Maximally Supersymmetric Gauge Theories: New Theoretical Playground

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in collaboration with L. Bork, A. Borlakov, M. Kompaniets, D.Tolkachev and D.Vlasenko

Motivation

Maximal SYM

Theories in D-dimensions where maximal possible number of super symmetries in realized

D=4 N=4
D=6 N=2
D=8 N=1
D=10 N=1

It is believed that these theories possess distinguished properties:

- **integrability,**
- **exact solutions,**
- **can provide break through into non-perturbative phenomena,**
- **can solve the problem of UV divergences in quantum gravity**

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All of them can be obtained from 10dim superstring by compactification on a torus

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- On-shell finite up to 8 loops
- Similar to higher dim SYM

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- **Similar to higher dim SYM**

Study of higher dim SYM gives insight into quantum gravity

Motivation

Maximal SYM

D=4 N=4
D=6 N=2
D=8 N=1
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- **Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)**
- **First UV divergent diagrams at L=6/(D-4)**
- **Conformal or dual conformal symmetry**
- **Common structure of the integrands**

Bern, Dixon &Co 10
Drummond, Henn, Korchemsky, Sokatchev 10
Arkani-Hammed 12

Object: Helicity Amplitudes on mass shell with arbitrary number of legs and loops

The case: Planar limit $N_c \to \infty,\ g^2_{YM} \to 0$ and $g^2_{YM} N_c$ - fixed

The aim: to get all loop (exact) result

New approach to gauge theories

Spinor-helicity formalism: S-matrix elements

Any light-like vector $p^2_{(i)} = 0$ **can be presented in the form**

Rev. in
BernDixonKosower 96

$$p^{(i)}_\mu \mapsto (\sigma^\mu)_{\alpha\dot{\alpha}} p^{(i)}_\mu = \lambda^{(i)}_\alpha \tilde{\lambda}^{(i)}_{\dot{\alpha}} \qquad \lambda_\alpha \in SL(2,\mathbb{C})$$

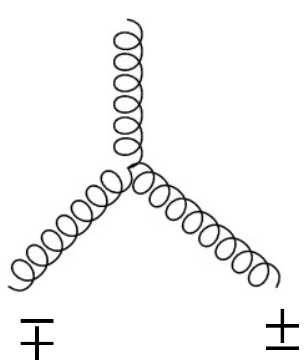
$$\epsilon^{\alpha\beta}\lambda^{(i)}_\alpha \lambda^{(j)}_\beta \equiv \langle ij\rangle = \sqrt{(p_i+p_j)^2}e^{i\phi_{ij}} = \sqrt{s_{ij}}e^{i\phi_{ij}},\ \phi_{ij}\in\mathbb{R} \quad (\langle ij\rangle)^* \equiv [ij]$$

Solutions to massless Dirac equation

Polarization vectors

Amplitudes

$$\epsilon^+_\mu(p) \mapsto \epsilon^+_{\alpha\dot{\alpha}}(p) = \frac{\lambda^k_\alpha \tilde{\lambda}^p_{\dot{\alpha}}}{\sqrt{2}\langle kp\rangle} \qquad \epsilon^-_\mu(p) \mapsto \epsilon^-_{\alpha\dot{\alpha}}(p) = \frac{\lambda^p_\alpha \tilde{\lambda}^k_{\dot{\alpha}}}{\sqrt{2}[kp]}$$



$p_i \in \mathbb{C}$

$$A_3(g_1^- g_1^- g_3^+) \sim \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

$$A_3(g_1^+ g_1^+ g_3^-) \sim \frac{[12]^4}{[12][23][31]}$$

Classification

$$\mathrm{MHV}_n = \lambda_{tot} = n-4$$

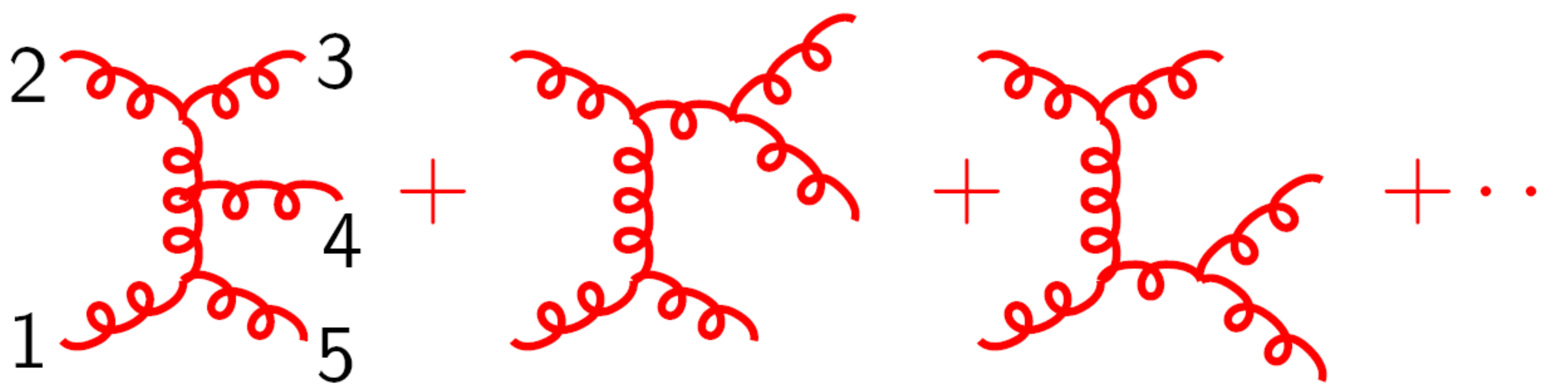
$$\mathrm{N}^k\mathrm{MHV}_n = \lambda_{tot} = n-2k$$

There is no need in Faddeev-Popov ghosts, gauge fixing, BRST, Batalin-Vilkovistsky formalism,etc in this approach!

Tree-level example: Five gluons

Force carriers in QCD are gluons. Similar to photons of QED except they self interact.

Consider the five-gluon amplitude:



Used in calculation of *pp* ⟶ 3 jets at CERN

If you evaluate this following textbook Feynman rules you find…

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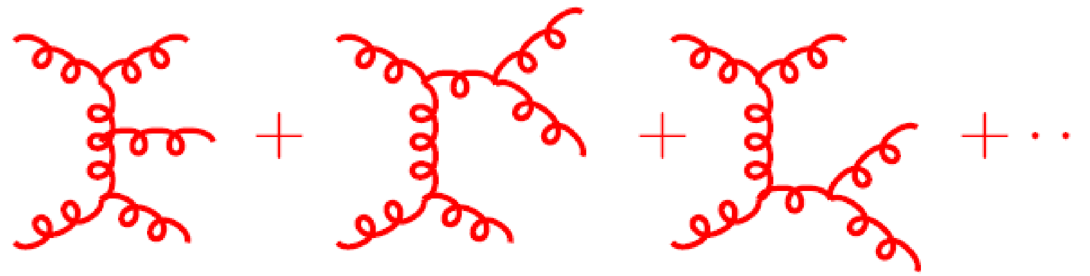
Result of evaluation (actually only a small part of it):

$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

Messy combination of momenta and gluon polarization vectors.

Reconsider Five-Gluon Tree



With a little Chinese magic, i.e. helicity states:

Xu, Zhang and Chang
and many others

$$A_5^{\text{tree}}(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0$$

$$A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$A_5^{\text{tree}}(1^-, 2^+, 3^-, 4^+, 5^+) = i\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

Use a better organization of color charges:

$$\mathcal{A}_5 = \sum_{\text{perms}} \text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})\, A_5(1,2,3,4,5)$$

Motivated by the color organization of open string amplitudes.

Mangano and Parke

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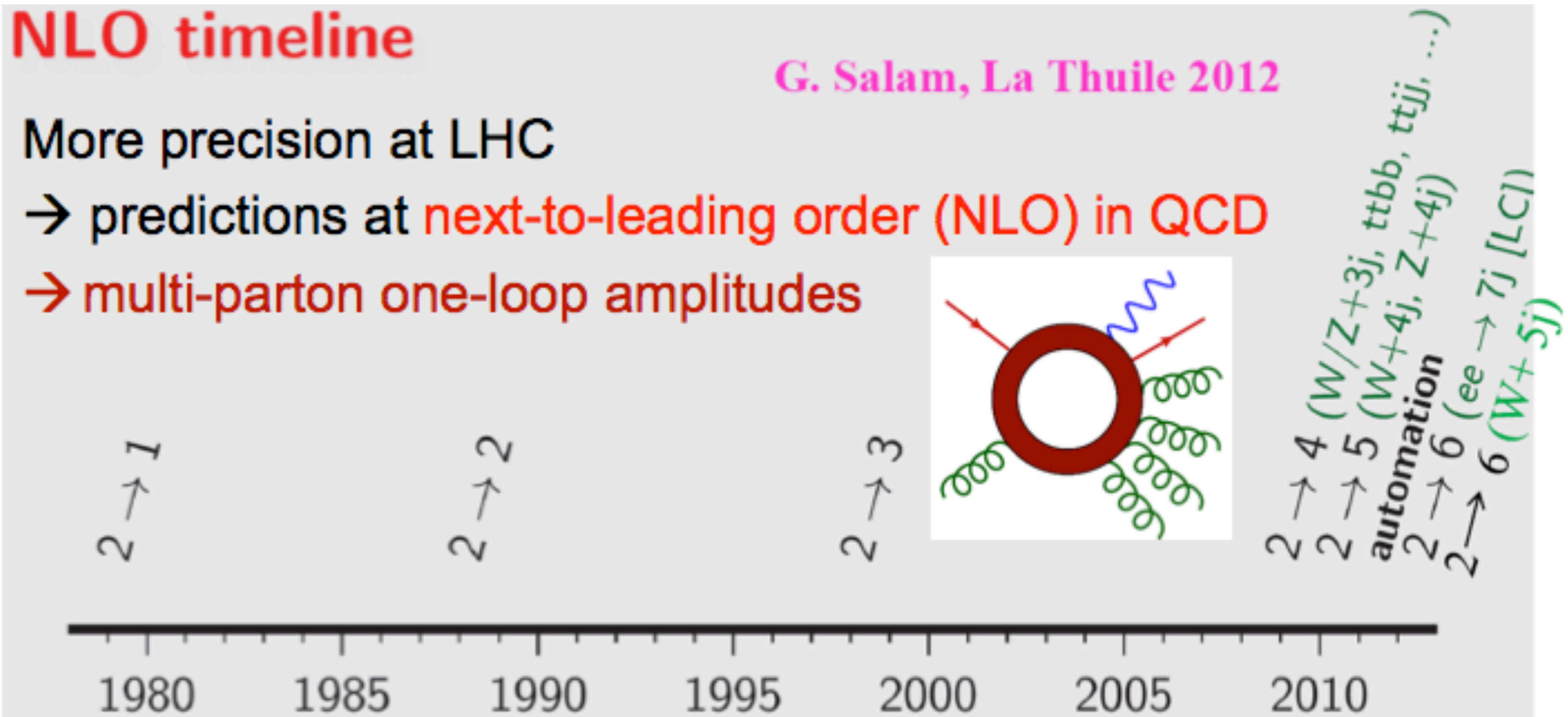
Recent progress in multi-leg FD calculations



Recent progress in multi-leg FD calculations

NLO timeline

G. Salam, La Thuile 2012

More precision at LHC

→ prec

→ mult

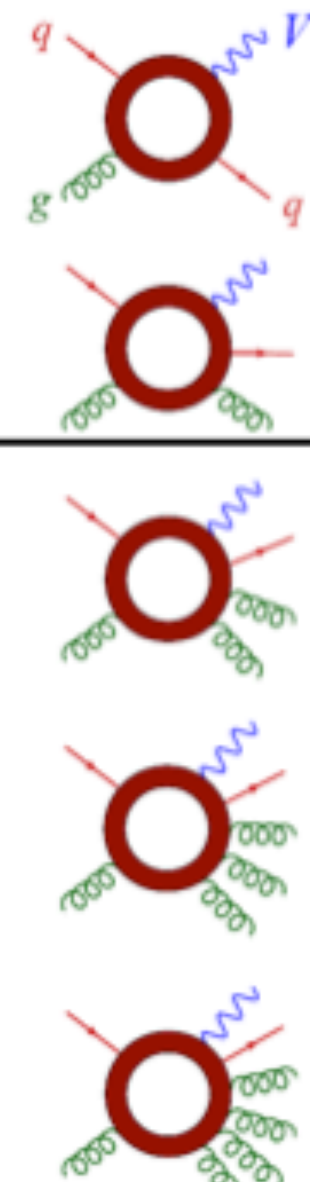
2 → 1

1980

b, ttjj, ...)

***pp* → *W* + *n* jets** (just amplitudes with most gluons)

# of jets	# 1-loop Feynman diagrams
1	11
2	110
3	1,253
4	16,648
5	256,265



Current limit with Feynman diagrams (below 2 jets)

Current limit with on-shell methods (below 5 jets)

UV divergences in all Loops

D=4 N=4	**No UV div**	**IR div on shell**
D=6 N=2	**UV div from 3 loops**	**No IR div**
D=8 N=1	**UV div from 1 loop**	**No IR div**
D=10 N=1	**UV div from 1 loop**	**No IR div**

All these theories are non-renormalizable by power counting

The coupling g^2 **has dimension** $[g^2] = \frac{1}{M^{D-4}}$

<u>The aim</u>: to get all loop (exact) result for the leading (at least) divs

Colour decomposition

Colour ordered amplitude

$$\mathcal{A}_n^{a_1\ldots a_n}(p_1^{\lambda_1}\ldots p_n^{\lambda_n}) = \sum_{\sigma\in S_n/Z_n} Tr[\sigma(T^{a_1}\ldots T^{a_n})]A_n(\sigma(p_1^{\lambda_1}\ldots p_n^{\lambda_n})) + \mathcal{O}(1/N_c)$$

This is what we calculate

Planar Limit $N_c \to \infty$, $g^2_{YM} \to 0$ and $g^2_{YM} N_c$ - fixed

Four-point amplitude

$$A_4^{(1),\mathrm{phys.}}(1,2,3,4) = T^1 A_4^{(0)}(1,2,3,4) M^{(1)}(s,t) + T^2 A_4^{(0)}(1,2,4,3) M^{(1)}(s,u) + T^3 A_4^{(0)}(1,4,2,3) M^{(1)}(t,u).$$

$$T^1 = \mathrm{Tr}(T^{a1}T^{a2}T^{a3}T^{a4})+\mathrm{Tr}(T^{a1}T^{a4}T^{a3}T^{a2}),$$
$$T^2 = \mathrm{Tr}(T^{a1}T^{a2}T^{a4}T^{a3})+\mathrm{Tr}(T^{a1}T^{a3}T^{a4}T^{a2}),$$
$$T^3 = \mathrm{Tr}(T^{a1}T^{a4}T^{a2}T^{a3})+\mathrm{Tr}(T^{a1}T^{a3}T^{a2}T^{a4})$$

Tree level amplitude usually has a simple universal form proportional to the delta function (conservation of momenta), in SUSY case - conservation of supercharge in on shell momentum superspace

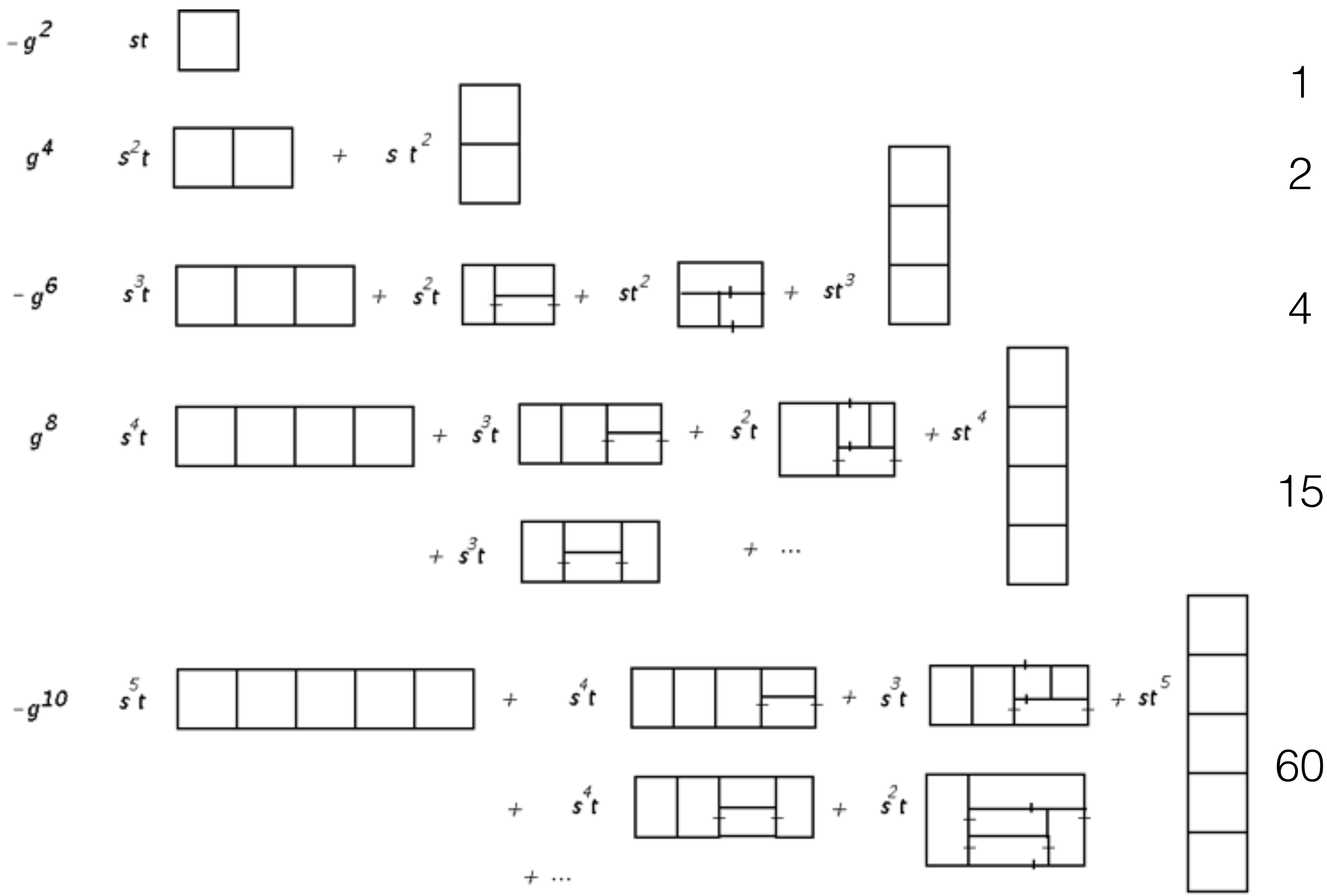
Perturbation Expansion for the Amplitudes for any D

A_4/A_4^{tree}

No bubbles
No Triangles

First UV div at
L=[6/(D-4)] loops

IR finite



T. Dennen Yu-yin Huang 10 ,
S.Caron-Huot D.O'Connell 10

Universal expansion for any D in maximal SYM due to Dual conformal invariance

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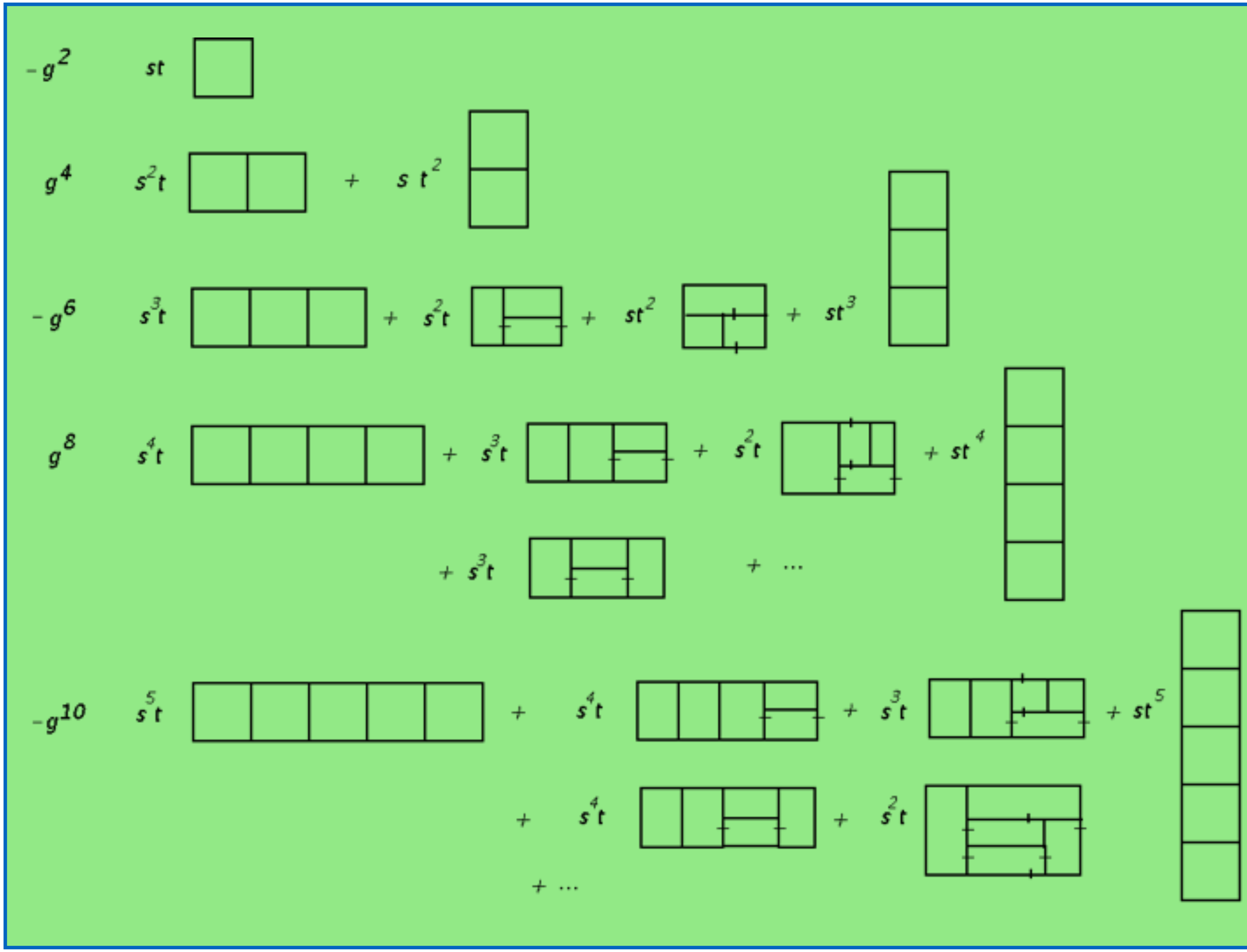

1
2
4
15
60

T. Dennen Yu-yin Huang 10 ,
S.Caron-Huot D.O'Connell 10

Universal expansion for any D in maximal SYM due to Dual conformal invariance

Leading Divergences from Generalized «Renormalization Group»

- **In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of** $1/\epsilon^n$ **in n loops in given by**

$$a_n^{(n)} = (a_1^{(1)})^n$$

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- **In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation**

$$\mathcal{R}'G = 1 - \sum_{\gamma} K\mathcal{R}'_{\gamma} + \sum_{\gamma,\gamma'} K\mathcal{R}'_{\gamma} K\mathcal{R}'_{\gamma'} - ...,$$

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$$\mathcal{R}'G = 1 - \sum_{\gamma} K\mathcal{R}'_{\gamma} + \sum_{\gamma,\gamma'} K\mathcal{R}'_{\gamma} K\mathcal{R}'_{\gamma'} - ...,$$

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + ... + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$
$$+ \frac{B_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^{n-1}} + \frac{B_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-1}} + ... + \frac{B_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^{n-1}}$$
$$+\text{lower order terms}$$

Leading pole

SubLeading pole

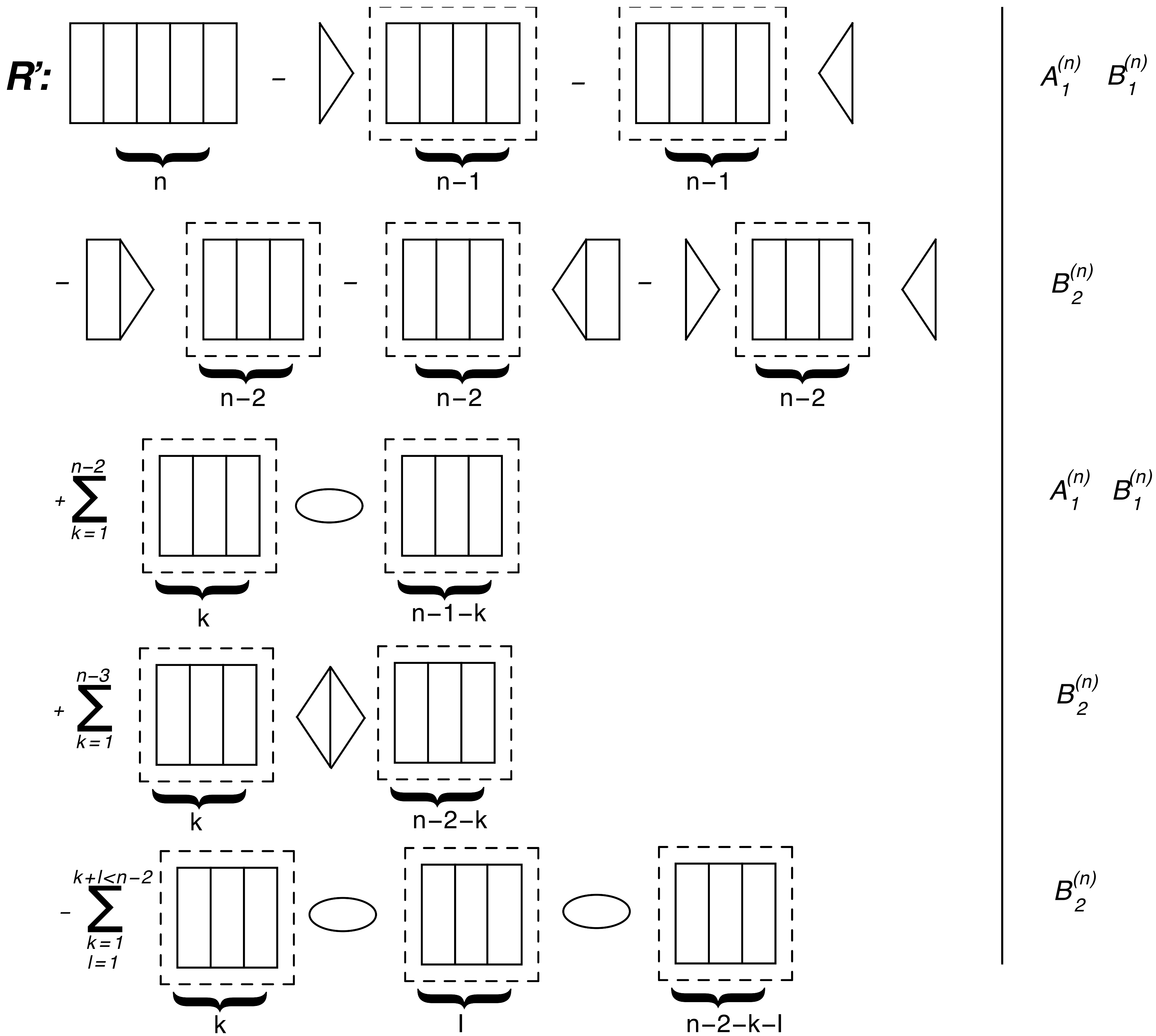
$A_1^{(n)},\ B_1^{(n)}$ **1-loop graph**

$B_2^{(n)}$ **2-loop graph**

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



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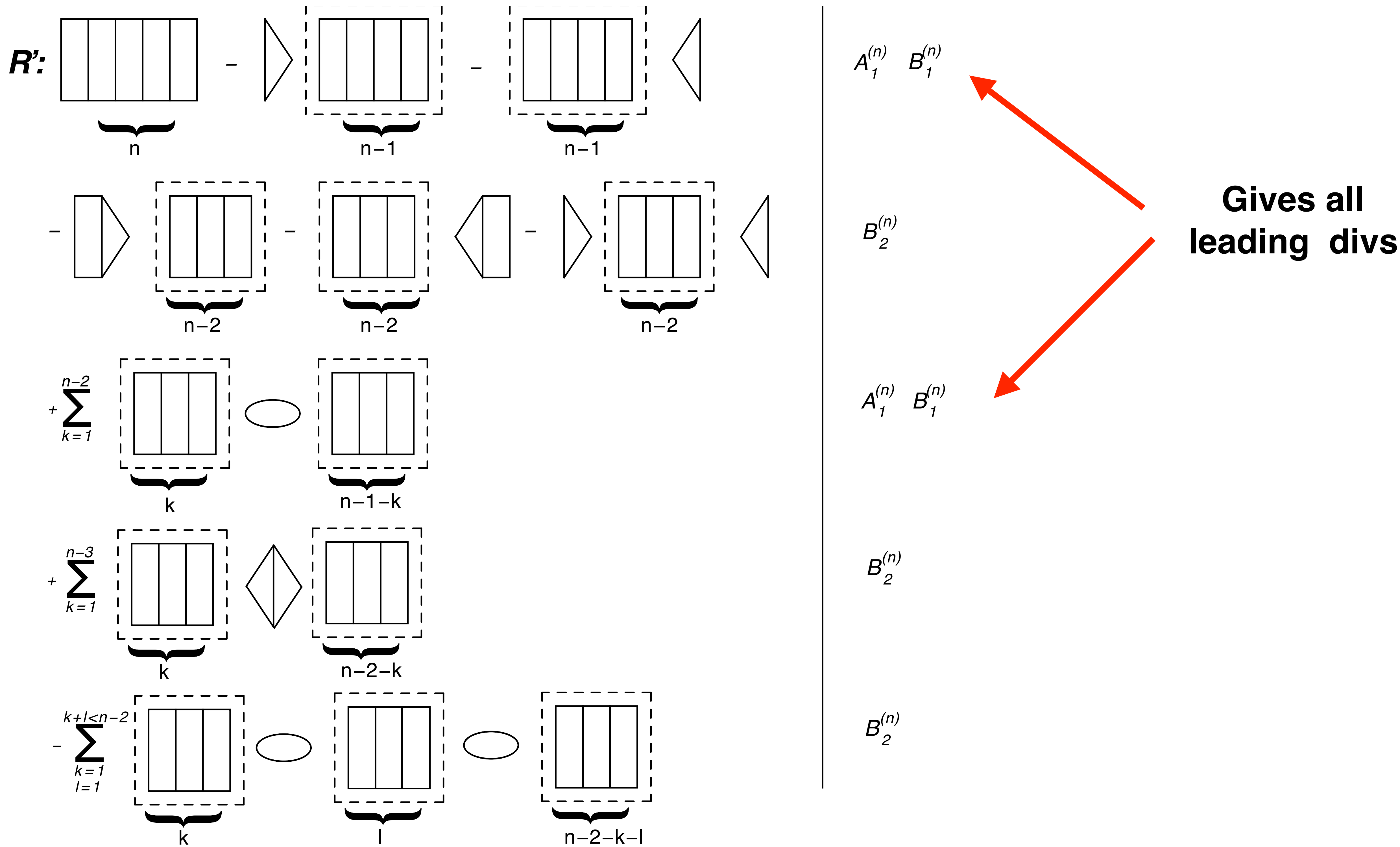

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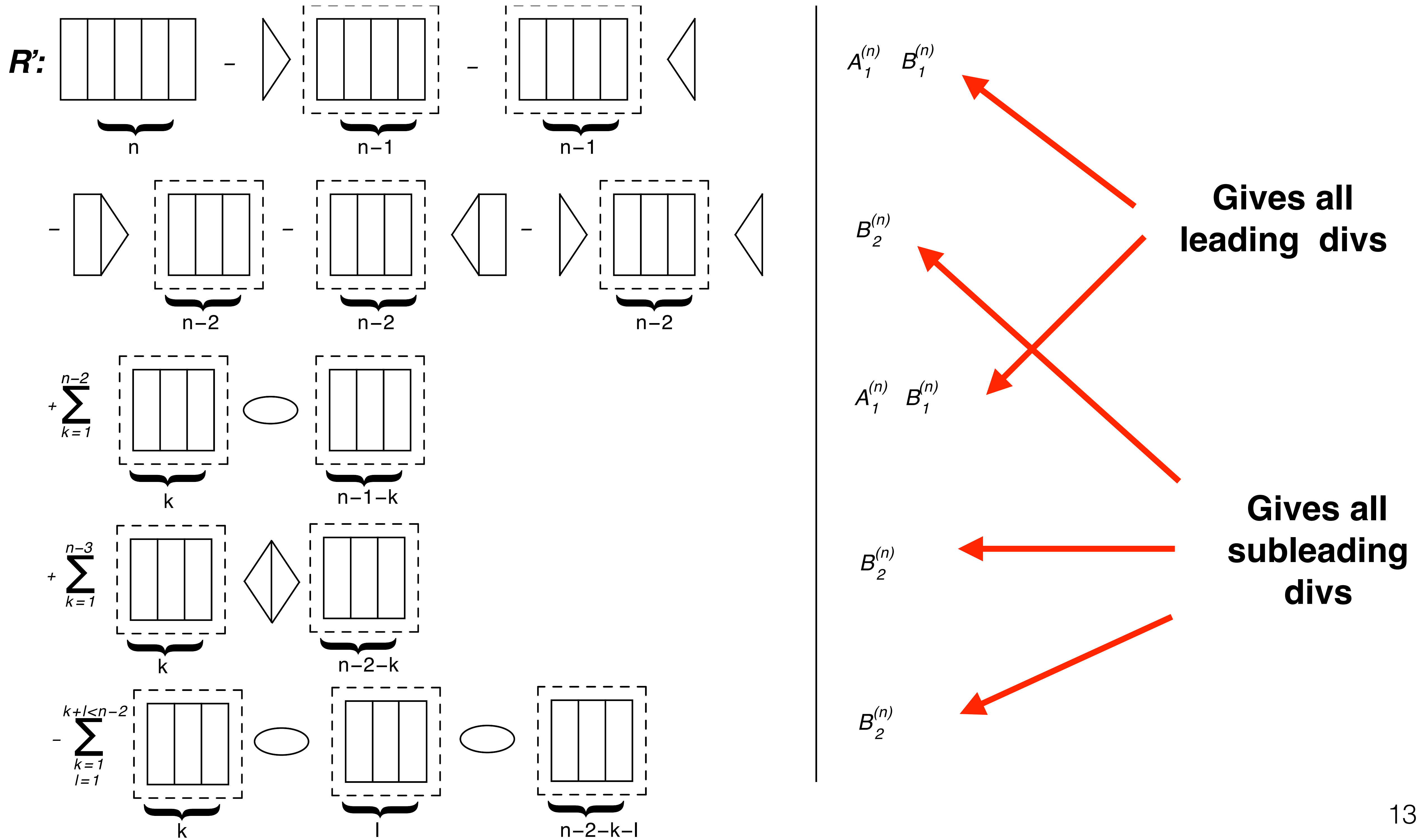

Perturbation Expansion for the Amplitudes

D=6 N=2

Result up to 5 loops

Leading Divergences

$$L.P. = 2stg^4\left[g^2\frac{s+t}{6\epsilon} + g^4\frac{s^2+st+t^2}{36\epsilon^2} + g^6\frac{s^3+\frac{2}{5}s^2t+\frac{2}{5}st^2+t^3}{216\epsilon^3}\right]$$

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Geom progression !?

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Leading powers of s > 0

$$\sum_{n=1}^{\infty}\left(\frac{g^2 s}{6\epsilon}\right)^n = \frac{\frac{g^2 s}{6\epsilon}}{1-\frac{g^2 s}{6\epsilon}}$$

Pole!

$\epsilon \to +0$

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Compare D=4 YM

$$g^2 = \frac{g_B^2}{1-\frac{11C_2}{3}\frac{g_B^2}{\epsilon}}$$

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General case will be given below

Perturbation Expansion for the Amplitudes

D=8 N=1 **Leading Divergences** **Result up to 4 loops**

$$L.P. = -st\Big[g^2\frac{1}{3!\epsilon} + g^4\frac{s^2+t^2}{3!4!\epsilon^2} + g^6\frac{4}{3}\frac{15s^4 - s^3t + s^2t^2 - st^3 + 15t^4}{3!4!5!\epsilon^3}$$

$$+ \quad g^8\frac{1}{63}\frac{16770s^6 - 536s^5t + 412s^4t^2 - 384s^3t^3 + 412s^2t^4 - 536st^5 + 16770t^6}{3!4!5!6!\epsilon^4}\Big].$$

Perturbation Expansion for the Amplitudes

D=8 N=1

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D=10 N=1

Leading Divergences **Result up to 4 loops**

$$L.P. = -st\left[g^2\frac{s+t}{5!\epsilon} + g^4\frac{8s^4 + 2s^3t + 2st^3 + 8t^4}{5!7!\epsilon^2}\right.$$
$$+ \quad g^6\frac{2(2095s^7 + 115s^6t + 33s^5t^2 - 11s^4t^3 - 11s^3t^4 + 33s^2t^5 + 115st^6 + 2095t^7)}{5!7!7!45\epsilon^3}$$
$$+ \quad g^8\frac{32(211218880s^{10} + 753490s^9t - 1395096s^8t^2 + 1125763s^7t^3 - 916916s^6t^4}{13!7!7!5!5\epsilon^4}$$
$$\left.\frac{+843630s^5t^5 - 916916s^4t^6 + 1125763s^3t^7 - 1395096s^2t^8 + 753490st^9 + 211218880t^{10})}{13!7!7!5!5\epsilon^4}\right].$$

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D=8 N=1

Leading Divergences

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$$L.P. = -st\left[g^2\frac{1}{3!\epsilon} + g^4\frac{s^2+t^2}{3!4!\epsilon^2} + g^6\frac{4}{3}\frac{15s^4 - s^3t + s^2t^2 - st^3 + 15t^4}{3!4!5!\epsilon^3}\right.$$
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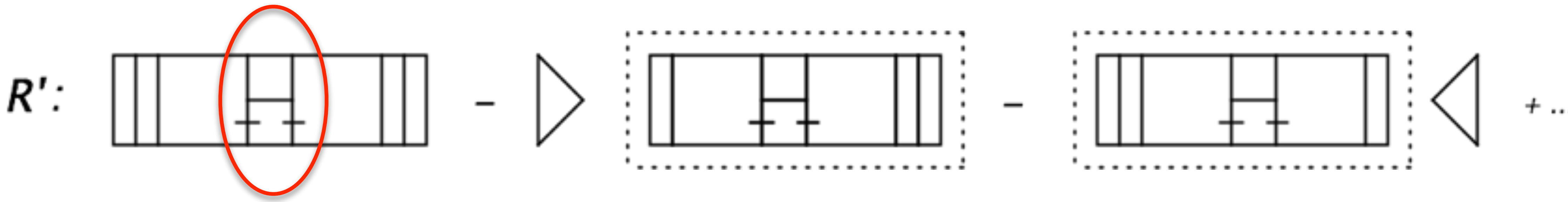
Doesn't look like Geom progression anymore, however, coefficients grow slowly

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



A_n $\qquad\qquad$ A_{n-1}

$nA_n = -A_{n-1}$ → $A_n = (-1)^n \frac{2}{n!}$ $\qquad (-g^2 s)^n$

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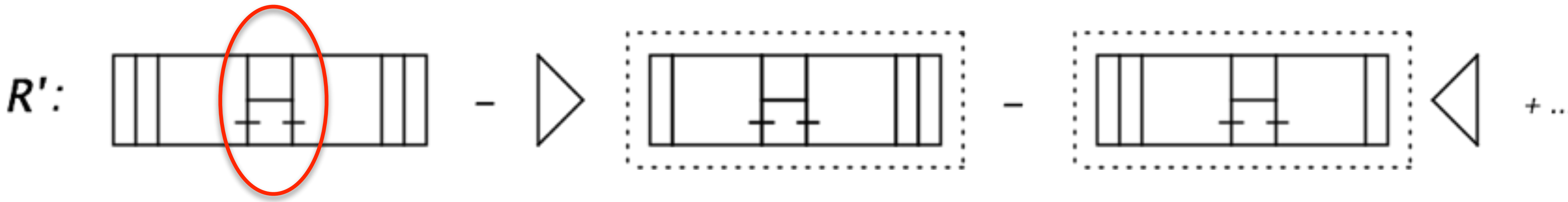
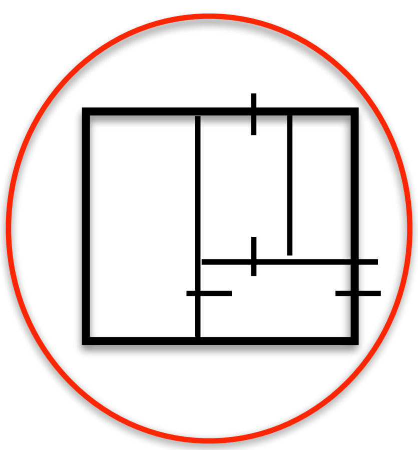


A_n A_{n-1}

$$nA_n = -A_{n-1} \quad \longrightarrow \quad A_n = (-1)^n \frac{2}{n!} \qquad (-g^2 s)^n$$

Horizontal boxes + double tennis court



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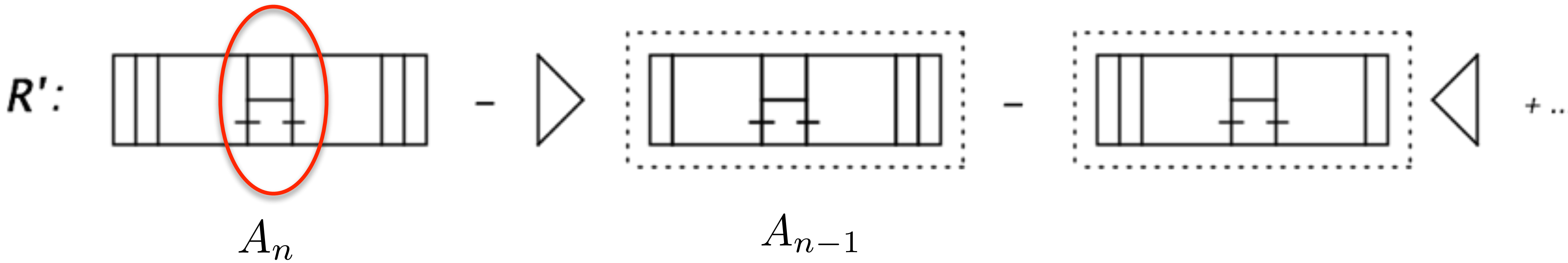
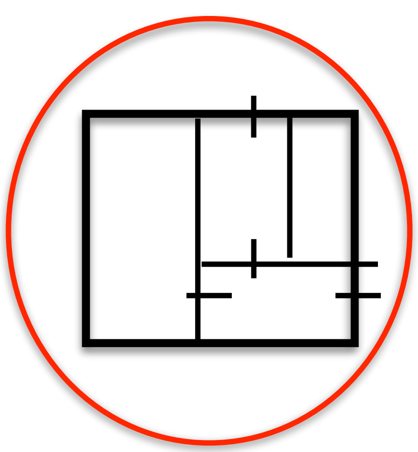
Horizontal boxes + tennis court



$$nA_n = -A_{n-1} \quad \rightarrow \quad A_n = (-1)^n \frac{2}{n!} \qquad (-g^2 s)^n$$

Horizontal boxes + double tennis court

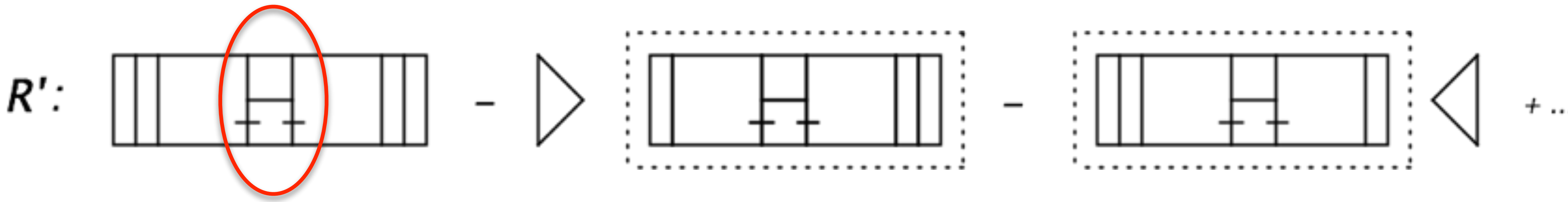


$$nA_n^t = -\frac{1}{3}A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3}A_{n-1}^t$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



A_n $\qquad$ A_{n-1}

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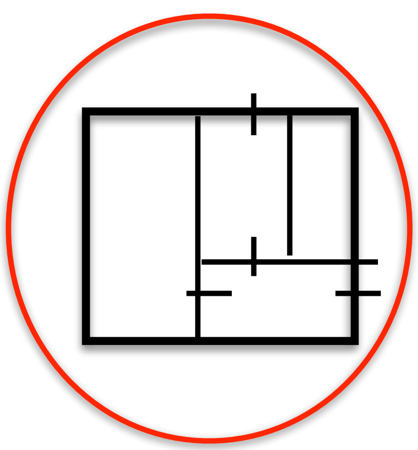


$$nA_n^t = -\frac{1}{3}A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3}A_{n-1}^t$$

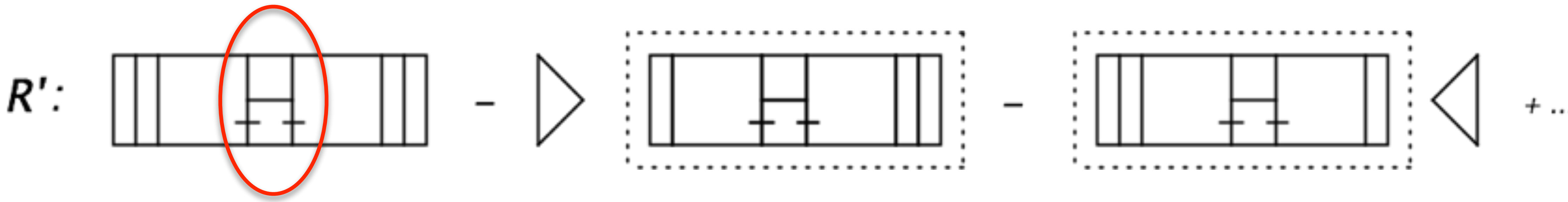
$$\downarrow$$

$$A_n^t = \frac{(-1)^n}{3^{n-3}}\frac{1}{n!}, \qquad A_n^s = \frac{1}{2}\frac{(-1)^n}{3^{n-3}}\frac{1}{n!} - \frac{1}{2}(-1)^n\frac{1}{n!}$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



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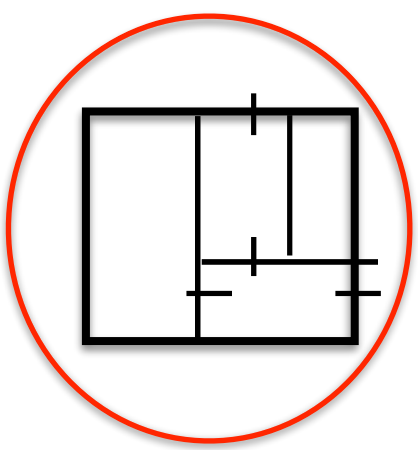

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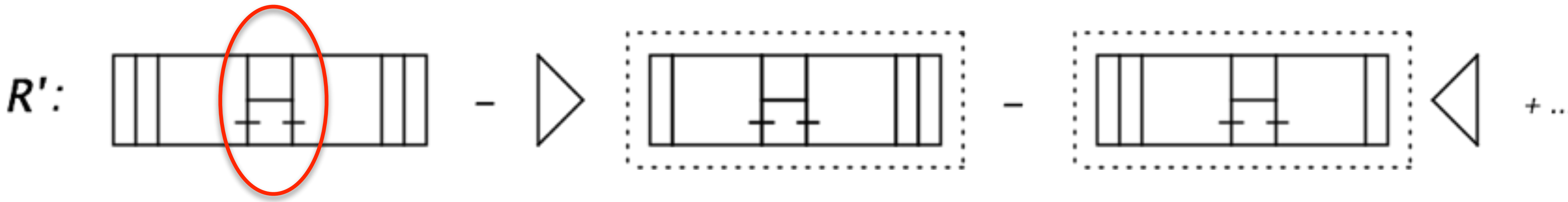
R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



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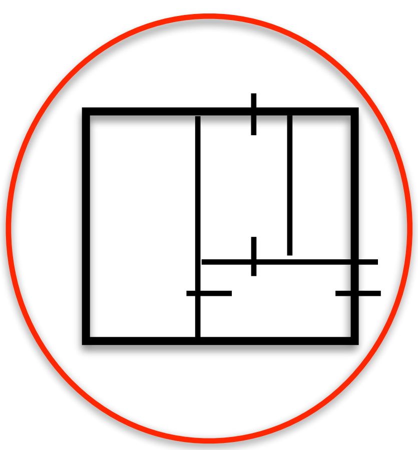

Horizontal boxes + double tennis court

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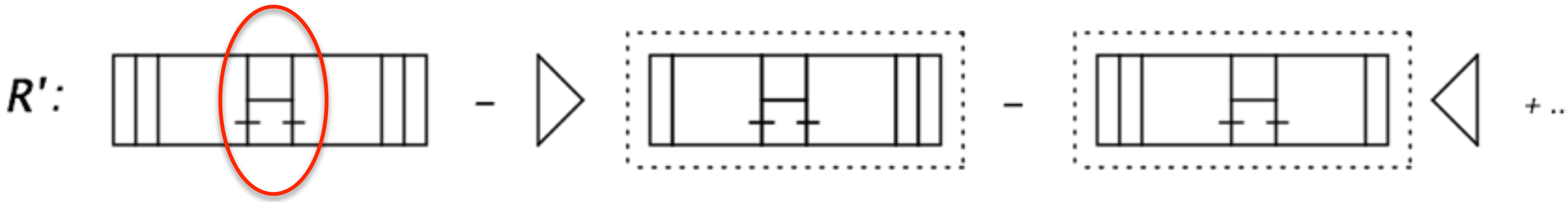
$$(-g^2 s)^{n-1} (-g^2 t) \qquad (-g^2 s)^n$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



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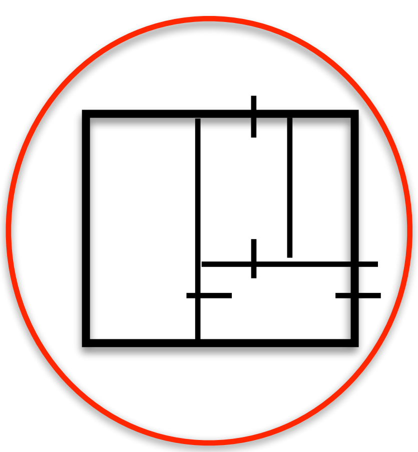

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$$(-g^2 s)^{n-1}(-g^2 t) \qquad (-g^2 s)^n$$

- **Similar relations one can get for all other series**
- **All of them have 1/n! behavior**
- **Number of these series group as n!**

Ladder diagrams (leading divs)

D=8 N=1

Horizontal boxes

$$A_n^{(n)} = s^{n-1} A_n$$

$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!}\sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3$$

$$A_1 = 1/6$$

1 loop box

Ladder diagrams (leading divs)

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Summation $$\Sigma_m(z) = \sum_{n=m}^{\infty} A_n(-z)^n$$

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$$-\frac{d}{dz}\Sigma_3 = -\frac{2}{4!}\Sigma_2 + \frac{2}{5!}\Sigma_1\Sigma_1.$$

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$\Sigma_A \equiv \Sigma_1$

Diff eqn

$$\frac{d}{dz}\Sigma_A = -\frac{1}{3!} + \frac{2}{4!}\Sigma_A - \frac{2}{5!}\Sigma_A^2$$

$z = g^2 s^2/\epsilon$

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$$\Sigma_A(z) = -\sqrt{5/3}\frac{4\tan(z/(8\sqrt{15}))}{1 - \tan(z/(8\sqrt{15}))\sqrt{5/3}} = \sqrt{10}\frac{\sin(z/(8\sqrt{15}))}{\sin(z/(8\sqrt{15}) - z_0)}$$

$$\Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \dots)$$

$$z_0 = \arcsin(\sqrt{3/8})$$

All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy \; (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$$n \geq 4$$

$$t' = t(x-y) - sy$$

$$S_3 = -s/3, \;\; T_3 = -t/3$$

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$$\Sigma_4(s,t,z) = \Sigma_3(s,t,z) + S_3(s,t)z^3 \qquad \Sigma(s,t,z) = z^{-2}\Sigma_3(s,t,z)$$

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$$\frac{d}{dz}\Sigma(s,t,z) = s - \frac{2}{z}\Sigma(s,t,z) + 2s \int_0^1 dx \int_0^x dy \; (\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=xt+yu}$$

All loop Exact Recurrence Relation

D=8 N=1

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Exact relation for ALL diagrams

$$nS_n(s,t) = -2s^2\int_0^1 dx \int_0^x dy\ y(1-x)\ (S_{n-1}(s,t')+T_{n-1}(s,t'))|_{t'=tx+yu}$$

$$+\quad s^4\int_0^1 dx\ x^2(1-x)^2\sum_{k=1}^{n-2}\sum_{p=0}^{2k-2}\frac{1}{p!(p+2)!}\ \frac{d^p}{dt'^p}(S_k(s,t')+T_k(s,t'))\times$$

$$S_1=\frac{1}{12},\ T_1=\frac{1}{12} \qquad \times\frac{d^p}{dt'^p}(S_{n-1-k}(s,t')+T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p$$

All loop Exact Recurrence Relation

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summation

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summation $\Sigma_3(s,t,z) = \Sigma_1(s,t,z) - S_2(s,t)z^2 + S_1(s,t)z, \;\; \Sigma_2(s,t,z) = \Sigma_1(s,t,z) + S_1(s,t)z$

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Diff eqn

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ (\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=tx+yu}$$

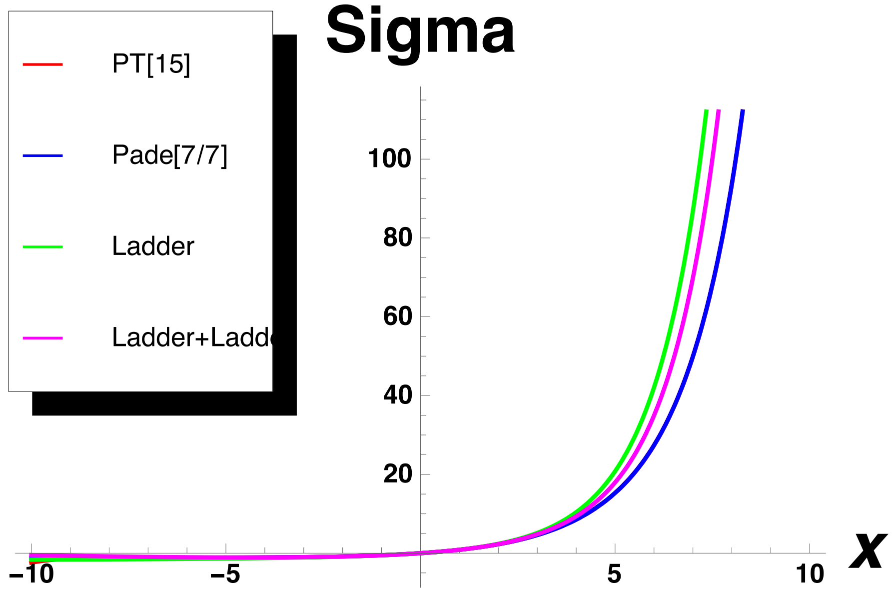
$$-s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{p=0}^{\infty} \frac{1}{p!(p+2)!}(\frac{d^p}{dt'^p}(\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=-sx})^2\ (tsx(1-x))^p.$$

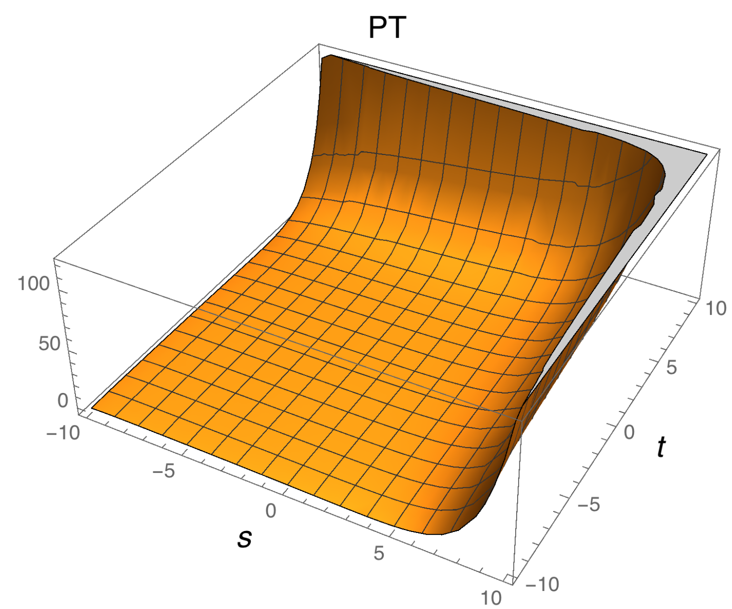
All loop Solution (leading divs)

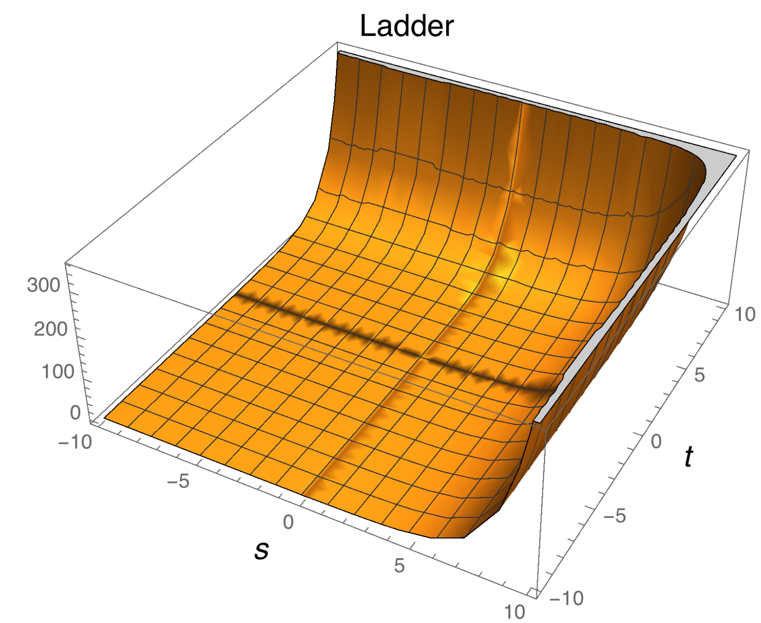
D=6 N=2

PT (15 terms)

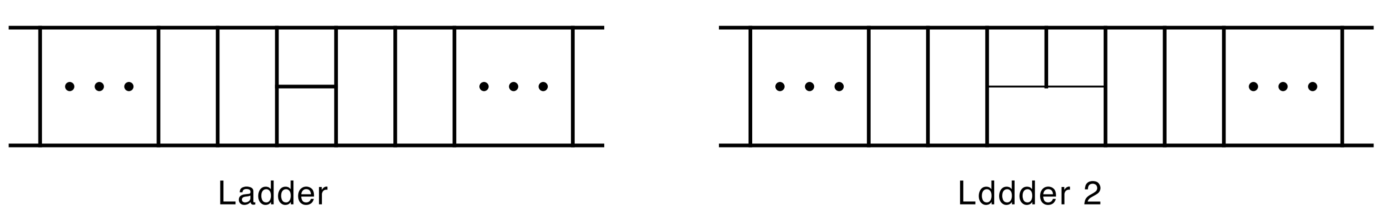
Sigma

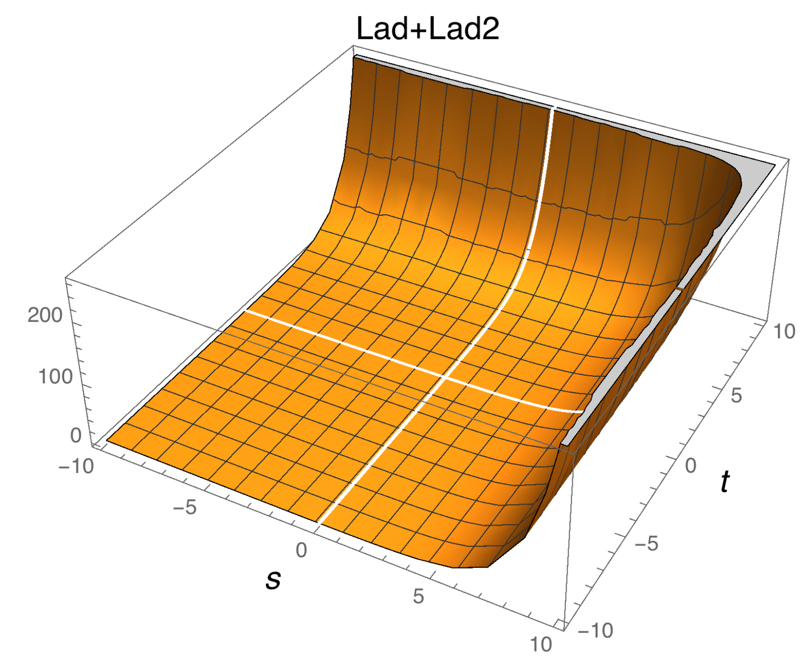






PT and Pade versus ladder for t=s





$$\Sigma_L(s,z) = \frac{2}{s^2 z^2}\left(e^{sz} - 1 - sz - \frac{s^2 z^2}{2}\right)$$

$$z = \frac{g^2}{\epsilon}$$

Numerical solution of the full equation is close to the ladder approx

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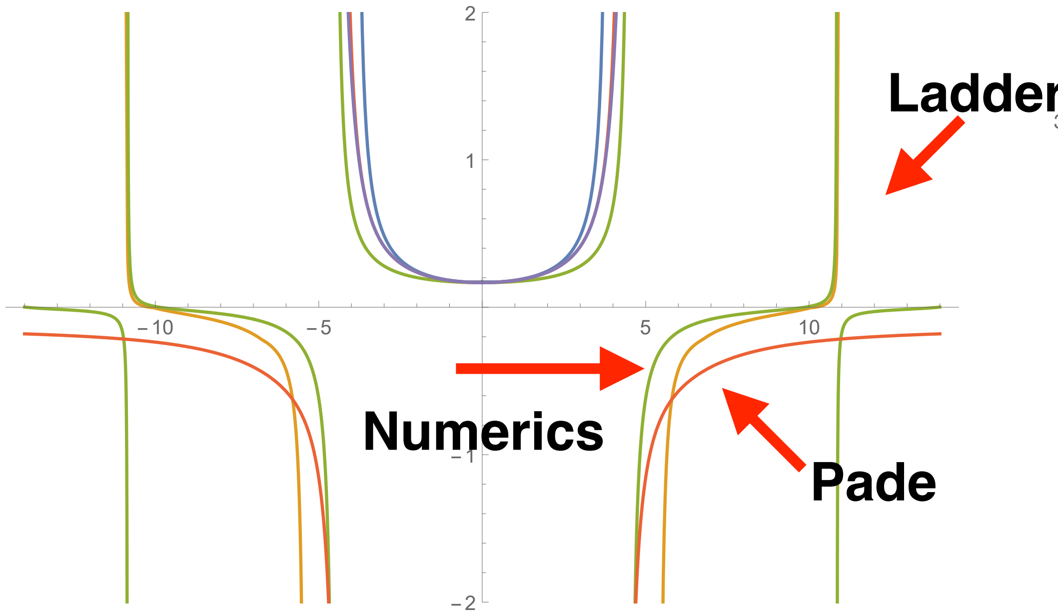
All loop Solution (leading divs)

D=8 N=1



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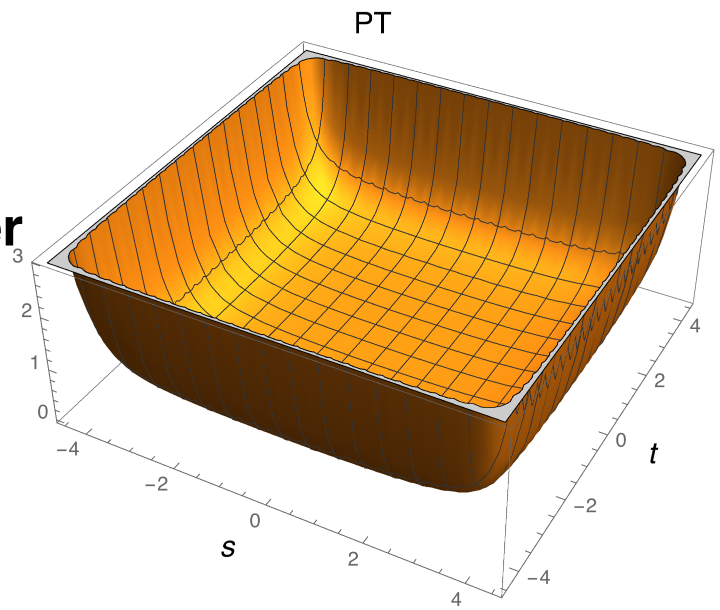

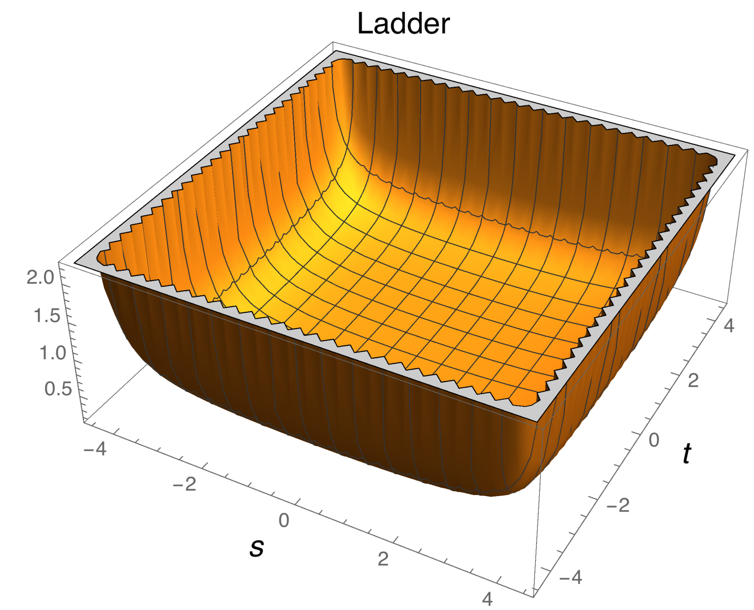

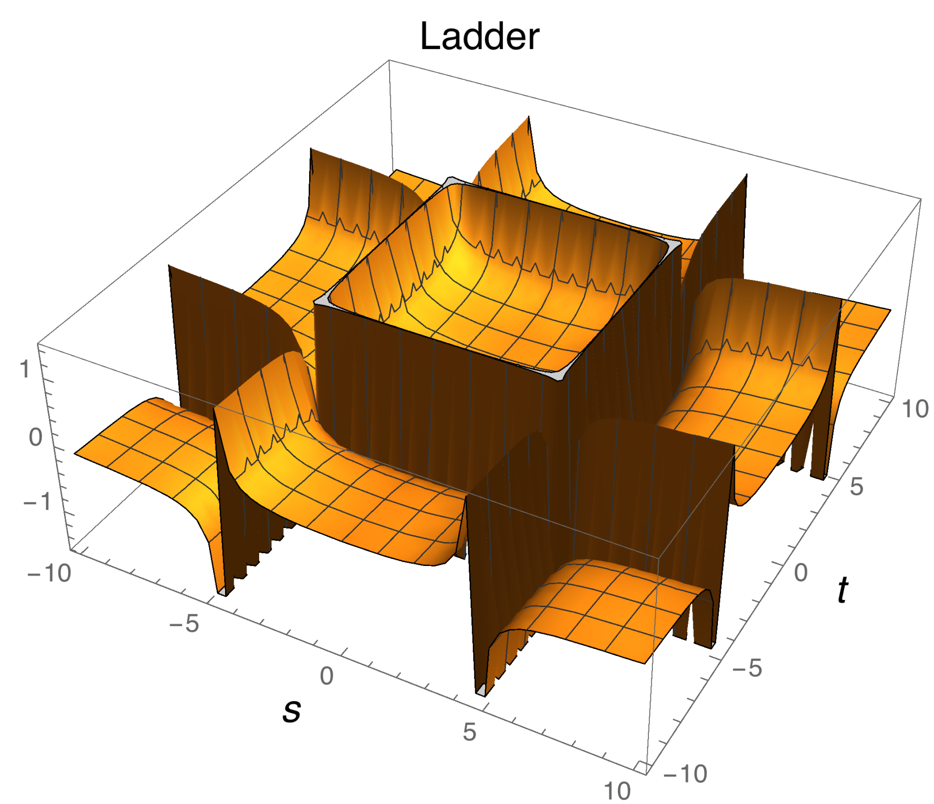

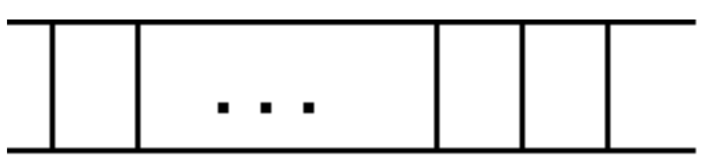

$$z = \frac{g^2}{\epsilon}$$

$$\Sigma_L(s,z) = -\sqrt{5/3}\,\frac{4\tan(zs^2/(8\sqrt{15}))}{1-\tan(zs^2/(8\sqrt{15}))\sqrt{5/3}}$$

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