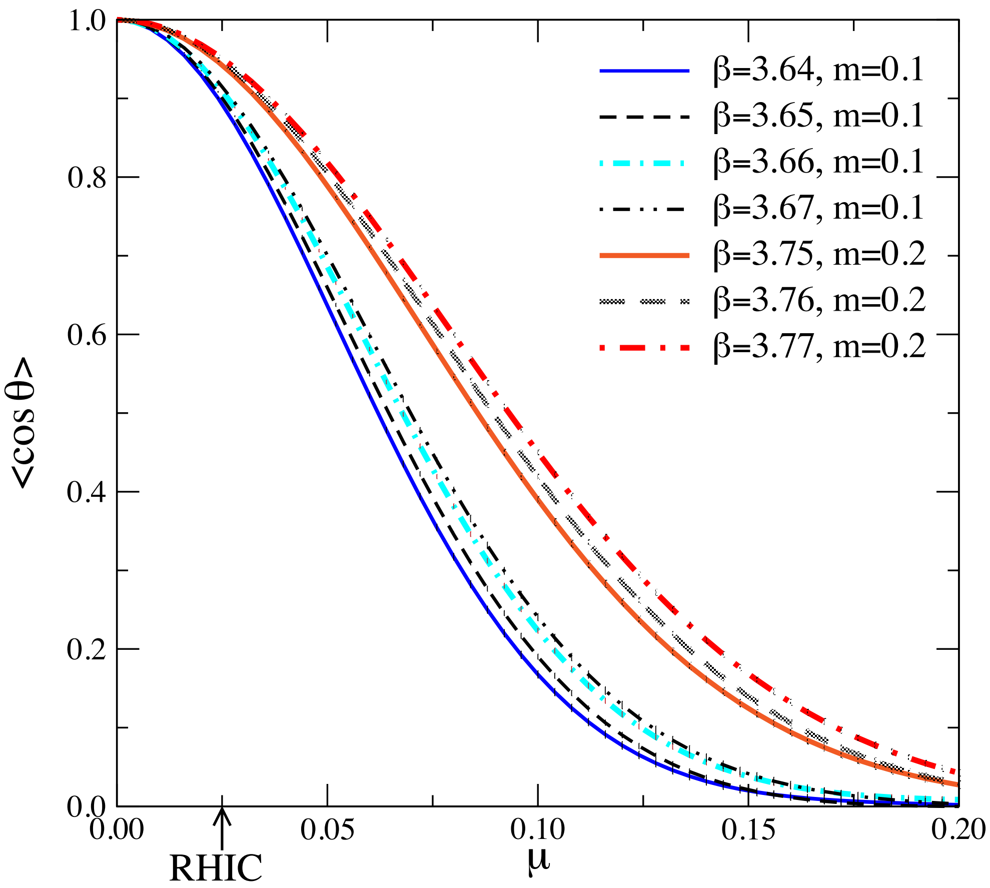
Sign Problem is sever

when μ is large

when T is low

Allton et al., Phys.Rev.D.66. 074507
(arXiv:hep-lat/0204010)

$$\det D = |\det D| e^{i\theta}$$



No Sign problem cases

1.Pure imaginary chemical potential

$$(\det \Delta(\mu))^* = \det \Delta(-\mu^*)$$

$$\mu = i\mu_I \quad \Rightarrow \quad (\det \Delta(\mu_I))^* = \det \Delta(\mu_I)$$

2.Color SU(2) $\quad U_\mu^* = \sigma_2 U_\mu \sigma_2$

$$\det \Delta(U, \gamma_\mu)^* = \det \Delta(U^*, \gamma_\mu^*) = \det \sigma_2 \Delta(U, \gamma_\mu^*) \sigma_2$$
$$= \det \Delta(U, \gamma_\mu)$$

3.Iso vector (finite iso-spin)

$$\mu_d = -\mu_u$$

$$\det \Delta(\mu_u) \det \Delta(\mu_d) = \det \Delta(\mu_u) \det \Delta(-\mu_u)$$
$$= \det \Delta(\mu_u) \det \Delta(\mu_u)^* = |\det \Delta(\mu_u)|^2$$

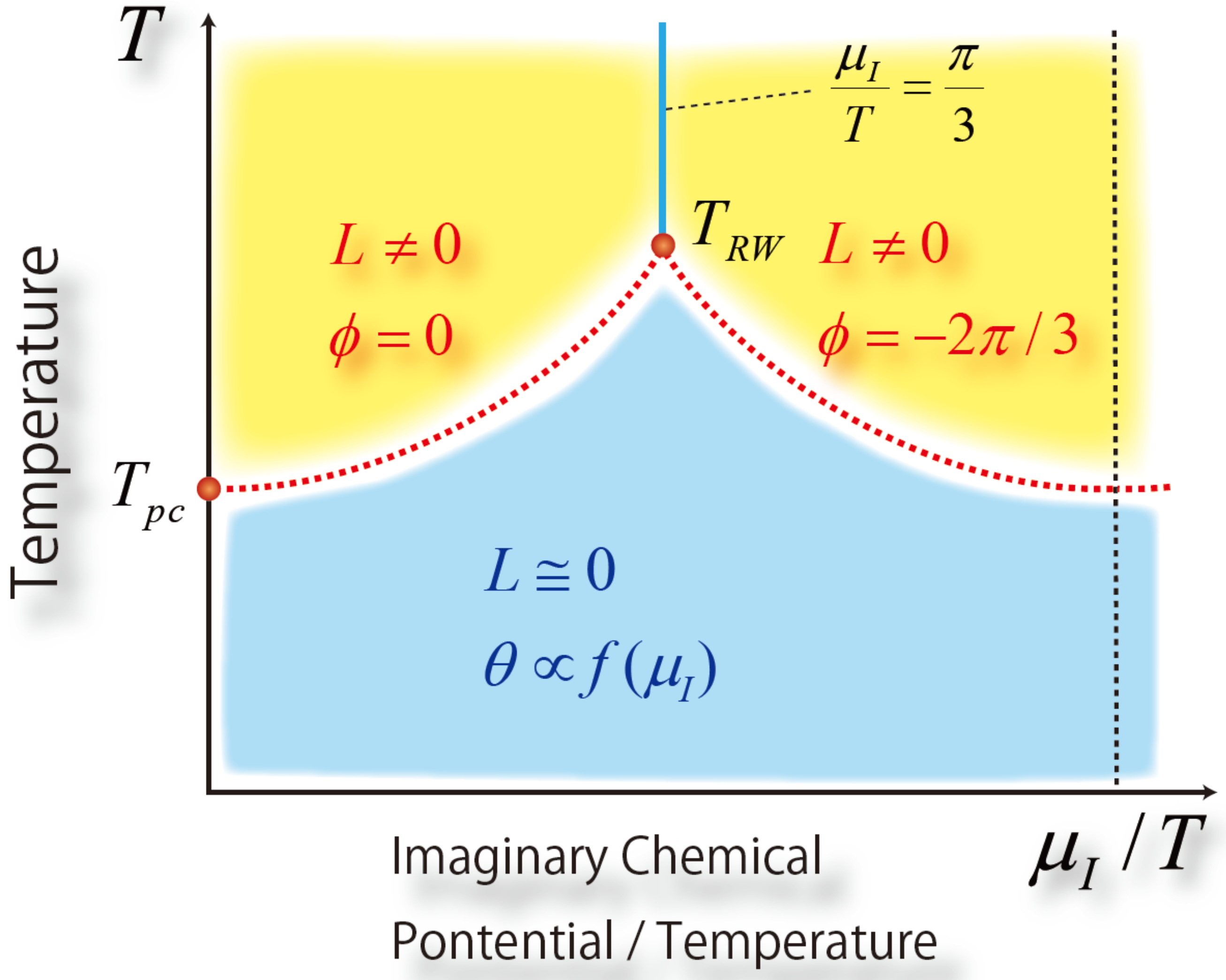
(Phase Quench)

Phase Structure in pure imaginary

$$(\det \Delta(\mu))^* = \det \Delta(\mu)^\dagger = \det \Delta(-\mu^*)$$

$$\mu = i\mu_I \;\Rightarrow\; \det \Delta\text{: Real !}$$

Phase diagram in μI region



Polyakov loop

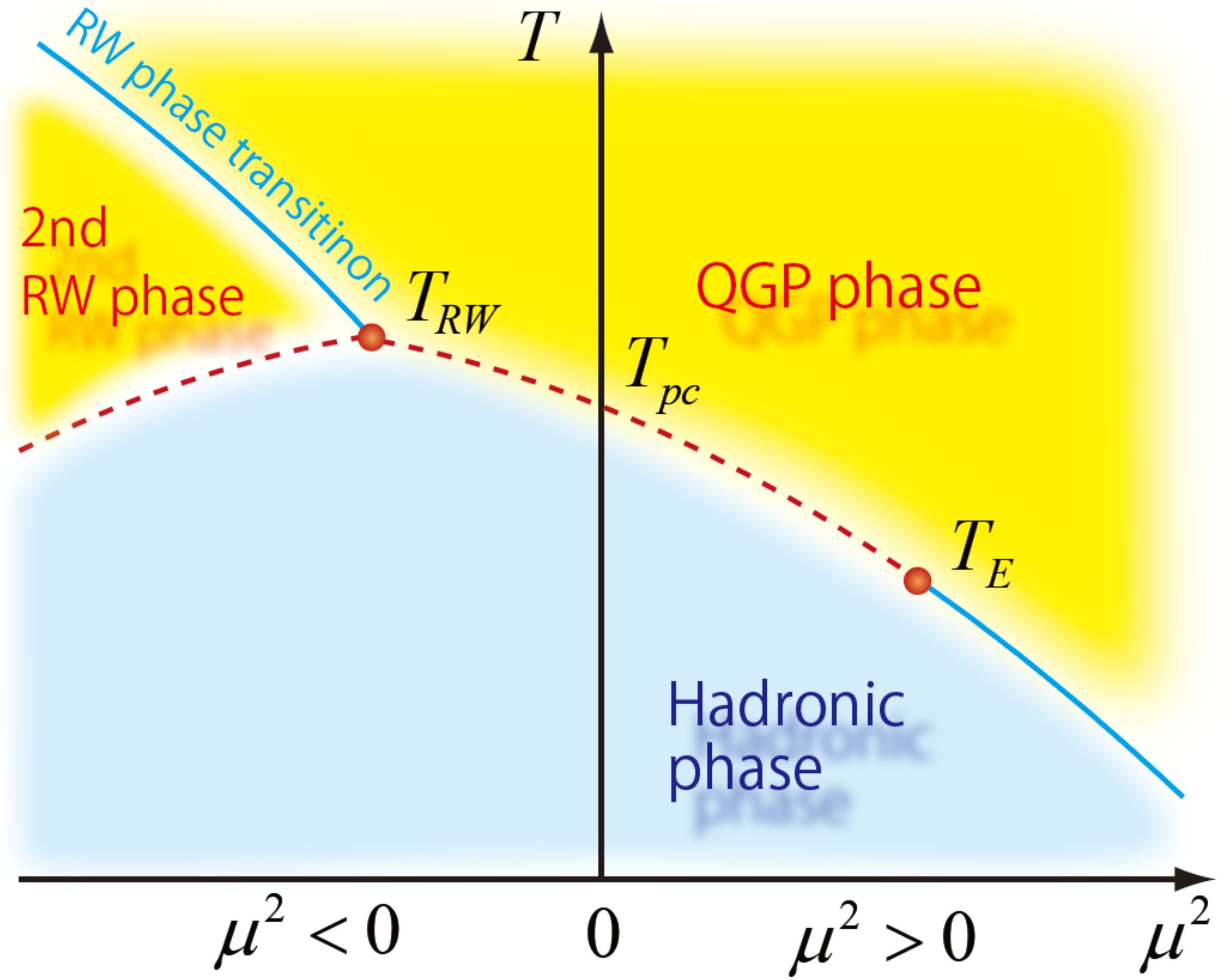
$$P = L_P \exp(i\phi_P)$$

If μ is pure imaginary

there is no sign problem.

$$(\det \Delta(\mu))^* = \det \Delta(-\mu^*)$$

Imaginary to real chemical potential



Many Approaches to Sign Problem

- Taylor Expansion
- Canonical Approach
- Density of State
- Complex Langevin

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Canonical Approach

proposed by
A.Hasenfratz and Toussaint in 1992
to solve the sign problem.
But it did not work.
We traced the cause and solve it with
multiple precision numerical calculations

Canonical Approach

$$Z(\mu, T) \Leftrightarrow Z_n(T)$$

Grand Canonical — **Canonical**

$$Z(\mu, T) = \mathrm{Tr}\, e^{-(H-\mu\hat{N})/T}$$

$$= \sum_n \langle n| e^{-(H-\mu\hat{N})/T} |n\rangle$$

If $[H, \hat{N}] = 0$

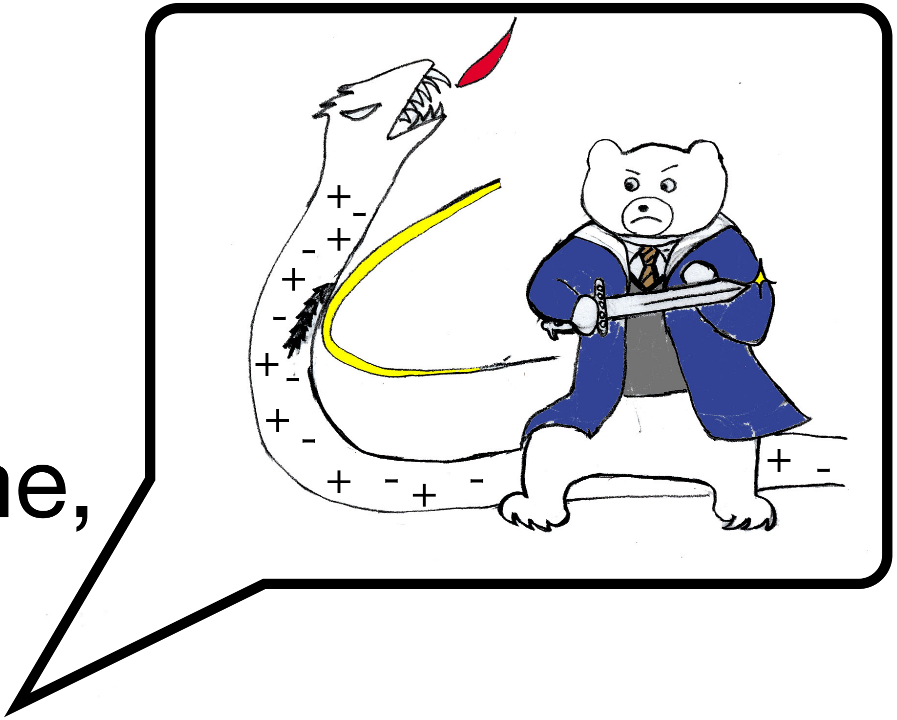
$$= \sum_n \langle n| e^{-H/T} |n\rangle\, e^{\mu n/T}$$

$$= \sum_n Z_n(T) \xi^n \qquad \left(\xi \equiv e^{\mu/T}\right)$$

Fugacity

Personal History about fighting against Sign Problem

Once upon a time,



We were looking for

A Reduction Formula for Wilson Fermions

$$\det \Delta = \det Q$$

Matrix Δ is smaller than Q

- ⭐Keitaro Nagata and Atsushi Nakamura
  Phys. Rev. D82,094027 (arXiv:1009.2149)
- ⭐A. Alexandru and U. Wenger
  Phys.Rev.D83:034502,2011 (arXiv:1009.2197)
- ⭐ One more group

For KS Fermions, the reduction formula was known.

- Gibbs Formula(*)
  - P.E.Gibbs, Phys.Lett. B172 (1986) 53-61

$$
\begin{aligned}
\det \Delta &= z^{-N} \begin{vmatrix} -B(-V) - z & 1 \\ -V^2 & -z \end{vmatrix} \\
&= \left| \begin{pmatrix} BV & 1 \\ -V^2 & 0 \end{pmatrix} - zI \right| \\
&= \det (P - zI) \\
&= \prod (\lambda_i - z)
\end{aligned}
$$

P

$$z \equiv e^{-\mu}$$

- P is $(2 \times N_c \times N_x \times N_y \times N_z)^2$ (Matrix Reduction)
- Determinant for any value of μ

*) A similar formula was developpped by Neuberger (1997) for a chiral fermion and applied by Kikukawa(1998).

The same matrix transformation like KS case cannot be employed, due to the fact that

$r \pm \gamma_4$ have no inverse, if the Wilson term $r = 1$.

Gibbs started to multiply V to the fermion matrix Δ.

Instead, we multiply $P = (c_a r_- + c_b r_+ V z^{-1})$

Here,

$$
V =
\left(\begin{array}{c|c|cccc|c}
0 & U_4(t=1) & 0 & \cdots & & & 0 \\ \hline
0 & 0 & U_4(t=2) & \cdots & & & 0 \\ \hline
0 & 0 & 0 & \cdots & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\
 & & & \cdots & U_4(t=N_t-2) & & 0 \\ \hline
0 & 0 & & \cdots & 0 & & U_4(t=N_t-1) \\ \hline
-U_4(t=N_t) & 0 & & \cdots & 0 & & 0
\end{array}\right)
$$

c_a and c_b are arbitary non-zero numbers.

$$\det P = (c_a c_b z^{-1})^{N/2}$$

if we take the following trick, Borici (2004)

$$r_+ r_- = \frac{r^2 - 1}{4} = \epsilon \to 0$$

where $r_\pm \equiv \dfrac{r \pm \gamma_4}{2}$

After very long calculation (See Nagata-Nakamura arXiv:1009.2149), we get

$$\det \Delta(\mu) = (c_a c_b)^{-N/2} z^{-N/2}$$

$$\times \left(\prod_{i=1}^{N_t} \det(\alpha_i) \right) \det \left(z^{N_t} + Q \right)$$

$$\frac{\det \Delta(\mu)}{\det \Delta(0)} = \frac{\det(\xi + Q)}{\det(1+Q)}$$

$$\xi \equiv e^{-\mu/T}$$
(fugacity)

Q is $(4N_cN_xN_yN_z) \times (4N_cN_xN_yN_z)$ matrix.

No N_t !

In case of KS matrix, the corresponding matrix is $(2N_cN_xN_yN_z) \times (2N_cN_xN_yN_z)$

Diagonalize Q,

$$Q \to \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_{N_{red}} \end{pmatrix}$$

$$\det(\xi + Q) = \prod(\xi + \lambda_n)$$

λ_n does not depend on μ.

Once we calculate λ_n,
we can evaluate $\det \Delta(\mu)$ for any μ.

$$\det(\xi + Q) = \prod(\xi + \lambda_k) = \sum C_n \xi^n$$

$$Z = \int \mathcal{D}U \det \Delta e^{-\beta S_G}$$

$$Z = \sum_n \left(\int \mathcal{D}U C_n e^{-\beta S_G} \right) \xi^n$$
$$= \sum_n z_n \, \xi^n$$

$$\xi \equiv e^{\mu/T}$$

Fugacity Expansion

$$Z(\mu, T) = \sum_n z_n(T)(e^{\mu/T})^n$$

$Z(\mu, T)$: Grand Canonical Partition Function

$z_n(T)$: Canonical Partition Function

Inverse transformation:

$$z_n = \int \frac{d\theta}{2\pi} e^{i\theta n} Z_{GC}(\theta \equiv \frac{\mathrm{Im}\mu}{T}, T)$$

A.Hasenfratz and Toussaint (1992)

z_n can be determined in imaginary μ regions.

This is Canonical approach by

A.Hasenfratz and Toussaint (1992)

$$Z_n = \int \frac{d\theta}{2\pi} e^{in\theta} Z(\theta = \frac{\mu_I}{T})$$

In pure Imaginary μ , there is no sign problem.

It was known that this method does not work.

Why ???

Check by an analytic method (Winding Number Expansion)

$$Z_n = \int \frac{d\theta}{2\pi} e^{in\theta} Z(\theta = \frac{\mu_I}{T})$$

A. Hasenfratz and D. Toussaint

$$Z(\mu) = \int DU \det \Delta(\mu) e^{-S_G}$$

Kentucky: Winding Number Expansion

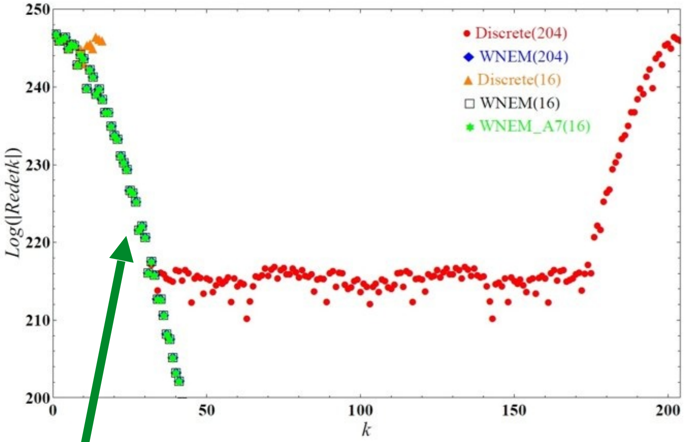
Meng et al., 2008

The original method does not work due to numerical errors.

$$\det \Delta = \exp(Tr \log \Delta)$$

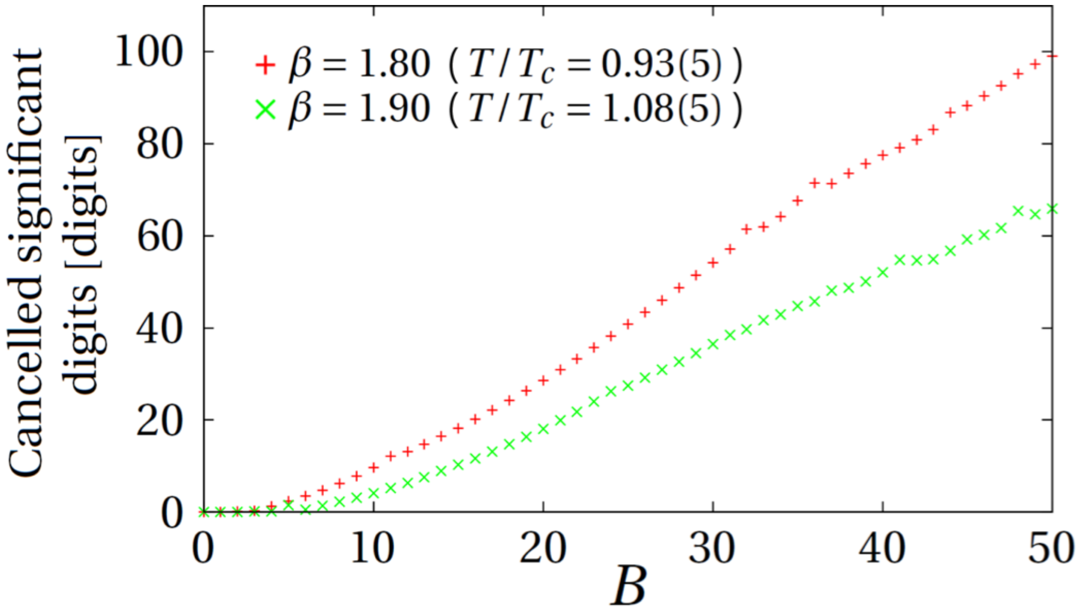
$$\log(I - \kappa Q) = -\sum_n \frac{\kappa^n Q^n}{n}$$

$$\det \Delta = \exp(\sum_n (W_n \xi^n + W_{-n} \xi^{-n}))$$



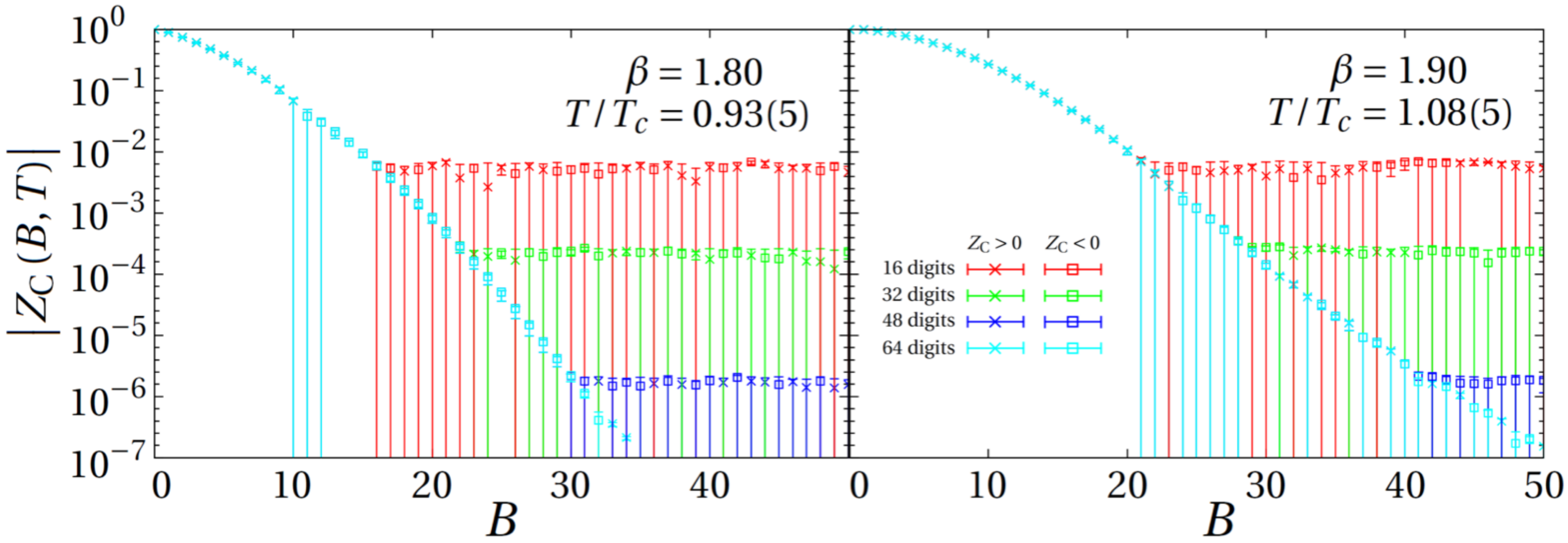
Take W_n for $|n| \leq 6$ and do the Fourier Trans. analytically.

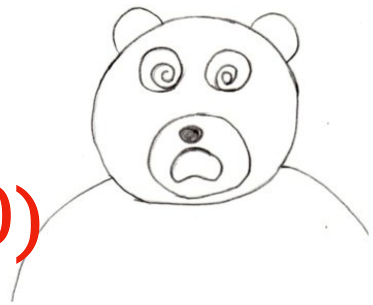
Big Cancellation in FFT !



S.Oka, arXiv:1511.04711
Talk at LATTICE 2015

Fukuda, Nakamura, Oka,
arXiv:1511.04711
Phys.RevD93, 094508 (2016)





θintegration → Multi-Precision (50 - 100)

$$z_n = \int \frac{d\theta}{2\pi} e^{i\theta n} Z_{GC}(\theta \equiv \frac{\mathrm{Im}\mu}{T}, T)$$

Using Multiple-precision, we have beaten Sign Problem.

But to make Canonical Approach workable,

we had to solve 2 problems:

1. Z_{GC} is not a direct observable in lattice QCD

2. We should perform simulations at many imaginary μ points.

Integration Method

Not Z_G but n_B in imaginary μ → z_n

$$n_B = \frac{1}{3V} T \frac{\partial}{\partial \mu} \log Z_G$$

$$= \frac{N_f}{3N_s^3 N_t} \int \mathcal{D}U e^{-S_G} \mathrm{Tr} \Delta^{-1} \frac{\partial \Delta}{\partial \mu} \det \Delta$$

(For pure imaginary μ, n_B is also imaginary)

Then, for fixed T

$$Z(\theta \equiv \frac{\mu}{T}) = \exp(V \int_0^{\theta} n_B d\theta')$$

$$z_k = \frac{3}{2\pi} \int_{-\pi/3}^{+\pi/3} d\theta \exp\left(i\, k\theta + \int_0^{\theta} n_B d\theta' \right)$$

Power of GPU !

Vladimir' Code has produced 11 papers in 2017 and 2018 on a small GPU machine.

V.G. Bornyakov, D. Boyda, V. Goy, A. Molochkov, A. Nakamura, A. Nikolaev and V.I. Zakharov
``Lattice Study of QCD Phase Structure by Canonical Approach'',
EPJ Web of Conferences 175, 07033 (2018)

M. Wakayama, V.G. Borynakov, D.L. Boyda, V.A. Goy, H. Iida, A.V. Molochkov, A. Nakamura, V.I. Zakharov
``Lee-Yang zeros in lattice QCD at finite baryon density from canonical approach'',
arXiv:1802.02014 [hep-lat]

``Lattice QCD at finite baryon density using analytic continuation'',
V.Bornyakov, D. Boyda, V. Goy, H. Iida, A. Molochkov, A. Nakamura, A. Nikolaev, M. Wakayama and V.I. Zakharov

EPJ Web of Conferences 18 (2018) 20201
V.Bornyakov, D. Boyda, V. Goy, A. Molochkov, A. Nakamura, A. Nikolaev and V.I. Zakharov
"Restoring canonical partition functions from imaginary chemical potential",
EPJ Web of Conferences 175, 07027 (2018)

D. Boyda, V.G. Bornyakov, V. Goy, A. Molochkov, A. Nakamura, A. Nikolaev, V.I. Zakharov
Lattice Study of QCD Phase Structure by Canonical Approach - Towards determining the phase transition line
arXiv:1704.03980 [hep-lat]

V. G. Bornyakov, D. L. Boyda, V. A. Goy, E.-M. Ilgenfritz, B. V. Martemyanov, A. V. Molochkov, Atsushi Nakamura, A. A. Nikolaev and V. I. Zakharov
Dyons and Roberge - Weiss transition in lattice QCD
EPJ Web of Conferences, Volume 137 (2017) 03002

V. G. Bornyakov, D. L. Boyda, V. A. Goy, A. V. Molochkov, Atsushi Nakamura, A. A. Nikolaev and V. I. Zakharov Study of lattice QCD at finite baryon density using the canonical approach EPJ Web of Conferences, Volume 137 (2017) 07017 DOI: https://doi.org/10.1051/epjconf/201713707017

V. A. Goy, V. Bornyakov, D. Boyda, A. Molochkov, A. Nakamura, A. Nikolaev, V. Zakharov
Sign problem in finite density lattice QCD Progress of Theoretical and Experimental Physics, 031D01 DOI: 10.1093/ptep/ptx018

D. L. Boyda, V. G. Bornyakov, V. A. Goy, V. I. Zakharov, A. V. Molochkov, Atsushi Nakamura, A. A. Nikolaev Novel approach to deriving the canonical generating functional in lattice QCD at a finite chemical potential JETP Letters 104, 657-661
V. G. Bornyakov, D. L. Boyda, V. A. Goy, E.-M. Ilgenfritz, B.V. Martemyanov, A. V. Molochkov, Atsushi Nakamura, A. A. Nikolaev, V. I. Zakharov Dyons and Roberge - Weiss transition in lattice QCD EPJ Web of Conferences, 137, 03002, (2016)

V.G. Bornyakov, D. Boyda, V. Goy, A. Molochkov, A. Nakamura, A. Nikolaev, V.I. Zakharov
Restoring canonical partition functions from imaginary chemical potential arXiv:1712.01515 [hep-lat]

GPU machine "Vostok 1" at Vladivostok

Hasenfratz-Toussant

+

- Multi-precision calculation
- Integration Method

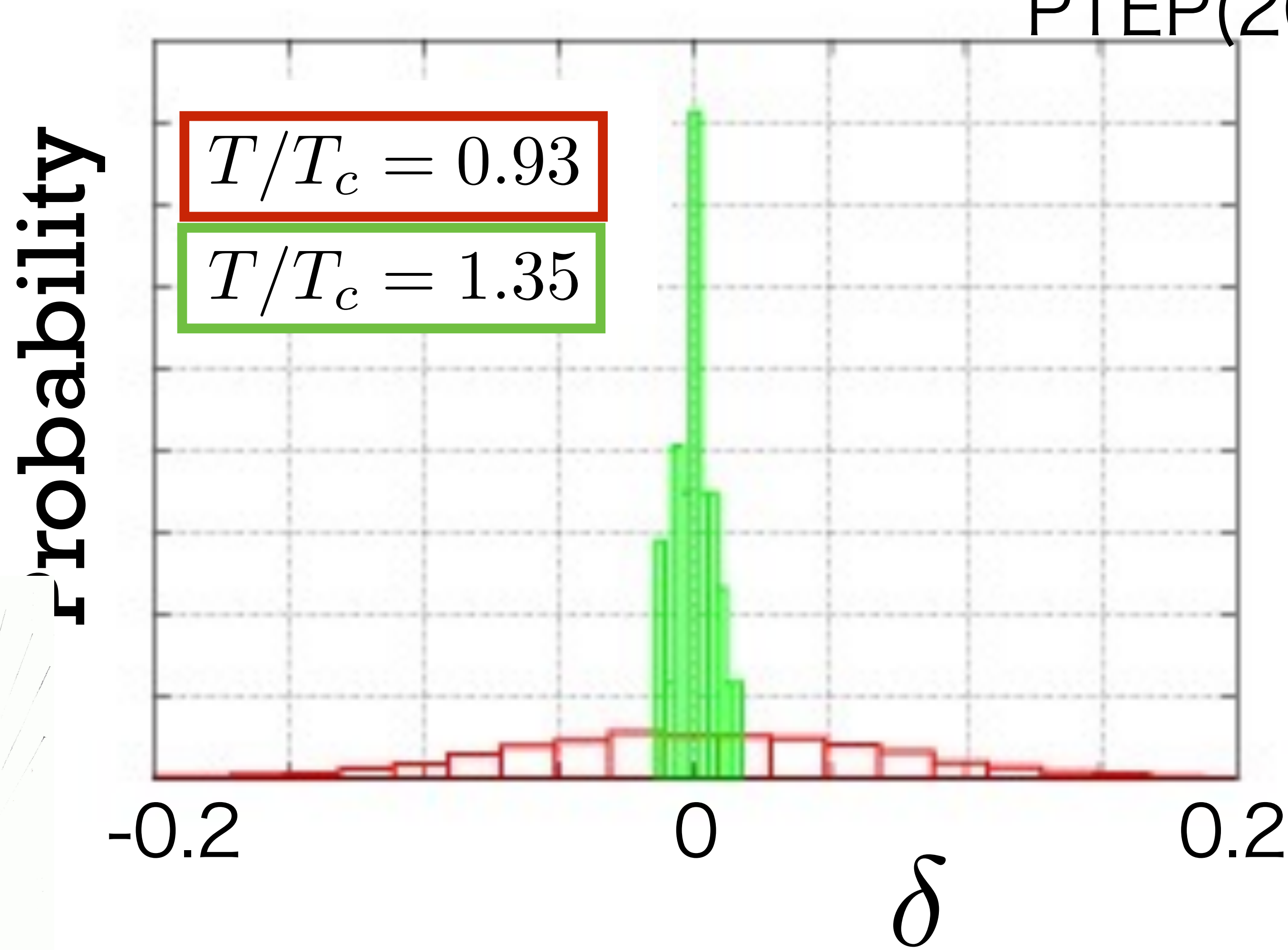
I thought we have beaten
Sign Problem.

But !

Hidden Sign Problem ?

Z_n **have phase on each configuration !**

V.Goy et al.,
PTEP(2017) 031D01



$$z_n \simeq |z_n| e^{in\delta}$$

$$Z_n = \langle z_n \rangle$$

are real positive.

References

A.Li et al.(Kentucky), Phys.Rev.D82:054502,2010, arXiv:1005.4158

A.Suzuki et al.(Zn Collaboration), Lattice 2016 Proceedings,

V.Goy et al.(Vladivostok), Prog Theor Exp Phys (2017) (3): 031D01,arXiv:1611.08093

Where comes the phase of z_n ?

A.Li et al.(Kentucky), Phys.Rev.D82:054502,2010, arXiv:1005.4158

$$Z = \int \mathcal{D}U \, (\det \Delta(\mu))^{N_f} \, e^{-S_G} \quad = e^{\log(1-\kappa Q)}$$

$$\begin{aligned}
\det \Delta(\mu) &= \det(1 - \kappa Q(\mu)) \\
&= \exp\left(A_0 + \sum_{n>0} [e^{in\phi} W_n + e^{-in\phi} W_n^\dagger] \right) \\
&= \exp\left(A_0 + \sum_n A_n \cos(n\phi + \delta_n) \right)
\end{aligned}$$

$$\begin{aligned}
A_n &\equiv 2|W_n| \\
\delta_n &\equiv arg(W_n)
\end{aligned}$$

We use $W_{-n} = W_n$

Then,

$$z_n \propto \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ik\phi} e^{A_0 + A_1 \cos(\phi+\delta_1) + A_1 \cos(2\phi+\delta_2) \cdots}$$

In the lowest order,

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ik\phi} e^{A_0 + A_1 \cos(\phi+\delta_1)} = e^{A_0} \int_{\delta_1}^{2\pi+\delta_1} \frac{d\phi'}{2\pi} e^{-ik(\phi'-\delta_1)} e^{A_1 \cos\phi'}$$

$$= e^{A_0+ik\delta_1} \int_{\delta_1}^{2\pi+\delta_1} \frac{d\phi'}{2\pi} e^{-ik\phi'} e^{A_1 \cos\phi'}$$

$$= e^{A_0+ik\delta_1} \int_0^{2\pi} \frac{d\phi'}{2\pi} e^{-ik\phi'} e^{A_1 \cos\phi'}$$

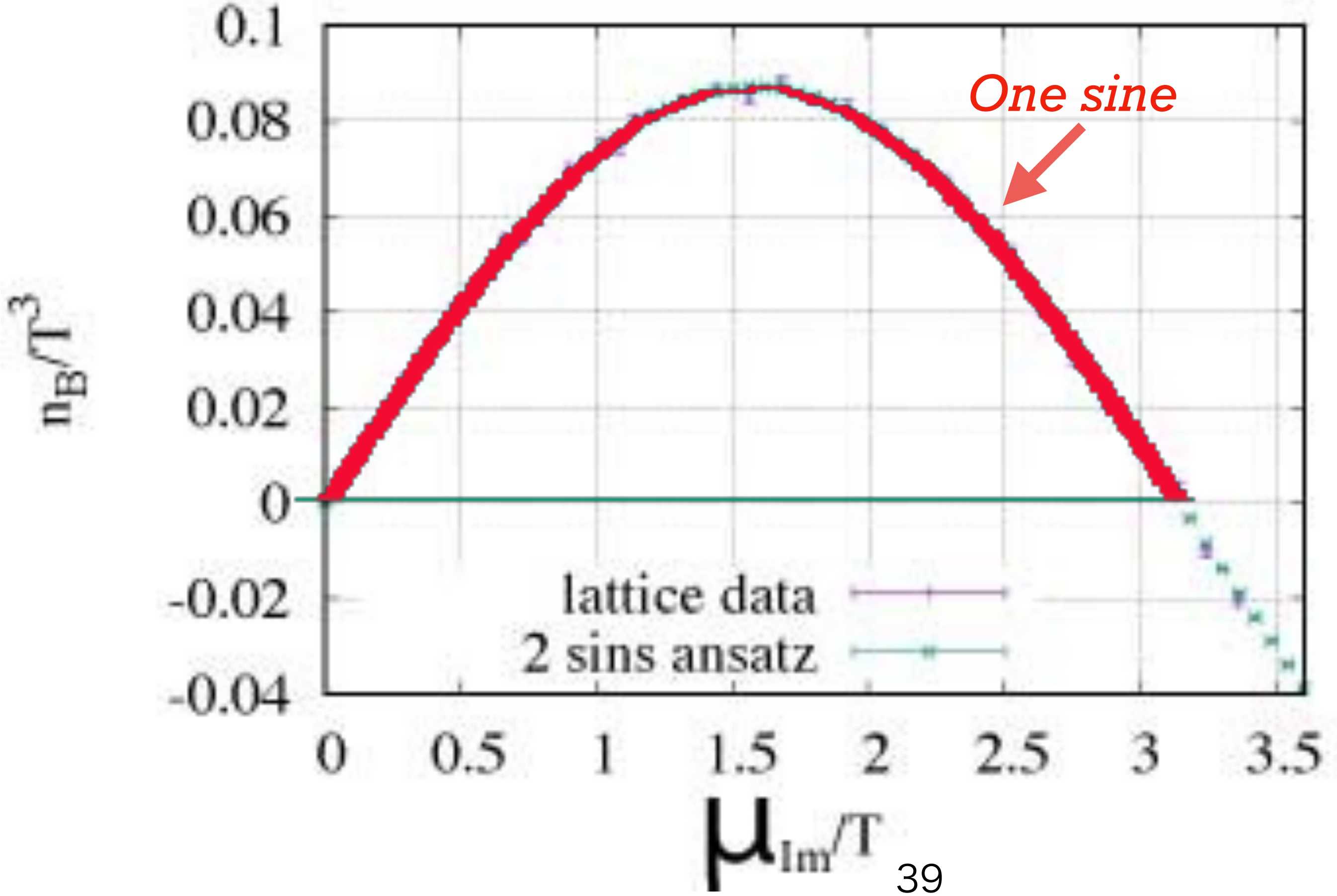
$$= e^{A_0+ik\delta_1} I_k(A_1)$$

$$\propto z_k$$

where we use

$$I_n(z) = \frac{(-1)^n}{2\pi} \int_0^{2\pi} e^{z\cos t} e^{-int} dt$$

A Remark of Function Form of $n_B(\mu_I)$



$n_B(\mu_I)$ is well approx-imated by sine function at *T*<*Tc*.

Takahashi et al. Phy. Rev. D 91 (1) (2015) 014501.
Bornyakov et al., Phys.Rev. D95, 094506 (2017)

Number density in Imaginary μ

We expand the number density as

$$n_B(\theta)/T^3 = \sum_{k=1}^{k_{max}} f_{3k} \sin(k\theta)$$

Confinement phase $T < T_c$

$$n_B(\theta)/T^3 = \sum_{k=1}^{k_{max}} a_{2k-1}\theta^{2k-1}$$

DeConfinement phase $T > T_c$

$$\theta \equiv \frac{\mu}{T}$$

Fitting functions are much more robust against the hidden sign problem, because a fitting curve include many points.

Of course they are correlated, therefore, the careful error estimation is important.

Hasenfratz-Toussant

+

- Multi-precision calculation
- Integration Method
- Fourier/Polynomial Expansion of $\langle n \rangle$

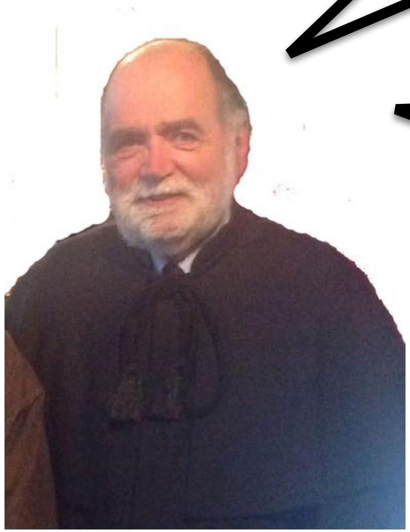
- Now we can say we have beaten Sign Problem for T>0 by Canonical Approach.

Now it's time for Champagne!

Sign Problem is now solved for T>0, and it is time to analyze the finite density QCD.

But people do not recognize it. Why ?

It takes very
long time until
your idea is understood.

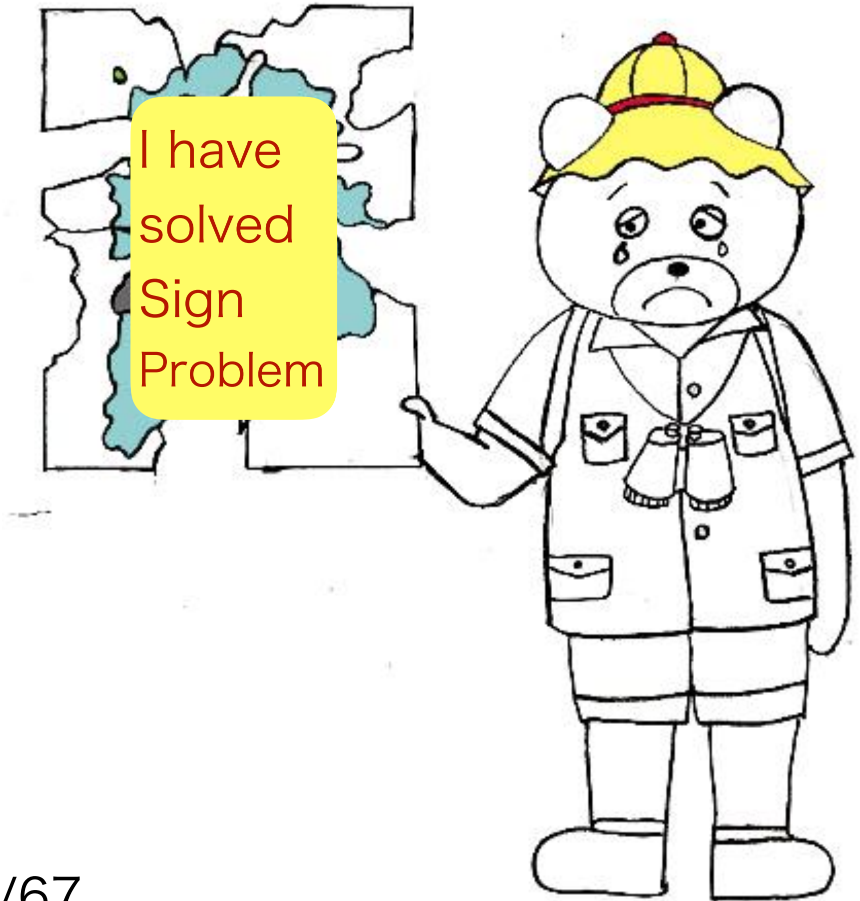


Because your
Approach is
different.

But I use
only Statistical
Mechanics and
Numerical
technique !?

People believe
Sign Problem will be solved
after several hundred years
by a genius.

Not now by you !



DISCLAIMER

Good. So I want to apply for T=0.

Great. I want to apply for mu/T=10^3.

Sorry,
not applicable.

$$Z(\mu, T) = \sum_n z_n(T)(e^{\mu/T})^n$$

d'Alembert:

$$\frac{z_{n+1} e^{(n+1)\mu/T}}{z_n e^{n\mu/T}} < 1 \Rightarrow z_{n+1}/z_n < e^{-\mu/T}$$

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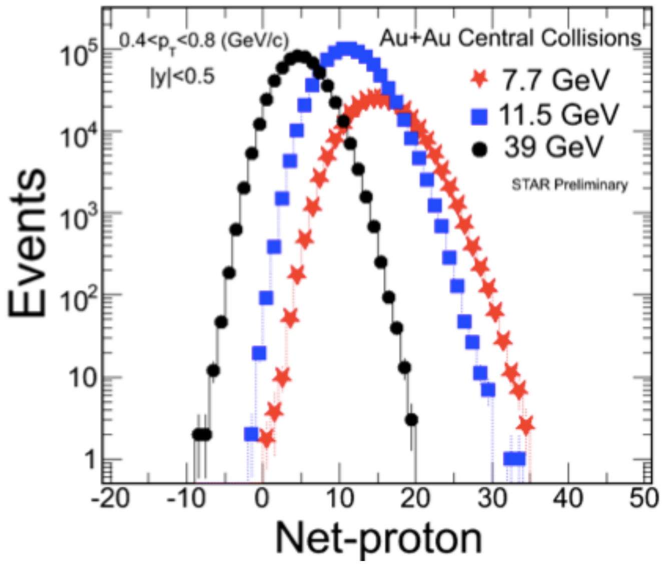
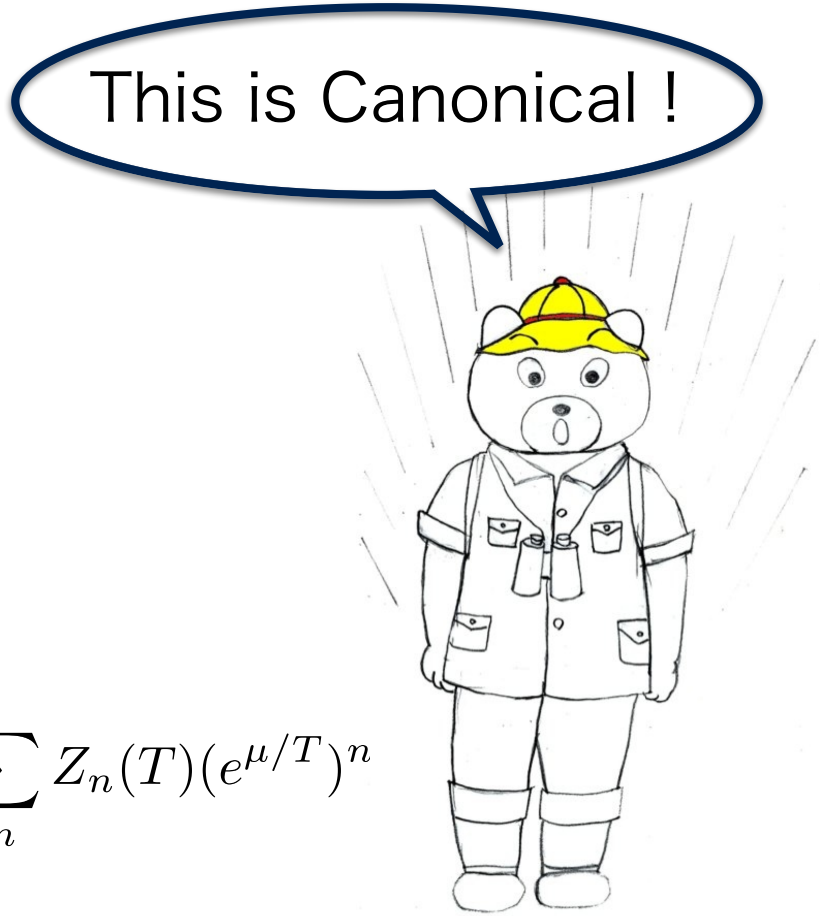
In 2012, at Wuhan

STAR@RHIC



Prof.Nu Xu

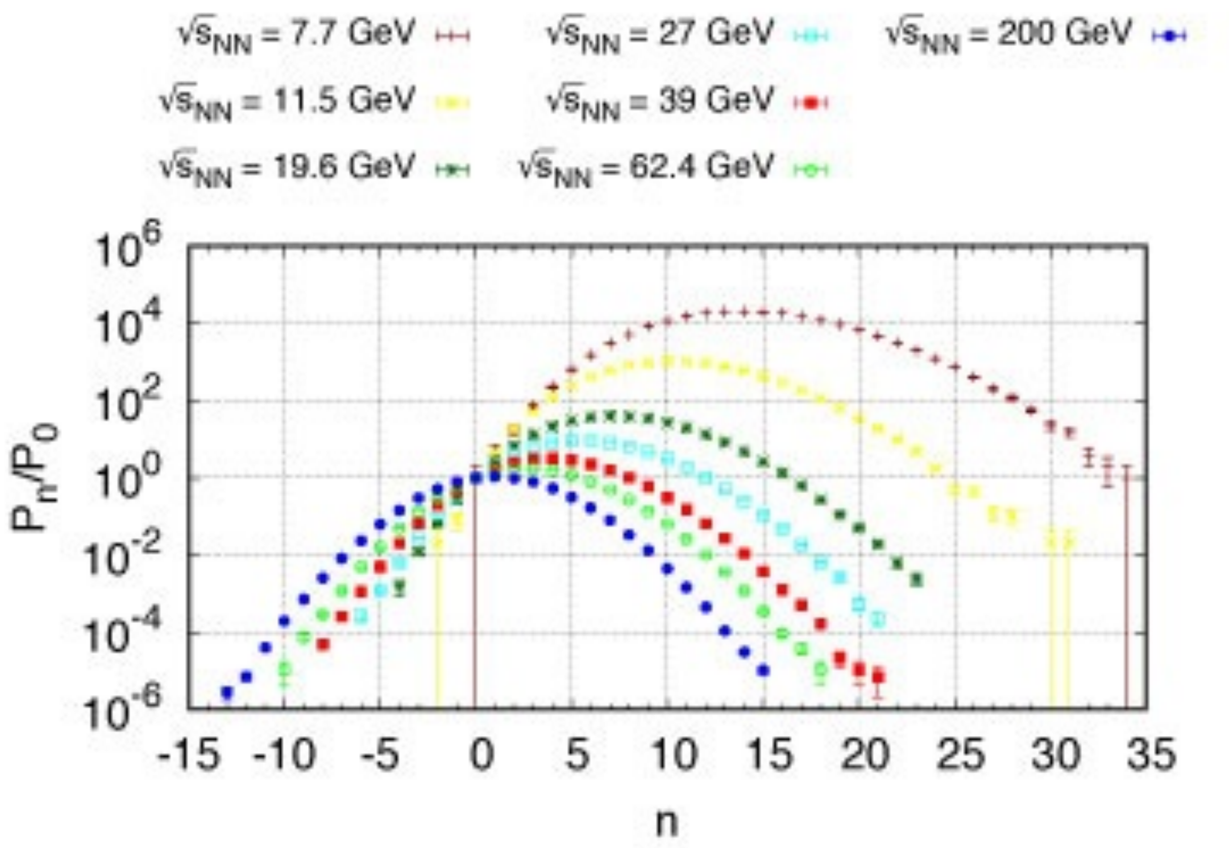


We thank Prof.Nu Xu
and Prof.Luo!

$$Z(\mu, T) = \sum_n Z_n(T)(e^{\mu/T})^n$$

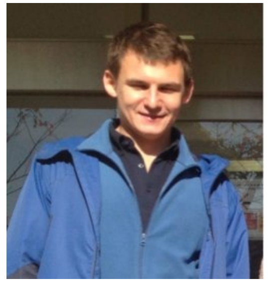
Experimental data and Fugacity Expansion

$$Z(\mu, T) = \sum_n z_n(T)(e^{\mu/T})^n$$



Proton multiplicity: Lattice data

D.Boyda noticed we can extract T,mu and V.

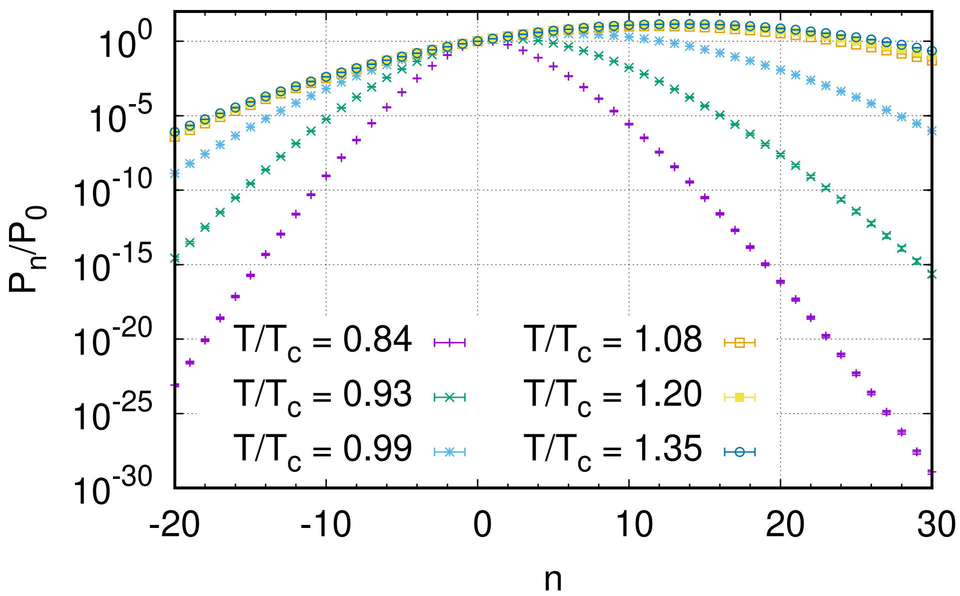


Probability interpretation:

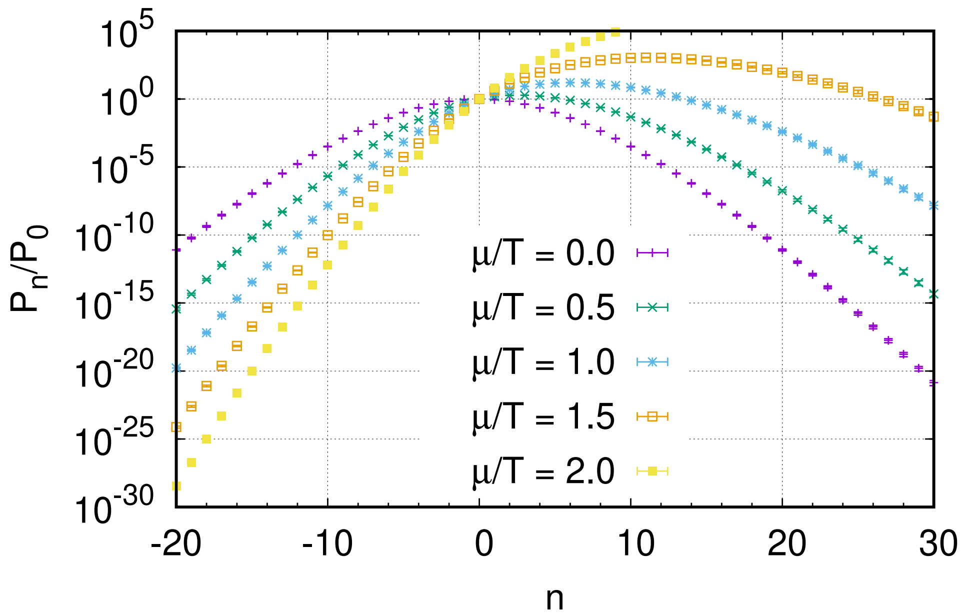
$$1 = \sum_n \frac{Z_n \xi^n}{Z_{GC}(\mu)}, \quad \xi = e^{\mu/T} \qquad \frac{N(n)}{N(0)} = \frac{N\, Z_n \xi^n / Z_{GC}(\mu)}{N\, Z_0 \xi^0 / Z_{GC}(\mu)} = Z_n \xi^n / Z_0$$

$\Rightarrow$ Proton multiplicity: $\frac{P_n}{P_0} = \frac{Z_n}{Z_0} e^{n\mu/T}$ ← Lattice QCD

Lattice data $\mu/T = 0.4$

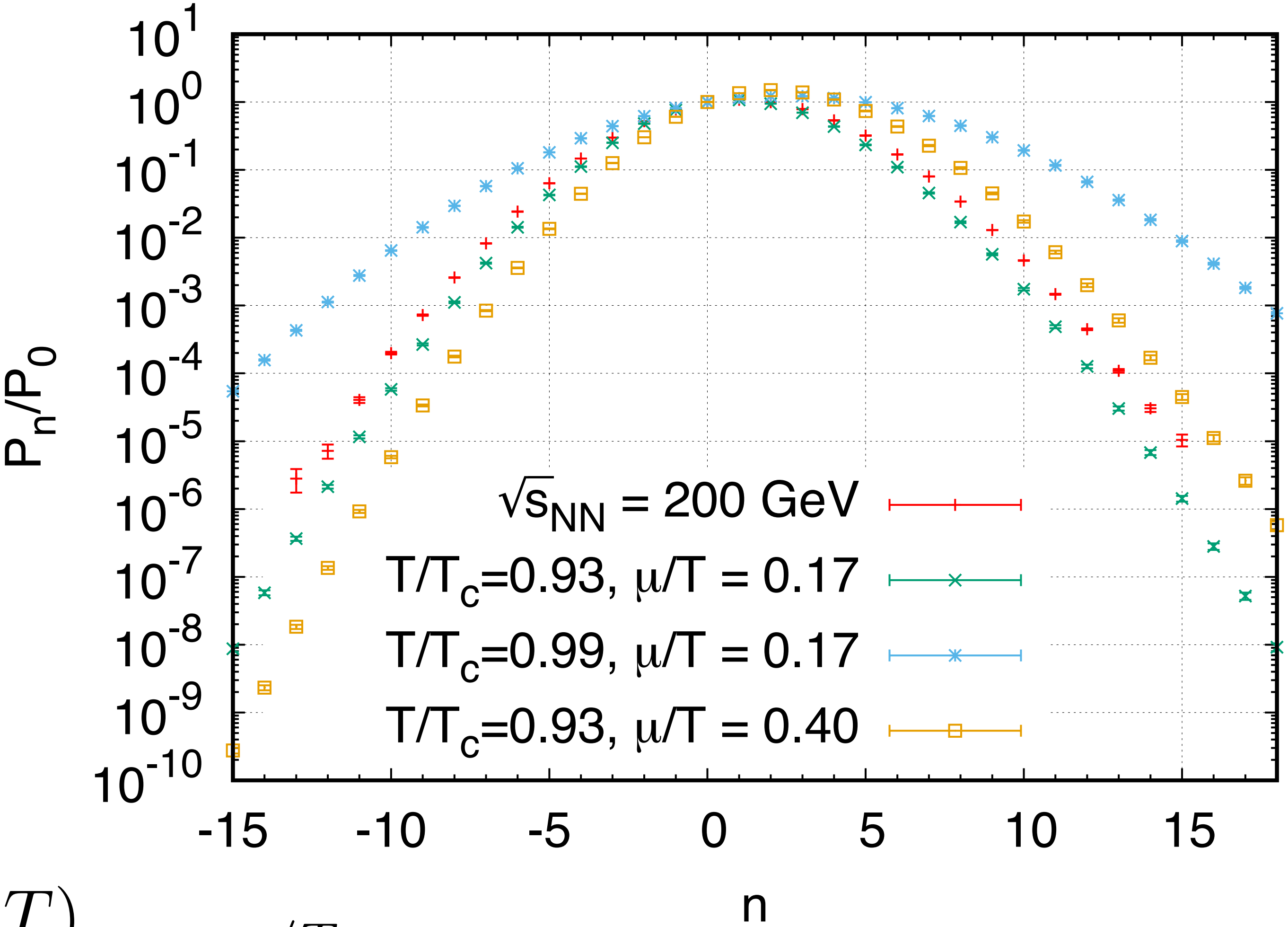


Lattice data $T/T_c = 0.93$



Proton multiplicity: RHIC experiment data and lattice

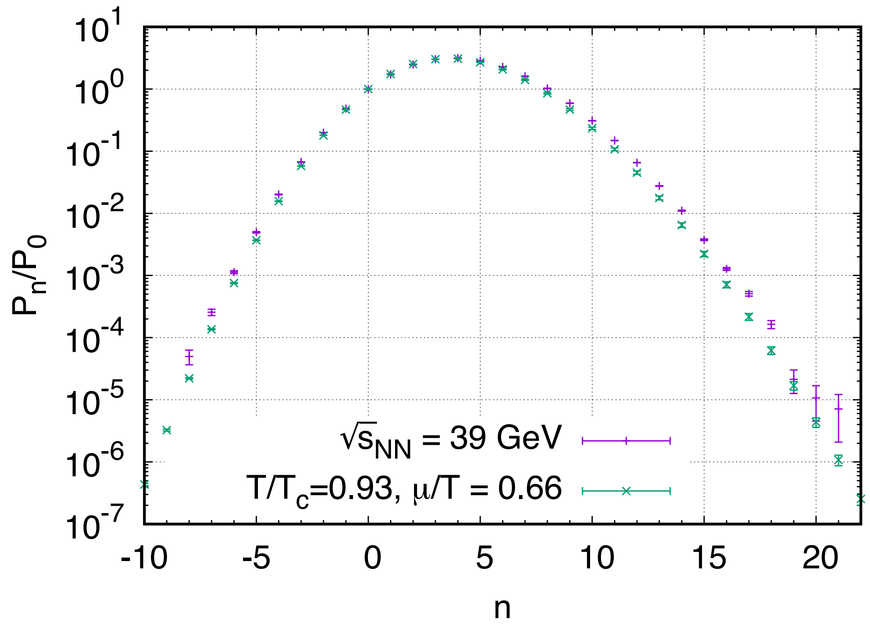
$$\sqrt{s}_{NN} = 200\mathrm{GeV}$$



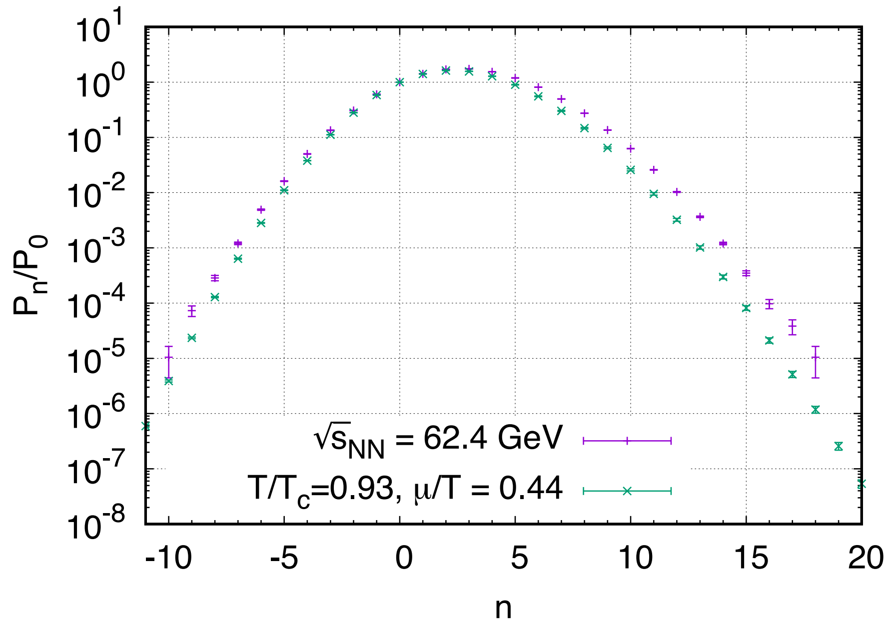
$$\frac{P_n}{P_0} = \frac{Z_n(T)}{Z_0(T)} \times (e^{\mu/T})^n$$

Proton multiplicity: RHIC experiment data and lattice

$$\sqrt{s}_{NN} = 39\mathrm{GeV}$$



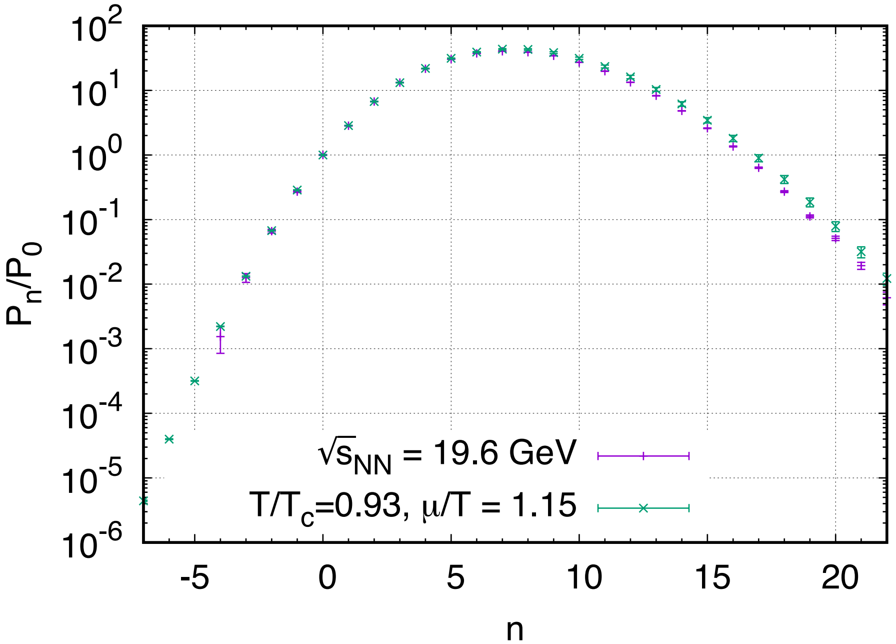
$$\sqrt{s}_{NN} = 62.4\mathrm{GeV}$$



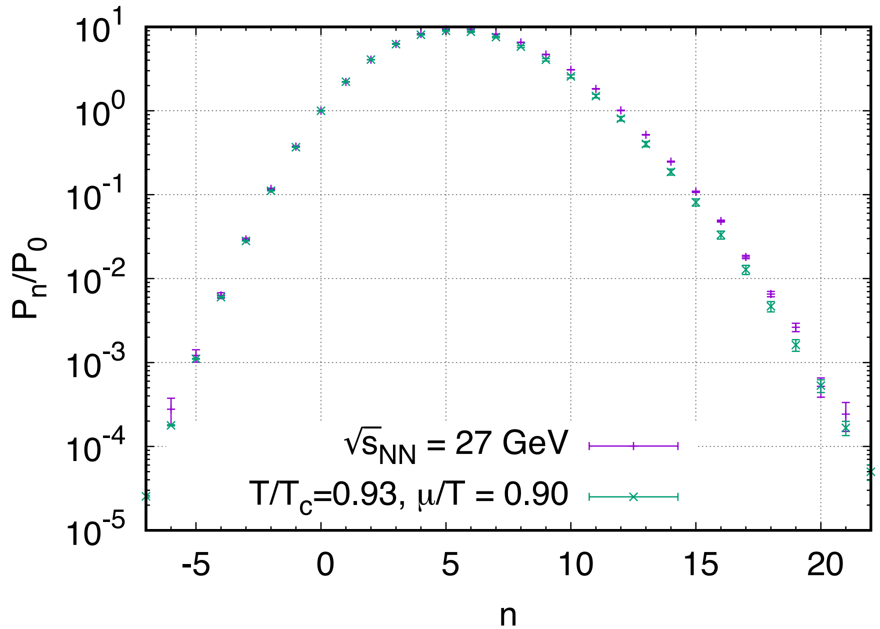
$$\frac{P_n}{P_0} = \frac{Z_n(T)}{Z_0(T)} \times (e^{\mu/T})^n$$

Proton multiplicity: RHIC experiment data and lattice

$$\sqrt{s}_{NN} = 19.6\text{GeV}$$



$$\sqrt{s}_{NN} = 27\text{GeV}$$

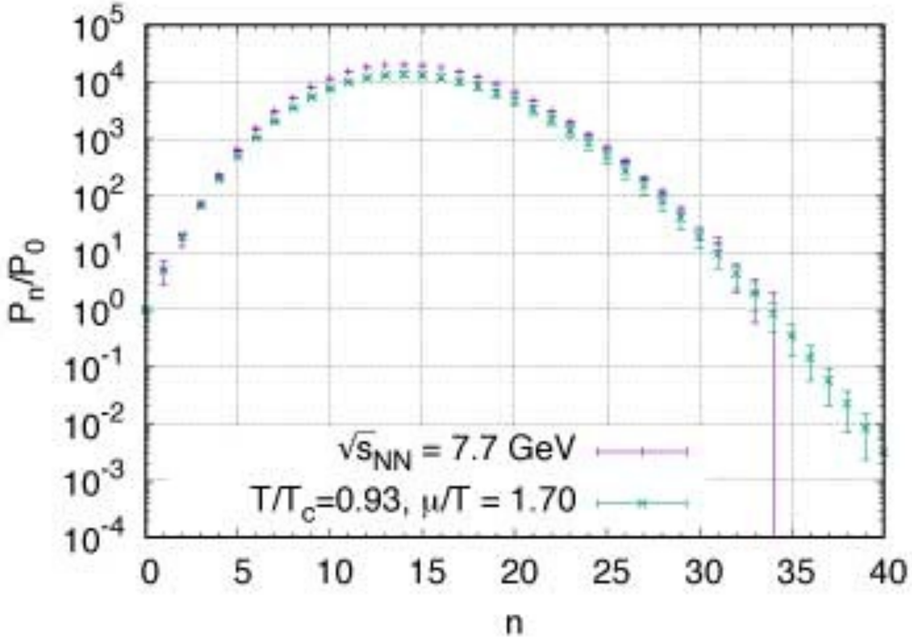


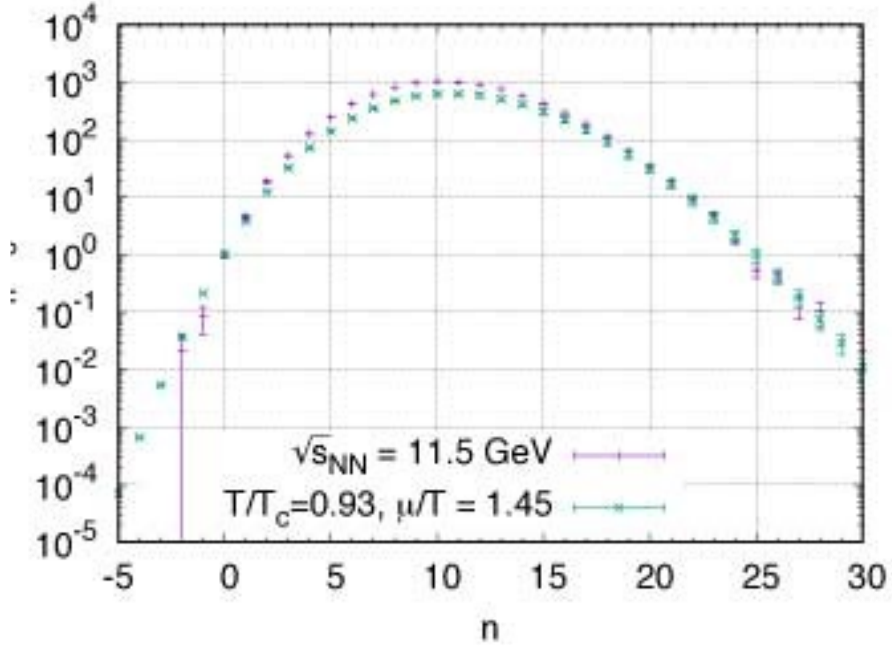
$$\frac{P_n}{P_0} = \frac{Z_n(T)}{Z_0(T)} \times (e^{\mu/T})^n$$

Proton multiplicity: RHIC experiment data and lattice

$\sqrt{s}_{NN} = 7.7\mathrm{GeV}$

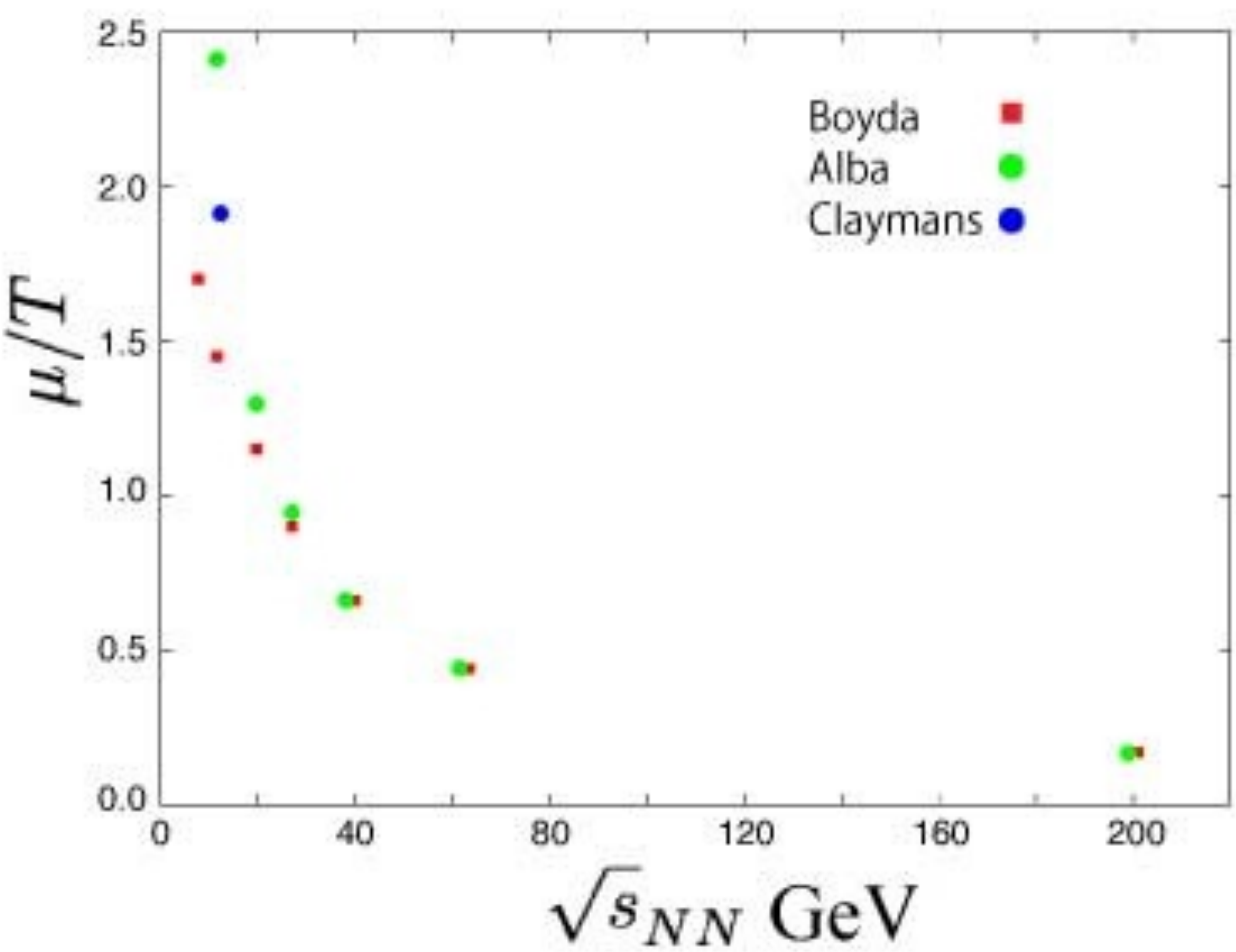
$\sqrt{s}_{NN} = 11.5\mathrm{GeV}$





$$\frac{P_n}{P_0} = \frac{Z_n(T)}{Z_0(T)} \times (e^{\mu/T})^n$$

Density vs Collision Energy



Boyda: Canonical method

Alba et al., Phys.Lett. B738, 305 (2014)

Cleymans et al., Phys. Rev. C, 73 (2006), p. 034905

Comparisson with RHIC experiment

A. Nakamura, K. Nagata PTEP, 033D01 (2016)
RHIC STAR data **(Luo X. CEJP 10, 1372 (2012))**

Probability interpretation:

$$1 = \sum_n \frac{Z_n \xi^n}{Z_{GC}(\mu)}, \quad \xi = e^{\mu/T} \qquad \frac{N(n)}{N(0)} = \frac{N\, Z_n \xi^n / Z_{GC}(\mu)}{N\, Z_0 \xi^0 / Z_{GC}(\mu)} = Z_n \xi^n / Z_0$$

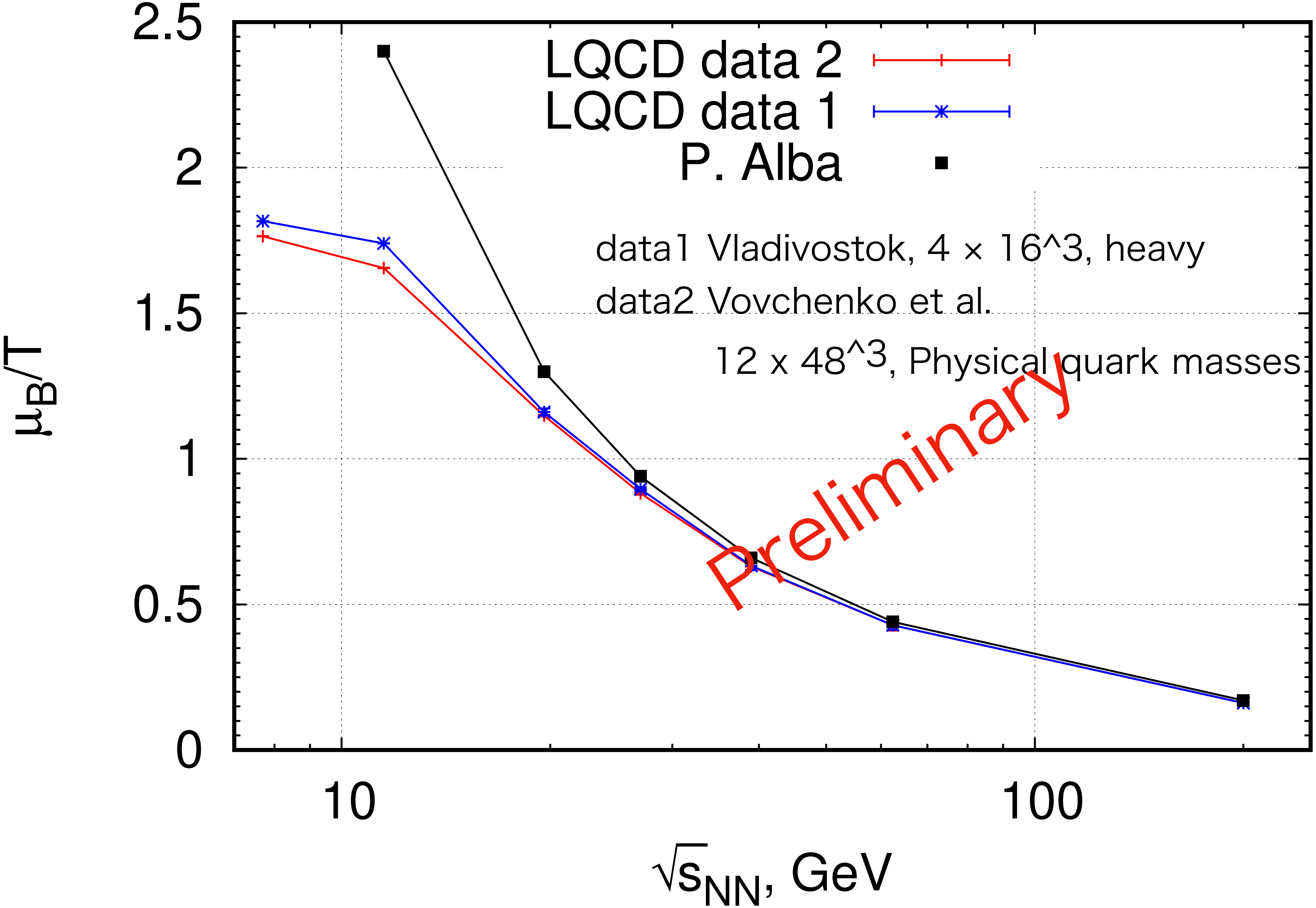
Multiplicity: $P_n = Z_n \xi^n \quad \Rightarrow \quad Z_n = P_n P_{-n}$ и $\xi = \sqrt[2n]{\frac{P_n}{P_{-n}}}$

Extracted fugacity $\xi (= e^{\mu/T})$ agreed with HRG model estimation

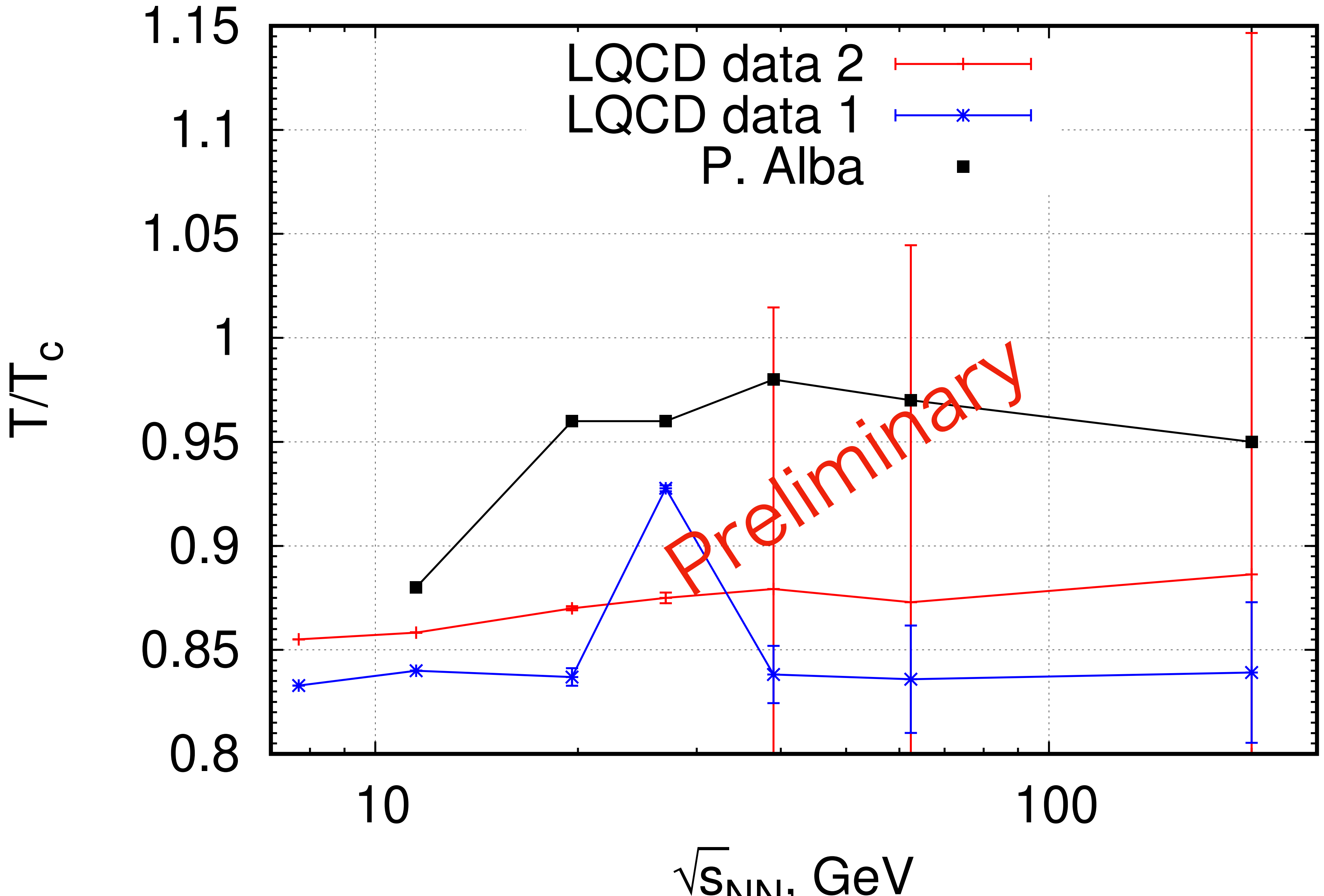
$\sqrt{s_{NN}}$, GeV	J. Cleynams	P. Alba	A. Nakamura
11.5	8.04	11.1	7.48
19.6	3.62	3.65	3.21
27.0	2.62	2.58	2.43
39.0	1.98	1.93	1.88
62.4	1.55	1.55	1.53
200.0	1.18	1.18	1.18

J. Cleaymans et al., Phys. Rev. C 73, 034905 (2006)
P. Alba et al., Phys. Let. B 738, 305 (2014)

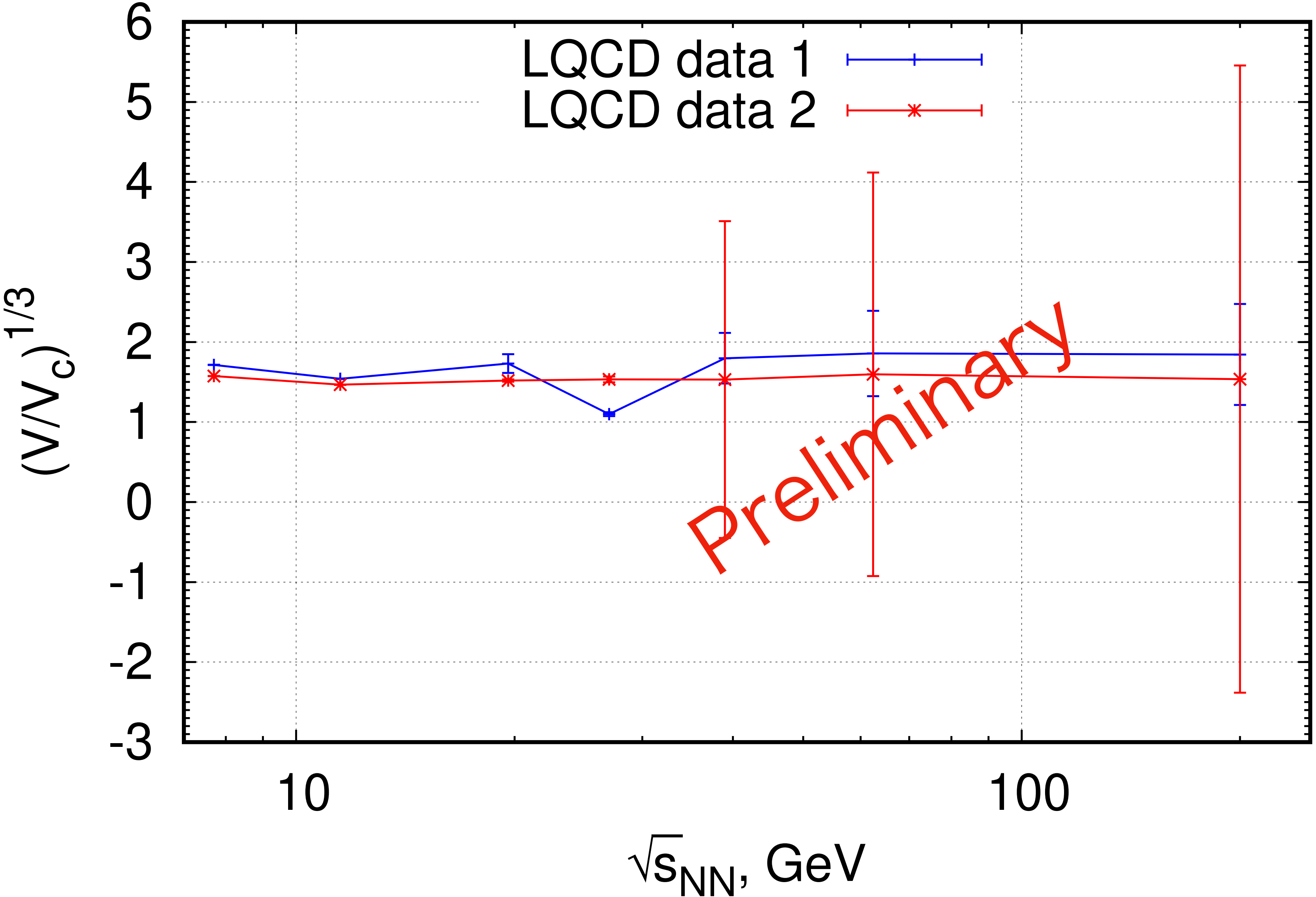
Multiplicity: RHIC experiment parameters - μ

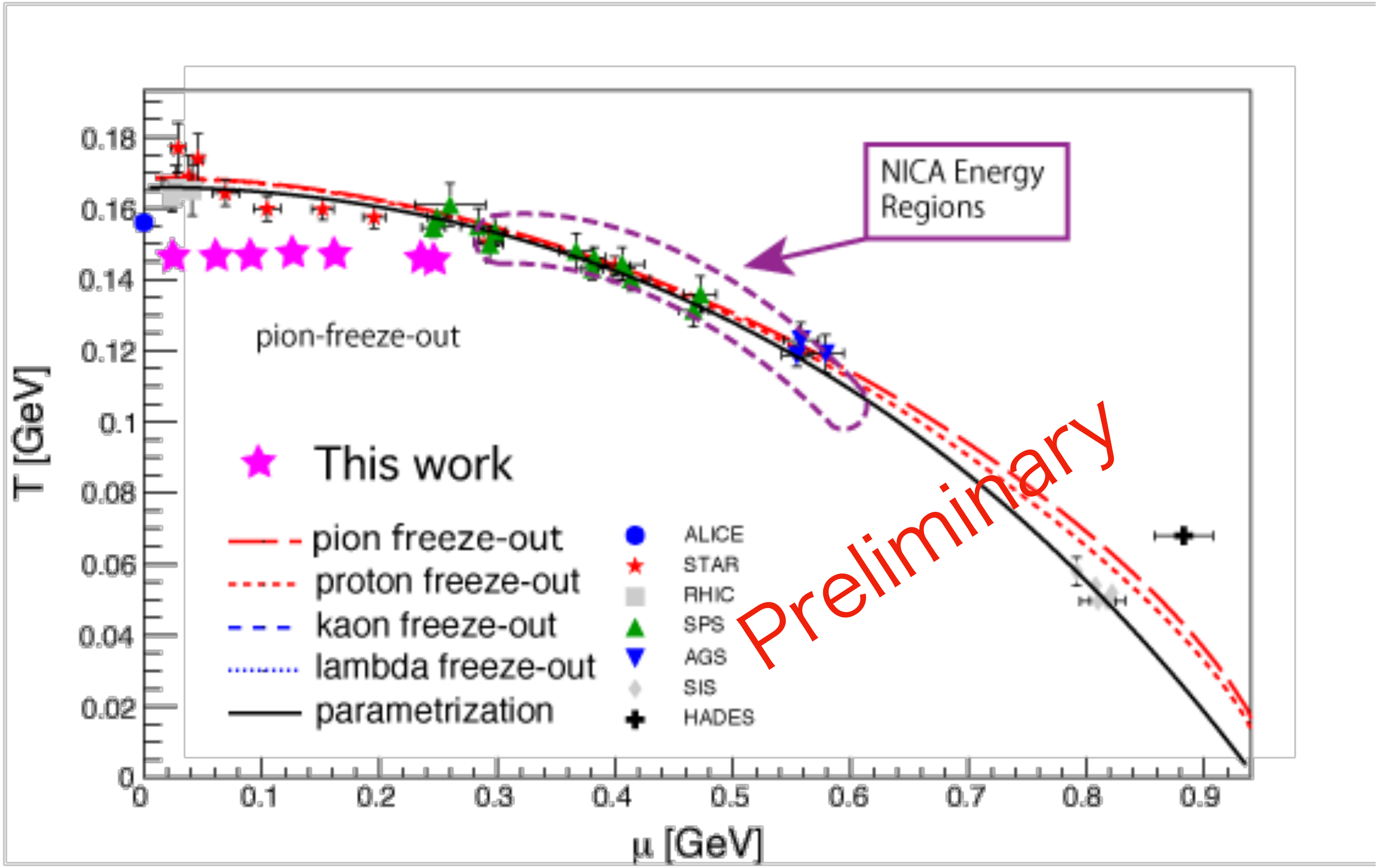


Multiplicity: RHIC experiment parameters - *T*



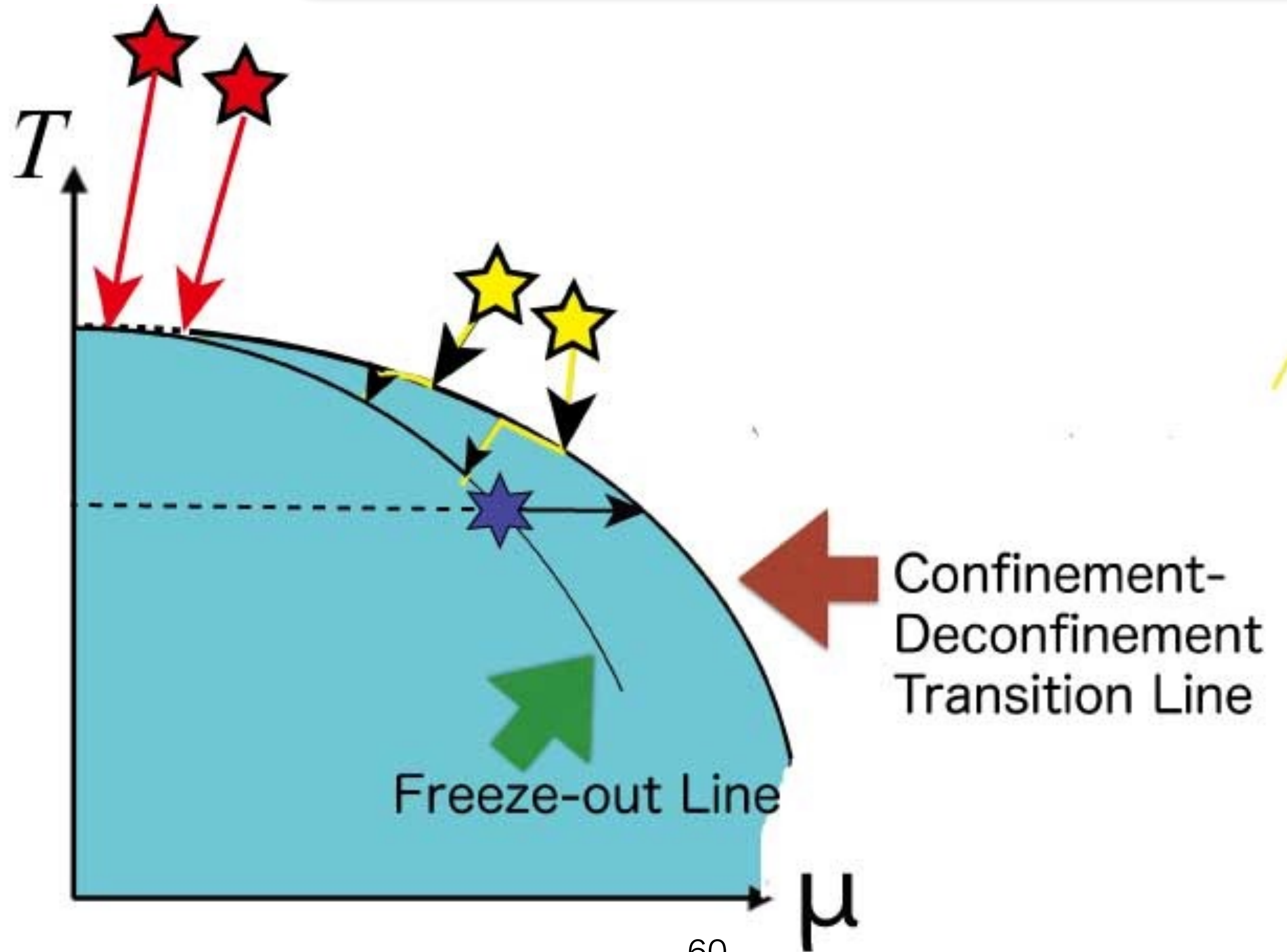
Multiplicity: RHIC experiment parameters - V




NICA Energy Regions
pion-freeze-out
This work
pion freeze-out
proton freeze-out
kaon freeze-out
lambda freeze-out
parametrization
ALICE
STAR
RHIC
SPS
AGS
SIS
HADES
Preliminary
T [GeV]
μ [GeV]
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

Content

$$Z(\mu, T) = \sum_n z_n(T)(e^{\mu/T})^n$$

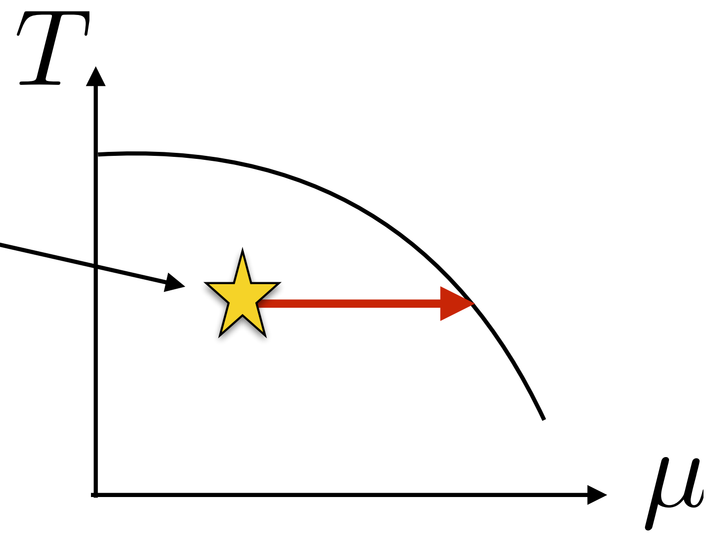


Information hidden in Fugacity Expansion ?

$$Z(\mu, T) = \sum_n z_n(T)(e^{\mu/T})^n$$

☆ Experimentally

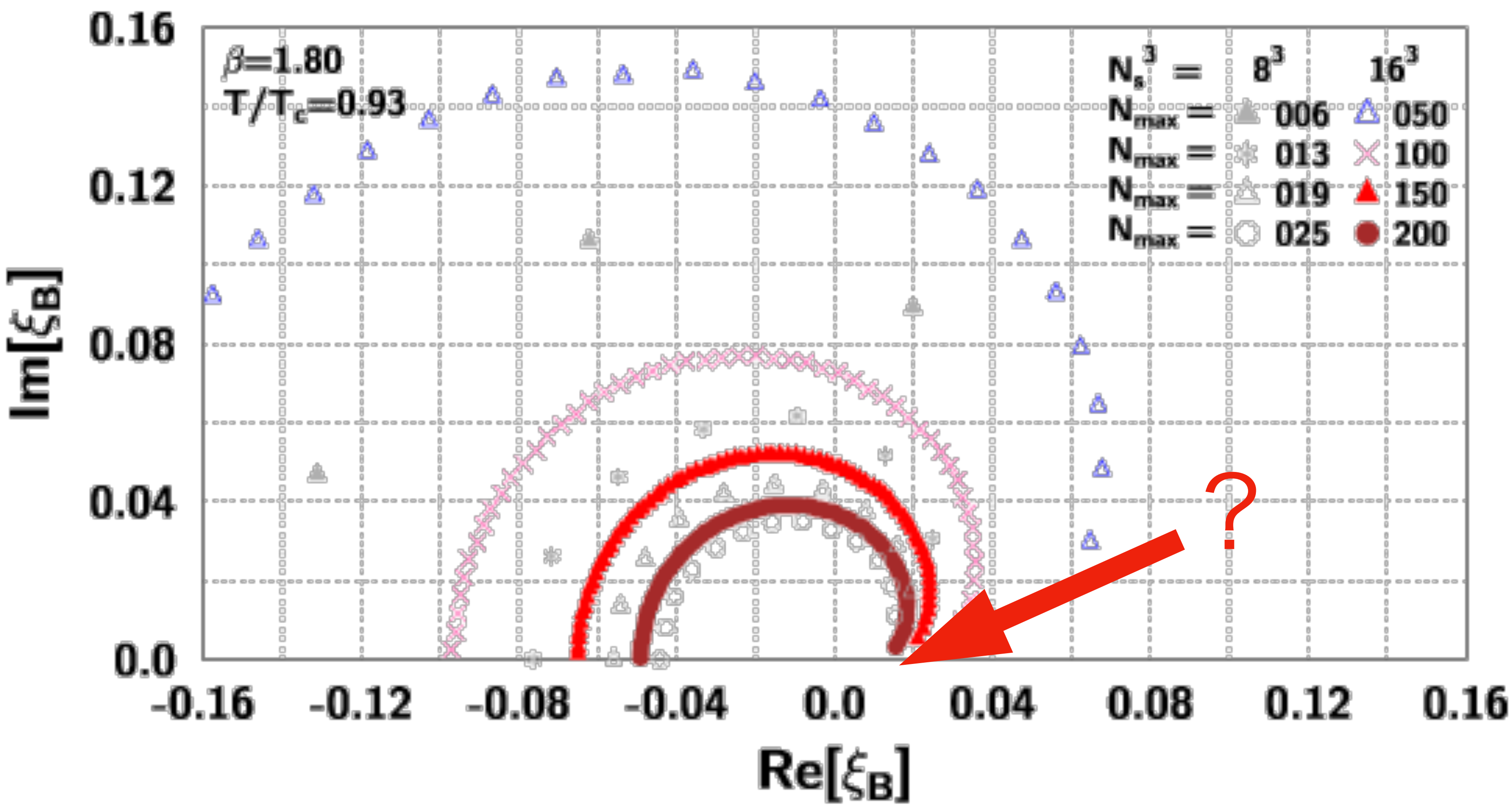
Determine $Z_n(T)$ here.
Then see QCD Phase
at higher density !



Lee-Yang zeros

Phys. Lett. B, 2019

Wakayama, V. Bornyakov, Boyda, Goy, Iida, Molochkov, Nakamura and Zakharov



Content

1. Introduction — Finite Density Regions
2. Sign Problem
3. Canonical Approach
4. Experimental data at RHIC — Higher Moments
5. How to find QCD phase transition line ?
6. **What should we do next ?**
7. Summary

What should we do next ?

- Let the world to know that the Sign Problem was solved by Vladivostok group
  - Canonical approach + Multiple precision beat the sign problem
- Quark mass in the present lattice QCD calculation is very heavy, and we want go to more realistic quark masses.
  - Physical quak mass lattice simulations have been done by several groups at zero density. (Algorithm is known)
- Study s-quark effects.
- Wigner function

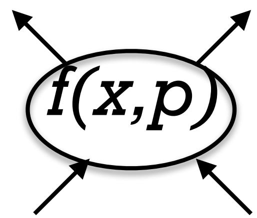
It is time to study QGP physics

Beyong Equilibrium

- Wigner function of QGP

$$W(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} < \bar{\psi}(x+\frac{y}{2})\psi(x-\frac{y}{2})) >$$

  - Quantum Analog of the distribution function, f(x,p)
  - Iida, Hatsuda, Baym



Gauge Invariant Wigner Function

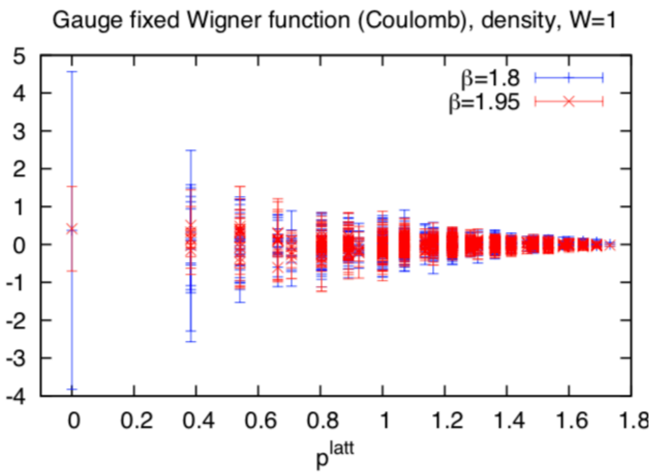
$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ipy} \langle \bar{\psi}_\beta(x_+) e^{-ig\int_{x_-}^{x_+} A_\mu(x)dz^\mu} \psi_\alpha(x_-) \rangle$$

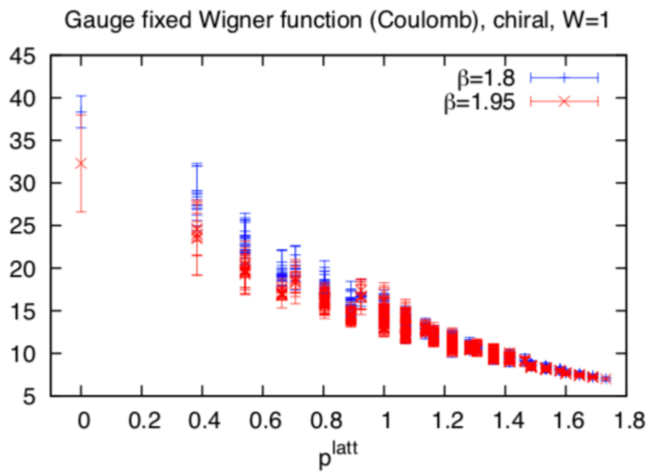
K.Yagi, T.Hatsuda andY.Miake, "Quark-Gluon Plasma" (Cambridge University Press)

On the lattice

$$f(\vec{p},\vec{x},\tau) = \int d\vec{y} e^{i\vec{p}\vec{y}} \langle 0|\bar{\psi}^a_\alpha(\vec{x}+\frac{\vec{y}}{2},\tau)\Gamma^{\alpha\beta}_\mu \times W^{ab}(\vec{x}+\vec{y}/2,\vec{x}-\vec{y}/2;\tau_0,\tau)\psi^b_\beta(\vec{x}-\frac{\vec{y}}{2},\tau)|0\rangle$$

W:Wilson line





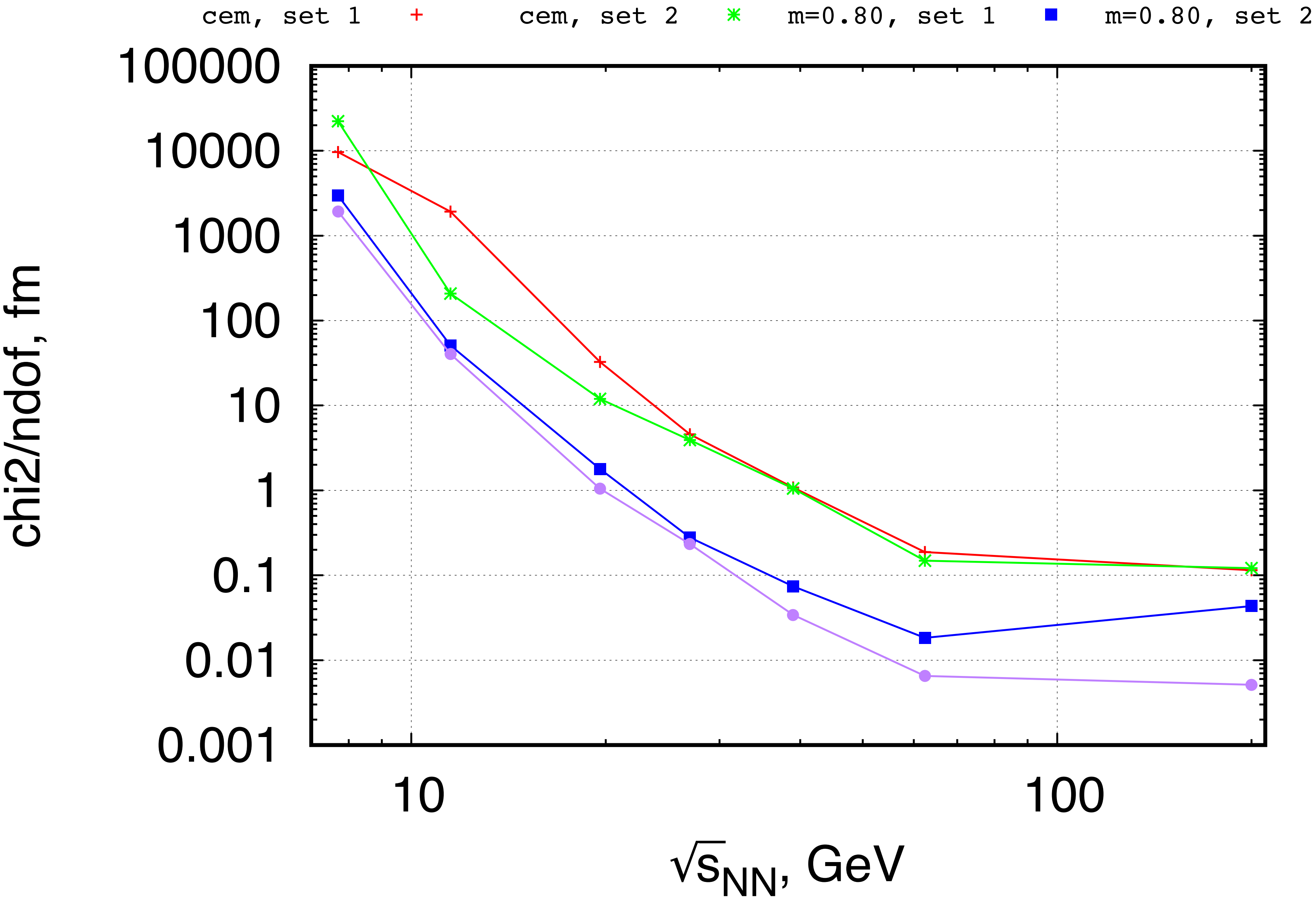
What should we do next ? (continue)

- Lattice Simulations in NICA /J-PARC (μ, T) regions, and study hadronic properties there.

Summary

- For T>0, Sign problem is beaten by Canonical approach with
  - Multi-precision calculation
  - Integration Method
  - Fourier/Polynomial Expansion of the number density.
- We can study Heavy Ion Collisions now.

Backup Slides


cem, set 1
cem, set 2
m=0.80, set 1
m=0.80, set 2
100000
10000
1000
100
10
1
0.1
0.01
0.001
chi2/ndof, fm
10
100
√s_NN, GeV

$$
\begin{aligned}
Z &= \sum_n z_n (e^{\mu/T})^n \\
&= z_0 + 2\sum_{n>0} z_n \cos(n\frac{\mu_I}{T}) \\
&= z_0 + 2\sum_{n>0} z_n \cosh(n\frac{\mu}{T})
\end{aligned}
$$

$$\mu = i\mathrm{Im}\,\mu$$