Nonequilibrium viscous correction to the phase space density and bulk viscosity in the relaxation time approximation

Alina Czajka

National Centre for Nuclear Research

in collaboration with S. Hauksson, C. Shen, S. Jeon, C. Gale

based on: Phys. Rev. C97 (2018) no. 4, 044914

The II International Workshop on Theory of Hadronic Matter under Extreme Conditions
Dubna, 16-19 September 2019

Outline

- Introduction and motivation

- Nonequilibrium deviation from the distribution function

- Equations of hydrodynamics with mean field effects
  - Equilibrium hydrodynamics
  - Nonequilibrium hydrodynamics: Landau condition and viscous corrections

- Bulk viscosity computation
  - Anderson-Witting model of the Chapman-Enskog approach
  - 14-moment approximation

- Summary

Hydrodynamics and heavy ion collisions

- ✓ Relativistic fluid dynamics describes very well the evolution of matter produced in heavy ion collisions after it achieves approximate local thermal equilibrium
- ✓ Hydrodynamics - macroscopic description of a system; transport coefficients - parameters

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - \Delta^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}$$

$\Pi \rightarrow \zeta, \tau_\Pi, \ldots$ $\pi^{\mu\nu} \rightarrow \eta, \tau_\pi, \ldots$

ϵ - energy density P - pressure u^μ - four-velocity

First applications of viscous hydro to model the nuclear matter evolution:

η/s - must be small

strongly supported by the AdS/CFT result

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

What do we know about bulk viscosity?

Related to conformal symmetry breaking

Nonconformality parameters:

- Microscopic:
  - m_0 - zero-temperature mass
  - β_λ - fixes the coupling as a function of the energy scale
- Macroscopic: $\epsilon - 3P, \ \frac{1}{3} - c_s^2$ $\qquad c_s^2 = \frac{dP/dT}{d\epsilon/dT}$

Extreme limits

- weak coupling - perturbative QCD: $\frac{\zeta}{\eta} \propto \left(\frac{1}{3} - c_s^2\right)^2$ *Arnold, Dogan, Moore (2006)*
- strong (infinite) coupling – string theories: $\frac{\zeta}{\eta} \propto \left(\frac{1}{3} - c_s^2\right)$ *Buchel (2008)*

the lower bound: $\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - c_s^2\right)$

more in *Czajka et al (2018)*

What do we know about bulk viscosity?

First-principle approaches:

- Ansatz for the bulk viscosity spectral function was proposed and bulk viscosity was computed

  Karsch, Kharzeev, Tuchin (2008)

  - the ansatz misses high-frequency contributions and is not reliable

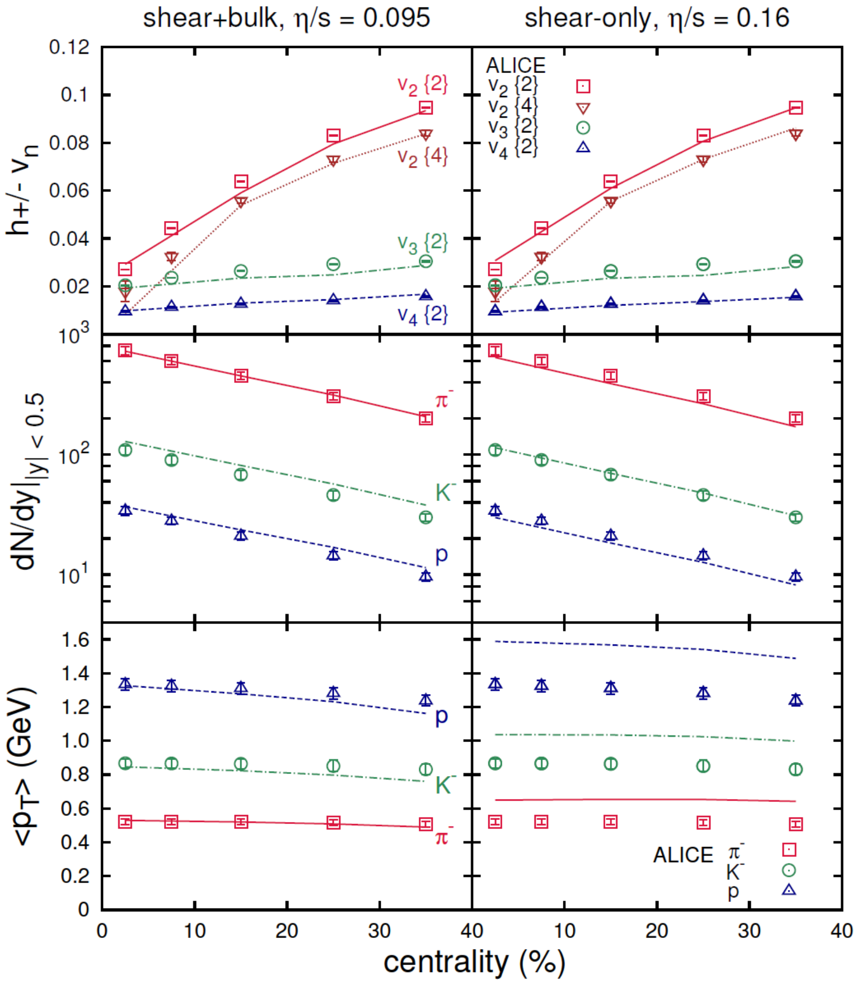
  Moore, Saremi (2008)

- Direct lattice computations – uncertaintities too large

Phenomenology:

Bulk viscosity over entropy density strongly peaked near critical temperature with $\zeta/s \simeq 0.3$ at the peak:
- simultaneous description of the multiplicity and the mean transverse momentum *Ryu et al (2015) and (2018)*
- relevant for IP-Glasma initial conditions

Schenke, Shen, Tribedy (2018)



Motivation

- need for more constraints on the physics induced by conformal anomaly
- need for consistent, well-defined equations of hydrodynamics, which include the effect of mean field and can provide reliable phenomenological estimates

  - ✓ QCD – complex, multiscale and multicomponent system
  - ✓ lack of systematic microscopic methods to compute the bulk viscosity near the phase transition region
  - ✓ **consequences of conformal anomaly can be understood by studying „simpler” systems in the perturbative regime where analytical methods can be applied**

Nonequilibrium vs. equilibrium - dictionary

The system under study is made of weakly interacting scalar particles, both classical and quantum statistics are considered

	Equilibrium (well defined state)	**Nonequilibrium** (small deviations, perturbative corrections to equilibrium quantities)
Quasiparticle thermal mass	$m_{\rm eq} \equiv m_{\rm eq}(x)$	$m_{\rm th} \equiv m_{\rm th}(x)$
Quasiparticle mass	$m_x = \sqrt{m_0^2 + m_{\rm eq}^2}$	$\tilde{m}_x = \sqrt{m_0^2 + m_{\rm th}^2}$
Quasiparticle energy	$E_k = \sqrt{\mathbf{k}^2 + m_x^2}$	$\mathcal{E}_k = \sqrt{\mathbf{k}^2 + \tilde{m}_x^2}$
Quasiparticle four-momentum	$k^\mu \equiv (k_0, \mathbf{k}) = (E_k, \mathbf{k})$	$\tilde{k}^\mu \equiv (\tilde{k}_0, \mathbf{k}) = (\mathcal{E}_k, \mathbf{k})$
Lorentz invariant measure	$dK = d^3\mathbf{k}/[(2\pi)^3 E_k]$	$d\mathcal{K} = d^3\mathbf{k}/[(2\pi)^3 \mathcal{E}_k]$
Distribution function	$f_0 = 1/[e^{\beta E_k} - 1]$	$f = f_0 + \Delta f$

Nonequilibrium deviation from the equilibrium distribution function

Boltzmann equation with the mean field contribution

$$(\tilde{k}^\mu \partial_\mu - \underbrace{\mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k}_{\text{term involving force}}) f = \underbrace{C[f]}_{\text{collision kernel}}$$

All quantities entering the equation are x-dependent

$$f(x,k) = \underbrace{f_{\rm th}(x,k)}_{\text{retains equilibrium form}} + \delta f(x,k) = f_0(x,k) + \delta f_{\rm th}(x,k) + \delta f(x,k) \qquad \Delta f(x,k) = \delta f_{\rm th}(x,k) + \delta f(x,k)$$

$$f_{\rm th}(x,k) \equiv f_0(x,k)|_{m_0^2+m_{\rm eq}^2(x)\to m_0^2+m_{\rm eq}^2(x)+\Delta m_{\rm th}^2(x)} = \left[\exp\left(\sqrt{\mathbf{k}^2+m_0^2+m_{\rm eq}^2(x)+\underbrace{\Delta m_{\rm th}^2(x)}_{\text{correction from the nonequilibrium thermal mass}}}\,\beta(x)\right)-1\right]^{-1}$$

$$\Delta f = \delta f - \beta f_0(1+f_0)\frac{\Delta m_{\rm th}^2}{2E_k}$$

Nonequilibrium deviation from the equilibrium distribution function

Equilibrium vs. nonequilibrium thermal mass

Equilibrium:

$$m_{\rm eq}^2 = \frac{\lambda(q_0)}{2} q_0$$

$$q_0 = \int dK f_0$$

Nonequilibrium:

$$m_{\rm th}^2(q) = m_{\rm th}^2(q_0 + \Delta q) = m_{\rm eq}^2(q_0) + \Delta m_{\rm th}^2 \qquad \Delta m_{\rm th}^2 = \frac{dm_{\rm eq}^2}{dq_0}\Delta q$$

$$q = \int d\mathcal{K} f \qquad \Delta q = \int dK \delta f + \frac{\partial q_0}{\partial m_{\rm eq}^2}\Delta m_{\rm th}^2$$

Solving self-consistently:

$$\Delta m_{\rm th}^2 = 2T^2 \frac{dm_{\rm eq}^2}{dT^2} \frac{\int dK \delta f}{\beta \int dK E_k f_0(1+f_0)}$$

Full form of the correction to the distribution function:

$$\Delta f = \delta f - T^2 \frac{dm_{\rm eq}^2}{dT^2} \frac{f_0(1+f_0)}{E_k} \frac{\int dK \delta f}{\int dK E_k f_0(1+f_0)}$$

Nonequilibrium deviation from the equilibrium distribution function

Thermal mass

For any theory with $m_{\rm eq}^2 = \mathcal{O}(\lambda T^2)$ $\lambda \ll 1$

$$m_{\rm eq}^2 = \frac{\lambda(q_0)}{2} q_0 \quad \Rightarrow \quad \frac{dm_{\rm eq}^2}{dT} = \frac{\lambda(q_0)}{2}\frac{dq_0}{dT} + \frac{q_0}{2}\frac{d\lambda(q_0)}{dT}$$

$$\beta_\lambda \equiv \beta(\lambda) = T\frac{d\lambda(q_0)}{dT} \qquad \beta_\lambda = \mathcal{O}(\lambda^2)$$

Temperature dependence of the thermal mass in weakly interacting systems:

$$T^2 \frac{dm_{\rm eq}^2}{dT^2} = m_{\rm eq}^2 + aT^2\beta_\lambda$$

Equations of hydrodynamics

Local equilibrium hydrodynamics

Energy-momentum tensor:

mean-field contribution

$$T_0^{\mu\nu} = \int dK k^\mu k^\nu f_0 - g^{\mu\nu} U_0$$

- thermodynamic consistency of hydrodynamic equations
- conservation of energy and momentum

$$dU_0 = \frac{q_0}{2} dm_{\rm eq}^2$$

$$T_0^{\mu\nu} = \epsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}$$

Energy density and pressure:

$$\epsilon_0 = \bar{\epsilon}_0 - U_0, \qquad \bar{\epsilon}_0 = \int dK (u_\mu k^\mu)^2 f_0$$

$$P_0 = \bar{P}_0 + U_0 \qquad \bar{P}_0 = -\frac{1}{3} \int dK \Delta^{\mu\nu} k_\mu k_\nu f_0$$

- Enthalpy not changed: $\bar{\epsilon}_0 + \bar{P}_0 = \epsilon_0 + P_0$
- Thermodynamic relation satisfied: $Ts_0 = T\frac{dP_0}{dT} = \epsilon_0 + P_0$

Equations of hydrodynamics

Nonequilibrium hydrodynamics

Energy-momentum tensor:

Non-equilibrium mean-field contribution

$$T^{\mu\nu} = \int d\mathcal{K} \tilde{k}^{\mu} \tilde{k}^{\nu} f - g^{\mu\nu} U$$

$$U = U_0 + \Delta U \qquad \Delta U = \frac{q_0}{2} \Delta m_{\mathrm{th}}^2$$

All quantities contain nonequilibrium thermal mass correction

$$T^{\mu\nu} = T_0^{\mu\nu} + \Delta T^{\mu\nu}$$

Particular components:

$$\Delta T^{00} = \int dK E_k^2 \Delta f$$

$$\Delta T^{0i} = \int dK E_k k^i \Delta f$$

$$\Delta T^{ij} = \int dK k^i k^j \Delta f - \frac{\Delta m_{\mathrm{th}}^2}{2} \int dK \frac{k^i k^j}{E_k^2} f_0 + \delta^{ij} \frac{\Delta m_{\mathrm{th}}^2}{2} \int dK f_0$$

Equations of hydrodynamics

Nonequilibrium hydrodynamics – local rest frame

Landau matching is defined by the eigenvalue problem $u_\mu T^{\mu\nu} = \epsilon u^\nu$

Local rest frame:

$$u^\mu = (1, 0, 0, 0) \quad \Rightarrow \quad T^{00} = \epsilon, \quad T^{0i} = 0 \quad \Rightarrow \quad \Delta T^{00} = 0, \quad \Delta T^{0i} = 0$$

Landau matching conditions:

$$\int dK E_k k^i \delta f = 0$$

$$\int dK \left[E_k^2 - T^2 \frac{dm_{\mathrm{eq}}^2}{dT^2} \right] \delta f = 0$$

Contains the medium correction

Viscous corrections:

$$\Delta T^{ij} = \int dK k^i k^j \delta f \quad \Rightarrow \quad \pi^{ij} = \int dK k^{\langle i} k^{j \rangle} \delta f, \quad \Pi = \frac{1}{3} \int dK \mathbf{k}^2 \delta f$$

Known structures but x-dependent mass enters the equations

Equations of hydrodynamics

Nonequilibrium hydrodynamics – general frame

Energy-momentum tensor:

$$T^{\mu\nu} = \int dK k^\mu k^\nu f_0 - g^{\mu\nu} U_0 + \int dK \left[k^\mu k^\nu - u^\mu u^\nu T^2 \frac{dm_{\rm eq}^2}{dT^2} \right] \delta f$$

Landau matching condition:

$$\int dK \left[(u_\mu k^\mu) k^\nu - u^\nu T^2 \frac{dm_{\rm eq}^2}{dT^2} \right] \delta f = 0$$

Viscous corrections:

$$\pi^{\mu\nu} = \int dK k^{\langle\mu} k^{\nu\rangle} \delta f$$

$$\Pi = -\frac{1}{3} \int dK \Delta_{\mu\nu} k^\mu k^\nu \delta f$$

$$A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$$

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - 2/3 \Delta^{\mu\nu} \Delta_{\alpha\beta})/2$$

On the way to compute bulk viscosity (and other transport coefficients)

Anderson-Witting model

Boltzmann equation in the Anderson-Witting model

$$\left(\tilde{k}^\mu \partial_\mu - \mathcal{E}_k \nabla \mathcal{E}_k \cdot \nabla_k\right) f = -\frac{(u \cdot \tilde{k})}{\tau_R} \Delta f$$

LHS of the Boltzmann equation dictates the form of RHS

$$\delta f(k) = f_0(k)(1 + f_0(k))\phi(k)$$

$$\Delta f(k) = f_0(k)(1 + f_0(k)) \left(\phi(k) - \frac{T^2}{E_k} \frac{dm_{\rm eq}^2}{dT^2} \frac{\int dK \phi(k) f_0(k)(1 + f_0(k))}{\int dK E_k f_0(k)(1 + f_0(k))}\right)$$

$\phi = \phi_{\rm s} + \phi_{\rm b}$ (shear part + bulk part)

Transport coefficients

Anderson-Witting model: shear viscosity

Solution of the A-W model for the shear part: $\phi_{\rm s}(k) = -\frac{\tau_R}{TE_k} k^{\langle j} k^{i\rangle} \partial_j u_i$ $\quad \delta f = f_0(1+f_0)\phi$

Shear viscosity can be computed using: $\pi^{ij} = 2\eta\sigma^{ij}$ $\quad \pi^{ij} = \int dK k^{\langle i} k^{j\rangle} \delta f$

$$\frac{\eta}{\tau_R} = \beta J_{3,2} \qquad \frac{\eta}{\tau_R} = \frac{\epsilon_0 + P_0}{5}$$

Shear viscosity is not influenced by the mean field in the leading order

For quantum gas: $J_{n,q} = a_q \int dK (u\cdot k)^{n-2q} (-\Delta_{\mu\nu} k^\mu k^\nu)^q \, f_0(k)(1+f_0(k))$

For classical gas: $J_{n,q} \to I_{n,q}$ $\quad I_{n,q} = a_q \int dK (u\cdot k)^{n-2q} (-\Delta_{\mu\nu} k^\mu k^\nu)^q f_{0,c}(k)$

Transport coefficients

Anderson-Witting model: bulk viscosity

Solution of the A-W model for the bulk part:

$$\phi_{\rm b}(k) = \beta\tau_R(\partial_i u^i)(c_s^2 - 1/3)\left(E_k - \frac{1}{E_k}\frac{J_{3,0} - T^2(dm_{\rm eq}^2/dT^2)J_{1,0}}{J_{1,0} - T^2(dm_{\rm eq}^2/dT^2)J_{-1,0}}\right)$$

$$\delta f = f_0(1+f_0)\phi$$

Bulk viscosity can be computed using: $\Pi = M\int dK\delta f$ and $\Pi = -\zeta\partial_i u^i$

Nonconformality parameter:

Microscopic: $M = -\frac{1}{3}\left(m_0^2 - a\beta_\lambda T^2\right)$

the consequence of mean field corrections

Macroscopic: $c_s^2 = \frac{dP_0/dT}{d\epsilon_0/dT}$ ➡ $\frac{1}{3} - c_s^2 = -\frac{MJ_{1,0}}{J_{3,0} - T^2(dm_{\rm eq}^2/dT^2)J_{1,0}}$

Transport coefficients

Anderson-Witting model: bulk viscosity

Bulk viscosity of the Boltzmann (classical) gas:

$$\frac{\zeta_{\text{Boltz}}}{\tau_R} \propto T^4 \left(\frac{1}{3} - c_s^2\right)^2$$

Bulk viscosity of the Bose-Einstein (quantum) gas:

$$\frac{\zeta}{\tau_R} \propto T^4 \left(\frac{1}{3} - c_s^2\right)^2 \underbrace{\frac{T}{m_x}}$$

Effect of the cut-off of infrared divergencies

Relaxation time approximation can be too crude
to obtain a reliable form of bulk viscosity

Transport coefficients

14-moment approximation

One deals with moments of the distribution function:

- 14-moment approximation – unknown moments are expressed in terms of Π and $\pi^{\mu\nu}$

$$\int dK (u^\alpha k_\alpha)^n \delta f \;\; \rightarrow \;\; \gamma_n^{(0)} \Pi$$

$$\int dK (u^\alpha k_\alpha)^n k^{\langle\mu} k^{\nu\rangle} \delta f \;\; \rightarrow \;\; \gamma_n^{(2)} \pi^{\mu\nu}$$

$\gamma_n^{(0)},\ \gamma_n^{(2)}$ - combinations of thermal integrals

- Equation of motion for Π

$$\Pi = \tilde{M} \int dK\, \Delta f \longrightarrow u^\mu \partial_\mu \Pi = u^\mu \partial_\mu \left[\tilde{M} \int dK\, \Delta f \right]$$

$$\tilde{M} = M \frac{J_{1,0}}{J_{1,0} - T^2 (dm_{\rm eq}^2/dT^2) J_{-1,0}}$$

The structure of EoM:

$$\dot{\Pi} + \frac{\Pi}{\tau_R} = -\frac{\zeta}{\tau_R}\theta - \frac{\delta_{\Pi\Pi}}{\tau_R}\theta\Pi + \frac{\lambda_{\Pi\pi}}{\tau_R}\pi^{\mu\nu}\sigma_{\mu\nu}$$

ζ/τ_R: the same as in the A-W model

$\delta_{\Pi\Pi}/\tau_R$: affected by higher powers of T/m_x

$\lambda_{\Pi\pi}/\tau_R$: $\propto 1/3 - c_s^2$

Conclusions

- The form of the nonequilibrium correction to the distribution function found
- Fully consistent incorporation of thermal mean field in the hydrodynamical description of the dynamics of one-component systems
- The physics of bulk viscosity studied for the Boltzmann and Bose-Einstein gases
- Bulk viscosity is of the expected parametric form for the classical gas in the relaxation time approximation
- Relaxation time approximation can be too crude to study bulk viscosity of the quantum gases with Bose-Einstein distribution