NEW TRENDS
IN HIGH-ENERGY PHYSICS

2 October 2016 – 8 October 2016

Montenegro, Budva, Becici

Twenty Years of the Analytic Perturbation Theory in QCD

Olga Solovtsova

*ICAS *, GSTU (Belarus)*

I

in collaboration with **S. Mikhavlov and O. Teryaev**

BLTP, Joint Institute for Nuclear Research

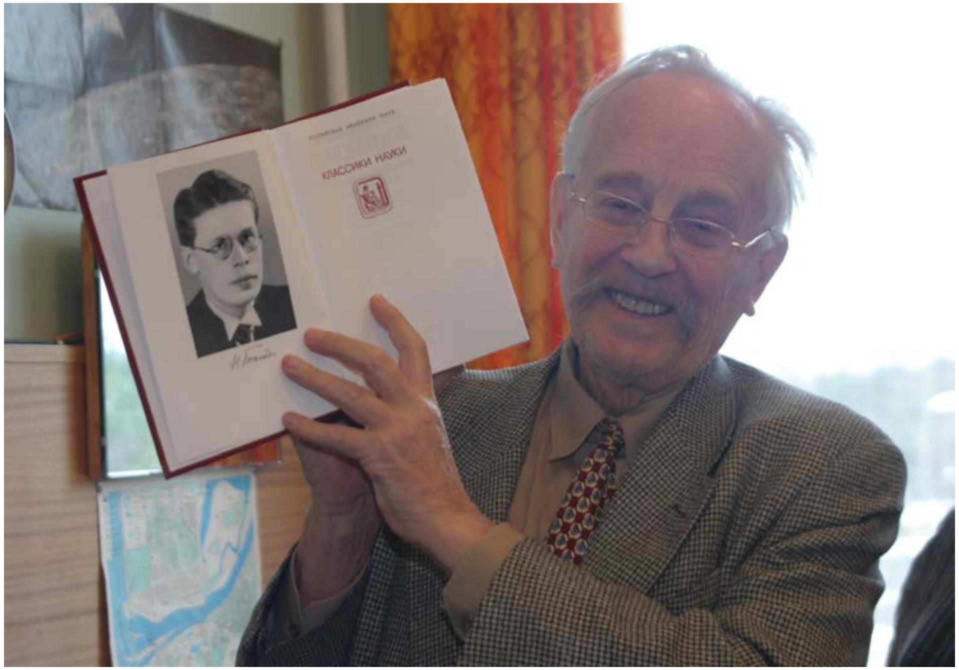
We present the main stages in the development of the Analytic Perturbation Theory (APT) in QCD. Advantages of the APT approach are illustrated by a number applications to hadron processes.

*ICAS

International Centre for Advanced Studies was created at Gomel State Technical University on the initiative of **D.V. Shirkov**.

Review-2016

Н. Н. Боголюбов. Собрание научных трудов, Том X



The renormalization-group method allows one to modify a perturbative expansions in accordance with the general principle of renormalization invariance.

The __analytic approach__ is the next step in the RG method:

This approach modifies the perturbative expansions so that the new approximations combine the renormalization invariance and the correct analytical properties of the series in the complex Q^2-plane.

Дополнение 1. Ренорм-группа Боголюбова 50 лет спустя

5.2. Аналитическая теория возмущений в КХД. Модификация ряда теории возмущений, выполненная методом ренорм-группы, позволяет улучшить свойства разложения в ультрафиолетовой области, однако приводит к нефизическим особенностям.

Так, в КХД сумму однопетлевых ультрафиолетовых (УФ) логарифмов для «инвариантного заряда» (57) обычно записывают в терминах

$$\overline{\alpha}_s^{(\ell=1)}(Q^2) = \frac{\alpha_\mu}{1+\alpha_\mu\beta_0\ln(Q^2/\mu^2)} = \frac{1}{\beta_0\ln(Q^2/\Lambda^2)}, \qquad \beta_0(n_f) = \frac{11-2n_f/3}{4\pi} \tag{70}$$

масштабного параметра КХД, определенного известным образом: $\Lambda = \mu\, e^{-1/(2\alpha_\mu\beta_0)}$.

Функция (70) имеет паразитную сингулярность в инфракрасной (ИК) области при $Q^2 = \Lambda^2$. Проблема особенностей такого сорта не может быть решена (см. § 50.2 основного текста) за счет учета любого *конечного* числа многопетлевых вкладов. Ложные сингулярности при этом не исчезают, а лишь меняют характер. Так, обычное пертурбативное двухпетлевое выражение для $\overline{\alpha}_s$

$$\overline{\alpha}_s^{(2)}(Q^2) = \frac{1}{\beta_0 l}\left[1-\frac{\beta_1}{\beta_0^2}\frac{\ln l}{l}\right] + O\left(\frac{\ln^2 l}{l^3}\right); \qquad l = \ln\frac{Q^2}{\Lambda^2},$$

представляющее собой разложение формулы (60), помимо полюса обладает нефизическим разрезом, обусловленным двойной логарифмической зависимостью от Q^2.

Подобная трудность впервые возникла в КЭД в 1950-х гг.[1] Вскоре в работе Боголюбова с соавторами [31] было предложено явное модельное решение проблемы на пути синтеза метода РГ и физического условия причинности в виде спектрального представления Челлена–Лемана для фотонного пропагатора.

Эта идея получила развитие [32, 33] в середине 1990-х гг. применительно к КХД, где в силу свойства асимптотической свободы нефизические особенности находятся в физически достижимой инфракрасной области (величина параметра Λ составляет несколько сотен МэВ) и существенно затрудняют обработку данных опыта.

Привлекая общее требование причинности в форме условий аналитичности пропагаторов (представление Челлена–Лемана) и амплитуд рассеяния (спектральное представление Йоста–Лемана[2]), удалось построить регуляр-

[1] Здесь призрачные сингулярности отвечают огромным масштабам, лишенным физического смысла.

[2] См. § 55.1 основного текста.

Motto of APT activity

Take care of Principles and
the Principles will take care of you.

D.V. Shirkov, I.L. Solovtsov

Method	Type of approximation	Properties		
		UV	IR	Anal
PT	Double set in powers of α_μ and $\ln Q^2/\mu^2$	$-$	$-$	$+$
PT + RG	Power series in invariant charge $\bar{\alpha}_s(Q^2)$	$+$	$-$	$-$
APT = PT + RG + analyticity	Nonpower expansions in $\mathcal{A}_k(Q^2)$ and $\mathfrak{A}_k(s)$			

Shirkov, Solovtsov 2007

I d e a:

Some initial approximation breaks general properties,
using a representation, which accumulates general properties,
restoration of these properties => improved approximation.

Outline

- History of the APT creation
- Overview of theoretical framework: low energy scales
- Some results of application of the APT
- From APT to FAPT
- DIS: FAPT: Q^2 – evolution SF
- Target mass corrections in DIS
- Conclusion

History of the APT creation

The Perturbation theory (PT) is a basic tool of calculation in quantum field theory.

1st step of improving **PT** is provided by **RG** Method (**Bogoliubov–Shirkov [1955-56]**).

In the infrared region the RG-modified PT series remains unstable.

2nd step in improving of **PT** solution is provided by the **analyticity** imperative, based on the **causality condition**.

The idea to combine the renormalization invariance and the Q^2 analyticity was generalized to the case of QCD twenty years ago by **Shirkov and Solovtsov.**

[**D.V. Shirkov and I.L. Solovtsov, «Analytic QCD running coupling with finite IR behaviour and universal $\alpha_S(0)$ value», JINR Rapid Comm. No.2[76], 1996, p. 5; Phys. Rev. Lett. 79 (1997) 1209**]

A further development of the analytic approach in QCD led to the formulation of the APT method and to its numerous applications to hadronic physics.

3rd step generalizes **APT** by including **fractional** powers of $\alpha_S(Q^2)$

This generalization has expanded the application of the APT to QCD analysis of the processes.

Bakulev-Mikhailov-Stefanis [2005-2010]

Overview of theoretical framework: Analytic approach

APT = PT + RG + Q^2 -analyticity

D.V. Shirkov, I.L.Solovtsov,
Theor. Math. Phys. 150(1) (2007) 132-152
"Ten years of the Analytic Perturbation Theory in QCD"

Well-known features of APT:

In the framework of APT the QCD running coupling (invariant charge)

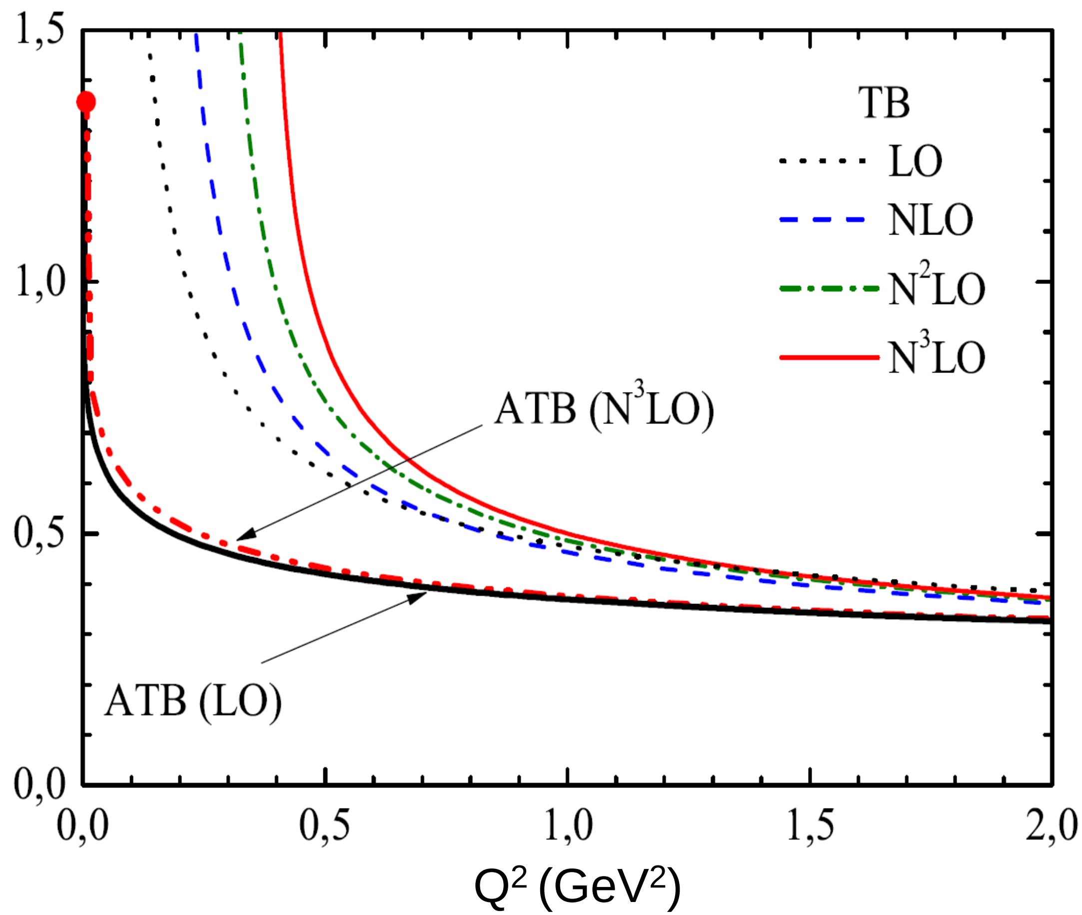
can be reconstructed via the Kallen-Lehmann representation in which the relevant spectral density is defined as the imaginary part of the perturbative invariant charge constructed by the RG method in the Euclidean domain.

$$\mathcal{A}\left(Q^2\right) \equiv \alpha_{APT}(Q^2) = \frac{1}{\pi}\int_0^{+\infty} \frac{\rho(\sigma)\,d\sigma}{\sigma + Q^2}$$

$$\rho(\sigma) = \mathrm{Im}\left(\left[\alpha_{PT}(-\sigma)\right]\right)$$

- **free from unphysical singularities and without additional parameters**
- **infrared stable point which is independent of the scale parameter Λ_{QCD} and higher-loop corrections**
- **the Euclidean and Minkowskian invariant charges are defined in a self-consistent way and** $\alpha^E_{APT}(0) = \alpha^M_{APT}(0) = 1/\beta_0$ $\qquad \beta_0 = 11 - 2n_f/3$
- **the better convergence properties of the APT nonpower expansions and stability with respect to higher-loop corrections, the theoretical ambiguity associated with the choice of renormalization scheme is diminished.**
- **leads to an essential change in the IR behavior, but APT -> PT at large Q^2**

QCD running coupling: PT and APT



The higher-loop behavior of PT and APT couplings

Overview of theoretical framework

Leading order

Euclidean region

$$\alpha_{\mathrm{E}}(Q^2) = \frac{1}{\pi}\int_0^\infty d\,\sigma \frac{\rho(\sigma)}{\sigma + Q^2} = \frac{4\pi}{\beta_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right]$$

Minkowskian region

$$\alpha_{\mathrm{M}}(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\rho(\sigma) = \frac{4\pi}{\beta_0 \pi} \arccos \frac{L}{\sqrt{L^2 + \pi^2}}\Bigg|_{L>0}$$

$$= \frac{4\pi}{\beta_0 \pi} \arctan \frac{\pi}{L}, \qquad L = \ln \frac{s}{\Lambda^2} \qquad \rho(\sigma) \equiv \rho_1(\sigma)$$

$$\rho_k(\sigma) = \mathrm{Im}\, \bar{\alpha}_s^k(-\sigma - i\epsilon)$$

IR stable points

$$\alpha_{\mathrm{E}}(0) = \alpha_{\mathrm{M}}(0) = \frac{4\pi}{\beta_0}$$

are independent of the loop level and Λ

Overview of theoretical framework

The main object in description of hadronic part of many physical processes is a $\Pi(q^2)$

$$\Pi_{\mu\nu}(q^2) = i\int d^4x\, e^{iqx} < 0|T V_\mu(x) V_\nu(0)^+|0 >$$
$$= (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)\ , \quad V_{ij}^\mu = \bar{\psi}_j \gamma^\mu \psi_i$$

It is useful to introduce an Eucledean characteristic, the so-called the Adler function

$$D(Q^2) = -Q^2 \frac{d\Pi(-Q^2)}{dQ^2}\ , \quad Q^2 = -q^2 > 0$$

[in Euclidean (spacelike) region]

The integral representation for the D-function is given in terms of the discontinuity of the correlator across the cut

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} R(s) \Rightarrow R(s) = \frac{1}{\pi} \operatorname{Im}\Pi(s)$$

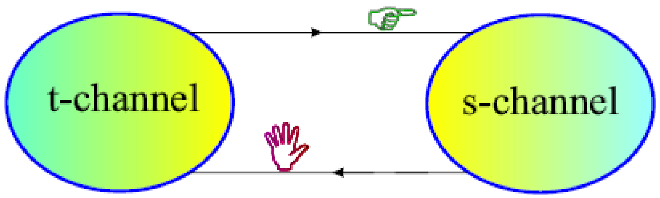
[in Minkowskian (timelike) region]

This representation defines the Adler function is an analytic function in the complex Q^2 plane with a cut along the negative real axis.
To parameterize R(s) in terms of QCD parameters a procedure of analytic continuation from Euclidean (s-channel) to Minkowskian (t-channel) region is required.

Minkowskian $\Longleftrightarrow$ Euclidean

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} R(s)$$

$$R(s) = -\frac{1}{2\pi i} \int_{s-i\varepsilon}^{s+i\varepsilon} \frac{dz}{z} D(-z)$$



$$D \propto 1 + d\,, \qquad R \propto 1 + r$$

$$d(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2}\ r(s)\,, \qquad r(s) = -\frac{1}{2\pi i} \int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z}\ d(-z)$$

$$d(Q^2, RS) = a(Q^2, RS)\left[1 + d_1(RS)\, a(Q^2, RS) + d_2\,(RS) a^2(Q^2, RS) + \ d_3\,(RS) a^3(Q^2, RS)\right] \qquad a \equiv \alpha_s/\pi$$

$$d_1^{\overline{\rm MS}} = 1.9857 - 0.1153\, n_f, \quad d_1^{\overline{\rm MS}} = 1.6398 \qquad (n_f = 3)$$

$$d_2^{\overline{\rm MS}} = 6.3710, \qquad d_3^{\overline{\rm MS}} = 49.08$$

Baikov, Chetyrkin, Kuhn, RPL (2008)

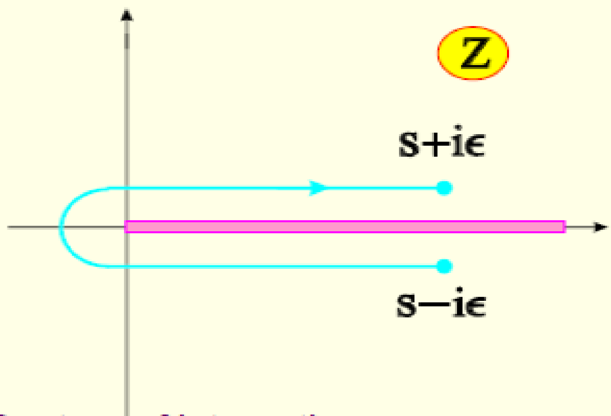
The D-function defined in the spacelike region is a smooth function without traces of the resonance structure and one can expect that reflects more precisely the quark-hadron duality and will be a convenient object for comparing theoretical predictions with experimental data.

Running coupling in the timelike region (a low-energy scales)

K.Milton, O.S.,Phys. Rev. D 57 (1998)

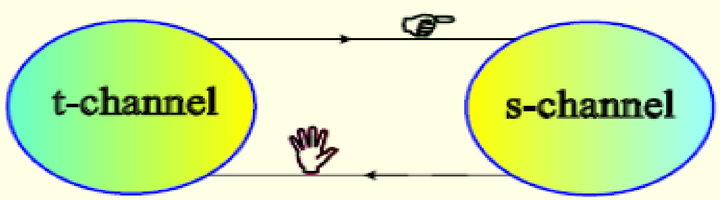
$$\alpha_{\rm E}(Q^2) = Q^2 \int_0^{\infty} \frac{d\,s}{(s+Q^2)^2}\,\alpha_{\rm M}(s), \quad \alpha_{\rm M}(s) = -\frac{1}{2\pi {\rm i}} \int_{s-{\rm i}\,\epsilon}^{s+{\rm i}\epsilon} \frac{d\,z}{z}\,\alpha_{\rm E}(-z).$$

A. Radyshkin, Preprint JINR E2-82-159 (1982)



Contour of integration

The perturbative approximation, in which the running coupling with unphysical singularities is used, breaks this connection between space and timelike quantities.



The leading order PT coupling (generates singularity)

$$\alpha_{PT}^{E}(z) = \frac{1}{\beta_0}\frac{1}{\ln(-z)}, \qquad \alpha_{PT}^{M}(s) = -\frac{1}{2\pi i}\int_{s-i\varepsilon}^{s+i\varepsilon}\frac{dz}{z}\,\alpha_{PT}^{E}(-z)$$

$$\alpha_{PT}^{E}(z) \neq Q^2 \int_0^{\infty} \frac{ds}{(s-z)^2}\,\alpha_{PT}^{M}(s)$$

Higher-loop PT orders not resolve this problem.

The APT leads to a self-consistent definition of analytic continuation.

RG+Analyticity

ghost-free $\alpha_{\rm E}(Q^2)$

Shirkov & Solovtsov 1996

The APT: R-quantities

We analyzed various physical quantities and functions generated by *R(s)* based on the APT. A common feature of all these quantities and functions is that they are defined through the function *R(s)* integrated with some other function.

$$R_\tau = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2} P(s) R(s)$$

The ratio of hadronic to leptonic tau-decay widths in the vector channel

$$a_l^{had} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s} K_l(s) R(s)$$

$$l = \mu,\ e,\ \tau$$

K(s) - known QED kernel

The hadronic contribution to the anomalous magnetic moment of the leptons (in the leading order in the electromagnetic coupling constant)

All these quantities include an infrared region as a part of the interval of integration and, therefore, they cannot be directly calculated within perturbative QCD.

Numerical results

The hadronic contribution to the anomalous magnetic moment:

✓ ***muon***

$$a_\mu^{had} = (694.9 \pm 3.7)\times 10^{-10} \text{ [one of set expt. result 2012]}$$

$$(702 \pm 16)\times 10^{-10} \text{ [APT]}$$

✓ ***electron***

$$a_e^{had} = (1.678 \pm 0.014)\times 10^{-12} \text{ [Nomura, Teubner 2013]}$$

$$(1.64 \pm 0.07)\times 10^{-12} \text{ [APT]}$$

✓ ***tau lepton***

$$a_\tau^{had} = (3.38 \pm 0.04)\times 10^{-6} \text{ [Passera'07, Nomura'2012]}$$

$$(3.28 \pm 0.05)\times 10^{-6} \text{ [APT]}$$

✓ ***fine structure***

$$\Delta\alpha_{had}^{(5)}(M_Z^2) = (276.26 \pm 1.38)\times 10^{-4} \text{ [Hagivara et al. 2011]}$$

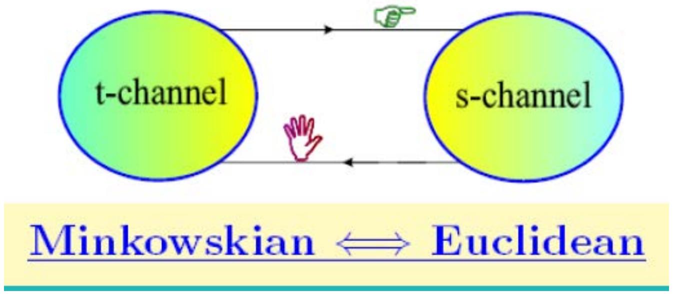
$$(279.9 \pm 4.0)\times 10^{-4} \text{ [APT]}$$

Good agreement for all considered quantities has been obtained.
The question : Wy?

Criterion of equivalence

$$Q_M = \int_0^\infty \frac{ds}{s} M(s) R(s)$$

$$Q_E = \int_0^\infty \frac{dt}{t} E(t) D(t)$$



$$Q_M = \int_0^\infty \frac{dt}{t} E(t) D(t) \equiv Q_E$$

(R-D self-duality)

When expressions for quantity Q in terms of $R(s)$ and $D(Q^2)$ are equivalent?
If one uses a method that does not maintain the required analytic properties of functions then these expressions are not equivalent.

The answer on the question about a simultaneous good agreement of various QCD observables is:
the APT approach used is support required analytic properties and gives good description of the D-function down to low energy scale
and a manifestation of quark-hadron duality (which establishes a bridge between quarks and gluons, and real measurements with hadrons) via the Adler D-function.

Example

R-D self-duality presentations

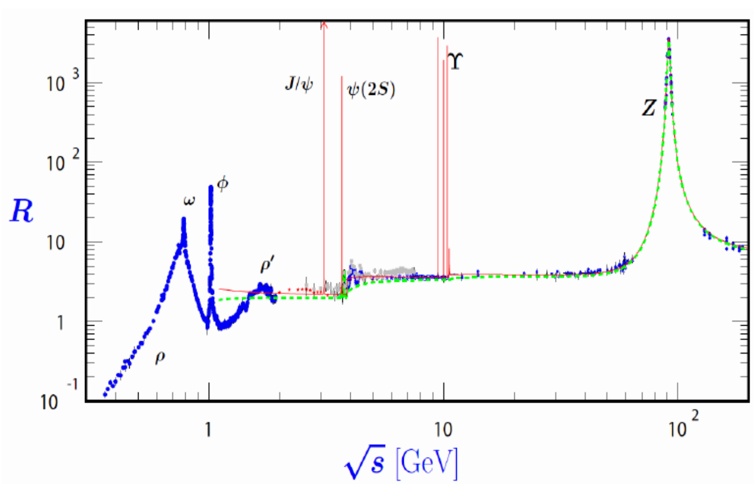
$$a_\mu^{had} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s} M(s) R(s) \qquad a_\mu^{had} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dt}{t} E(t) D(t)$$

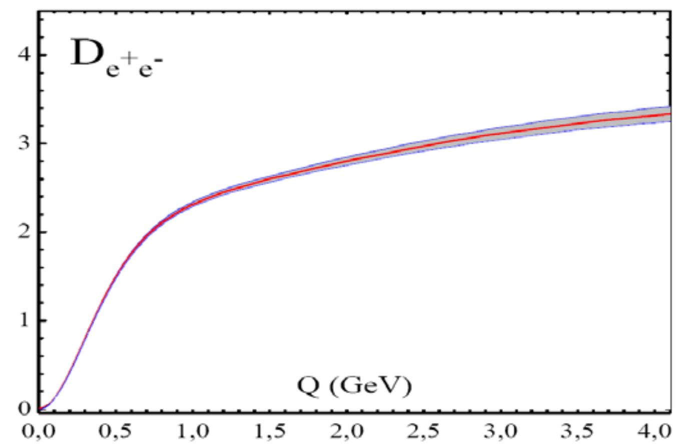
$$E(t) = -\frac{1}{2\pi \mathrm{i}} \int_{t-\mathrm{i}\,\epsilon}^{t+\mathrm{i}\,\epsilon} \frac{dz}{z} M(-z).$$

$$E(t) = \frac{1}{2}\left[\frac{\sqrt{1+4m^2/t}-1}{\sqrt{1+4m^2/t}+1}\right]^2,$$

$$M(s) == \int_0^1 dx \frac{x^2}{x^2+(1-x)s/m^2}$$

$$\eta = \frac{1-v}{1+v}, \qquad v = \sqrt{1-\frac{4m^2}{s}}.$$

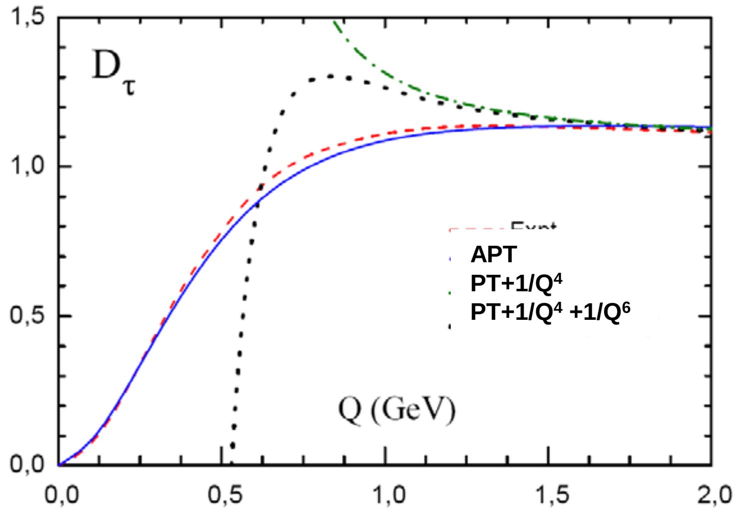


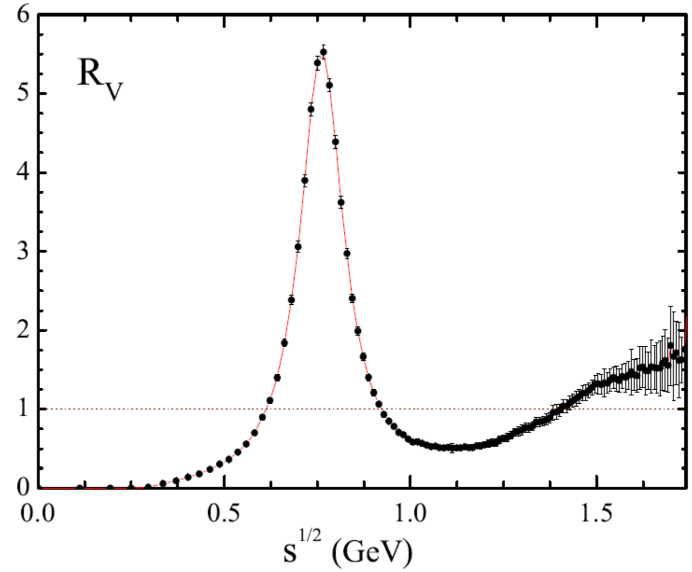


From F. Jegerlehner (2008) data

D-function

The `light' Adler function constructed from ALEPH tau-decay data





ALETH data

The experimental D-function (dashed curve) turned out to be a smooth and monotone function without traces of the resonance structure.

The theoretical approach (APT) which we used to describe the experimental curve works well (solid line) for the whole interval, including the infrared region.

Note that any finite order of the operator product expansion (OPE) fails to describe the infrared tail of the D-function (dotted curve) .

The polarized Bjorken Sum Rule

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 \left[g_1^p(x,Q^2) - g_1^n(x,Q^2) \right] dx = \frac{|g_A|}{6} C_{Bj}(Q^2) + \sum_{i=2}^{\infty} \frac{\mu_{2i}^{p-n}}{Q^{2i-2}}$$

$$|g_A| = 1.2701 \pm 0.0025$$

The pQCD correction Δ_{Bj} defined by the coefficient function $C_{\mathrm{Bj}}(Q^2) \equiv 1 - \Delta_{\mathrm{Bj}}(Q^2)$ has a form of the power series in α_S and at the N³LO (in the massless case) reads as

$$\Delta_{\mathrm{Bj}}^{\mathrm{PT}}(Q^2) = 0.318\, \bar{\alpha}_{\mathrm{s}}(Q^2) + 0.363\, \bar{\alpha}_{\mathrm{s}}^2(Q^2) + 0.652\, \bar{\alpha}_{\mathrm{s}}^3(Q^2) + 1.804\, \bar{\alpha}_{\mathrm{s}}^4(Q^2)$$

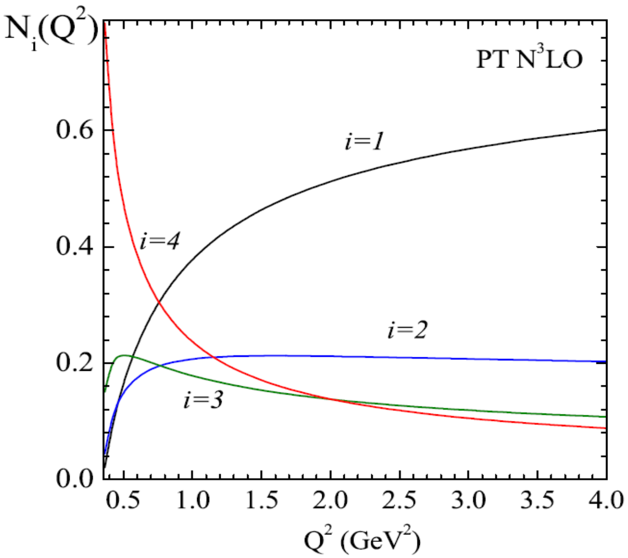
[Baikov, Chetyrkin, Kühn (2010)]

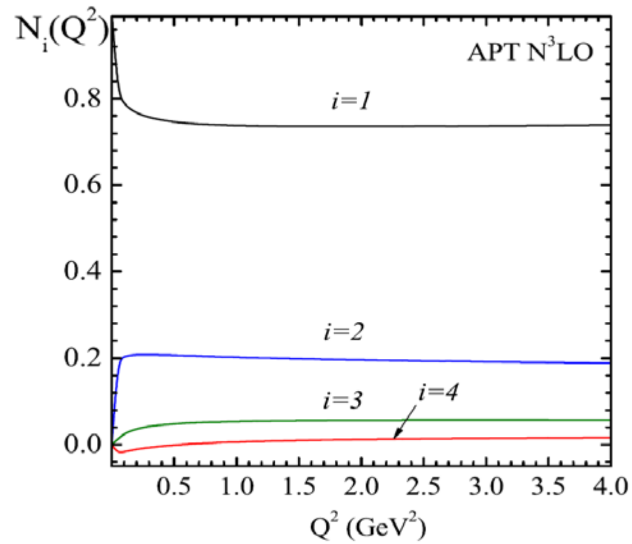
$$\Delta_{\mathrm{Bj}}^{\mathrm{PT}} = \sum_{k \le 4} c_k \left(\bar{\alpha}_{\mathrm{s}}(Q^2) \right)^k \quad \Rightarrow \quad \Delta_{\mathrm{Bj}}^{\mathrm{APT}}(Q^2) = \sum_{k \le 4} c_k\, \bar{\mathcal{A}}_k(Q^2)$$

$$\mathrm{N}_i(Q^2) = \frac{c_i \alpha_S^i(Q^2)}{\Delta_{Bj}(Q^2)}$$

The polarized Bjorken Sum Rule

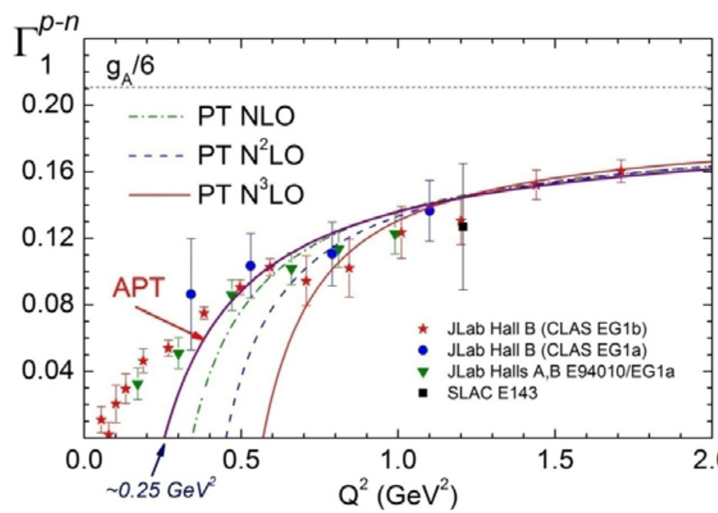
R.S. Pasechnik, D.V. Shirkov, O.V.Teryaev, O.P.Solovtsova, V.L.Khandramai,
Phys. Lett. B 706 (2012) 340





In the APT case:
one-loop ~ 70 %,
two-loop ~ 20 %,
three-loop ~ 5 %,
four-loop ~ 1 %.

Order	Q^2_{min}	μ_4, GeV^2	χ^2
2 in PT	0.5	-0.025±0.004	0.80
3 in PT	0.66	-0.012±0.006	0.59
4 in PT	0.71	0.005±0.008	0.51
in APT	0.47	-0.043±0.002	0.82



- **3rd step** generalizes **APT** by including fractional powers of coupling

Global Fractional APT (FAPT)

Analytization of α_s^{ν}: $\mathcal{A}_{\nu}(Q^2) \Leftrightarrow \mathfrak{A}_{\nu}(s)$

A.Bakulev & Mikhailov & Stefanis

$$\left[\alpha_{PT}(Q^2)\right]^{\nu} \Rightarrow A_{\nu}\left(Q^2\right) = \frac{1}{\pi}\int_0^{+\infty} \frac{\rho_{\nu}(\sigma)\,d\sigma}{\sigma + Q^2} \quad ,$$

$$\rho_{\nu}(\sigma) = \mathrm{Im}\left(\left[\alpha_{PT}(-\sigma)\right]^{\nu}\right)$$

Reference

- A. B., Mikhailov, Stefanis — **PRD 72 (2005) 074014; PRD 72 (2005) 074014; PRD 75 (2007) 056005**
- A. B. — **Phys. Part. Nucl. 40 (2009) 715**
- A. B., Mikhailov, Stefanis — **JHEP 1006 (2010) 085**

In the description of Q^2 -evolution of the structure function (SF) moments the generalized powers (anomalous dimensions) for the running coupling appear.
In the leading order (LO) the nonsinglet moments evolve as

$$M_N(Q^2) = \frac{[\alpha_S(Q^2)]^{\nu}}{[\alpha_S(Q_0^2)]^{\nu}} M_N(Q_0^2) \ , \quad \nu(N) \equiv \gamma_{NS}^{(0)}(N)/2\beta_0$$

$$M_N(Q^2) = \int_0^1 x^{N-1} F(x, Q^2) dx$$

nonsinglet one-loop anomalous dimensions

In the framework of the FAPT this expression transforms as follows:

$$M_N^{APT}\left(Q^2\right) = \frac{A_\nu\left(Q^2\right)}{A_\nu\left(Q_0^2\right)} M_N^{APT}\left(Q_0^2\right)$$

FAPT ➔ $\bar{\mathcal{A}}_\nu^{LO}(Q^2) = \left[\bar{a}_{\rm PT}^{LO}(Q^2)\right]^\nu - \frac{{\rm Li}_\delta(t)}{\Gamma(\nu)} \,, \qquad \bar{\mathcal{A}}_\nu = \beta_0 \mathcal{A}_\nu/(4\pi)$

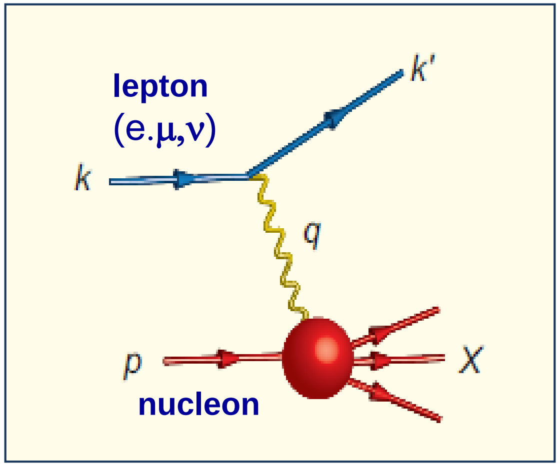
$${\rm Li}_\delta(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^\delta} \,, \quad t = \frac{\Lambda^2}{Q^2} \,, \quad \delta = 1 - \nu$$

${\rm Li}_\delta$ is the polylogarithm function

$$\alpha_{\rm APT}^{LO}(Q^2) = \alpha_{\rm PT}^{LO}(Q^2) + \frac{4\pi}{\beta_0} \frac{\Lambda^2}{\Lambda^2 - Q^2} \quad (\nu = 1)$$

additional term $-\frac{4\pi}{\beta_0}\frac{\Lambda^2}{Q^2}$

DIS: xF_3



Deep inelastic scattering

How the APT approach works in comparison with the ordinary PT?

Considering a combined set of the F_3-data:

the kinematic region of combined set of data is

289

$0.015 < x < 0.8$ and $0.5\ \text{GeV}^2 < Q^2 < 196\ \text{GeV}^2$

we extracted values of the scale parameter Λ_{QCD}, the parameters of the form of the xF_3, and compared the difference of the results of the PT and APT analysis with the corridor of experimental uncertainties.

The APT/(Fractional)APT has been applied to DIS in a set of works:

- ✓ G.Cvetic, A.Y. Illarionov, B.A. Kniehl, A.V.Kotikov, Phys. Lett. B 679 (2009) 350
- ✓ R.S. Pasechnik, D.V. Shirkov, O.V.Teryaev, O.P.Solovtsova, V.L.Khandramai, Phys. Rev. D81 (2010) 016010; Phys. Lett. B 706 (2012) 340
- ✓ A.V. Kotikov, V.G. Krivokhizhin, B.G. Shaikhatdenov, Phys. Atom. Nucl. 75 (2012) 507
- ✓ A.V. Sidorov, O.P. Solovtsova,, Mod. Phys .Lett. A29 (2014) no.36, 1450194;
- ✓ C. Ayala, S. Mikhailov, Calculation of Nucleon Structure Function in APT Phys. Rev. D92 (2015) 014028.

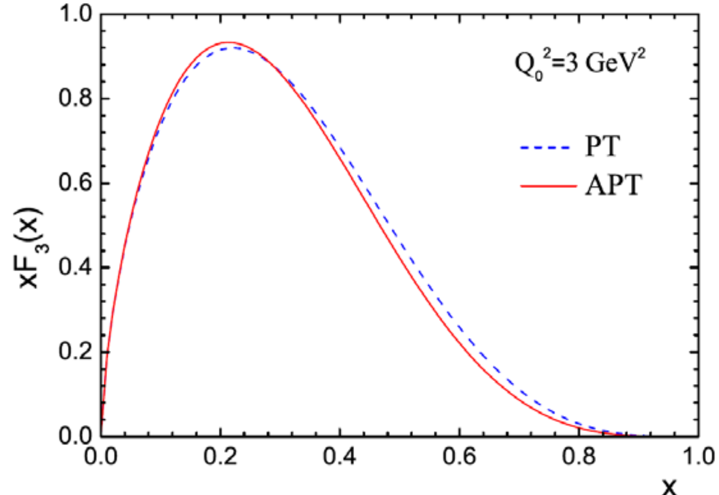
We have found, that in the region $Q^2 > 1\ \text{GeV}^2$ difference between APT and PT approaches not so big and is obviously shown only at large x-values.

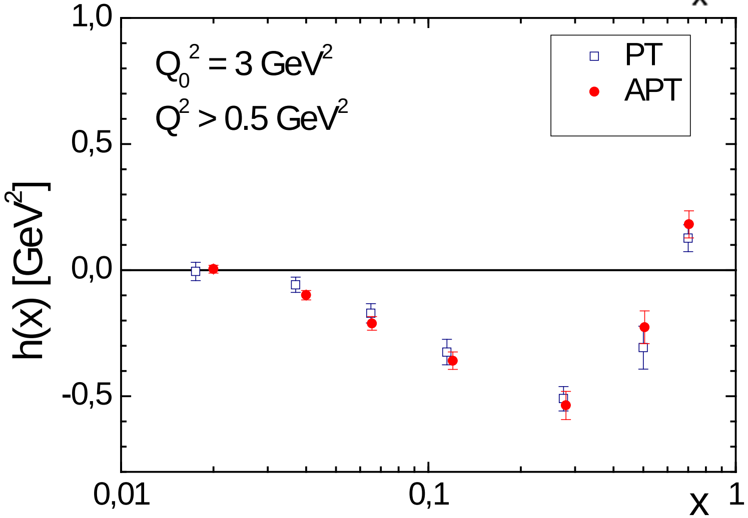
Fitting result

$$xF_3(x,Q_0^2) = Ax^{\alpha}(1-x)^{\beta}(1+\gamma x)$$

A, α, β, γ, Λ - free parameters

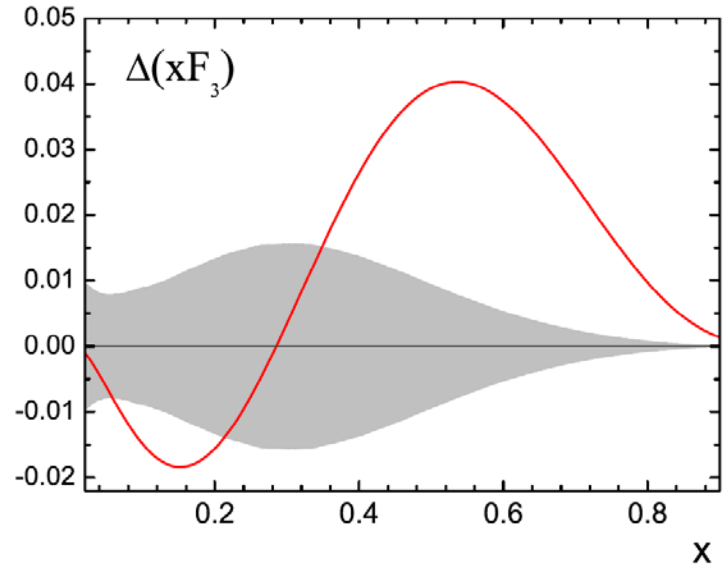
The QCD analysis of $F_3(x,Q^2)$ data is simplified because one does not need to parameterize gluon and see quark contributions and can parameterize the shape of the $F_3(x,Q^2)$ structure function itself at some Q^2- value.





$$\Lambda_{APT} = 407 \pm 75 \text{ MeV}$$

$$\Lambda_{PT} = 363 \pm 49 \text{ MeV}$$



$$\Delta(xF_3) = xF_3^{PT}(Q_0^2) - xF_3^{APT}(Q_0^2)$$

We also obtained that APT Q^2-evolution could be apply for the analysis of data for the polarized NS combination $x\Delta q_3$, $x\Delta q_8$ and NS fragmentation function $D_{u_V}^{\pi^+}$.

On the threshold problem

The operator product expansion method is powerful tool to study properties of DIS structure functions.

The OPE expansion was derived in the massless limit. If a finite mass for the nucleon target is considered, the new terms arise: leading to additional power terms of kinematical origin called the target mass corrections (TMC).

The TMC become larger and larger in the range of low Q^2 and approaching to the kinematic limit as the Bjorken variable x tends to unity.

The OPE was first used to study target mass effects by Georgi and Politzer

[H.Georgi and H.D.Politzer Phys. Rev. D 14 (1976)].

Georgi and Politzer (GP) approach (or ξ -scaling approach because it was formulated through the Nachtmann ξ-variable) suffer the so-called **threshold problem:**
for the structure functions obtained by using this method have a difficulty arising from the violation of the spectral condition. It hence became a problem to describe the structure functions as the Bjorken variable x tends to unity.

For the structure functions the general quantum field theory principles, including covariance, Hermiticity, spectrality, and causality, are expressed by the Jost-Lehmann-Dyson (JLD) integral representation.

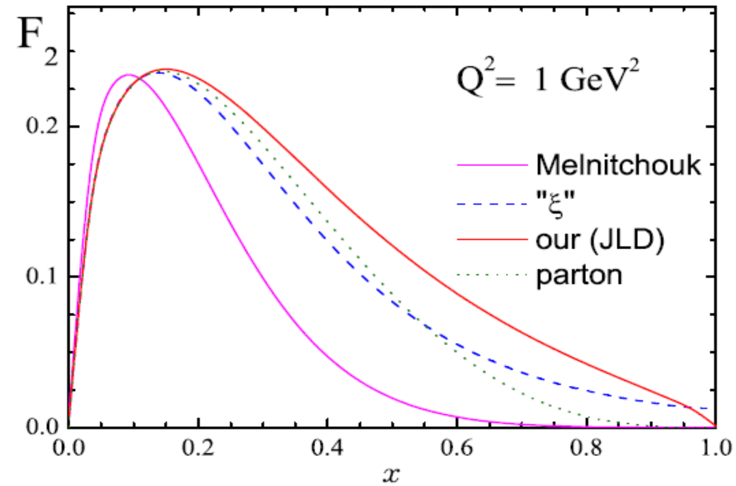
It has been argued by using the JLD representation, it is possible to get an expressions for the structure functions in terms of the quark distribution incorporating the target mass effects and having the correct spectral property.

On the threshold problem

Reference

1. Georgi, Politzer, Phys. Rev. D 14 (1976)

$$\xi(x) = \frac{2x}{1+\sqrt{1+4\varepsilon x^2}} \qquad \varepsilon = \frac{M^2}{Q^2}$$



XXI century

2. Solovtsov, Part. Nucl. Lett. (2000);

$$\xi_S = x\frac{\sqrt{1+4\varepsilon}}{\sqrt{1+4\varepsilon x^2}}$$

(JLD)

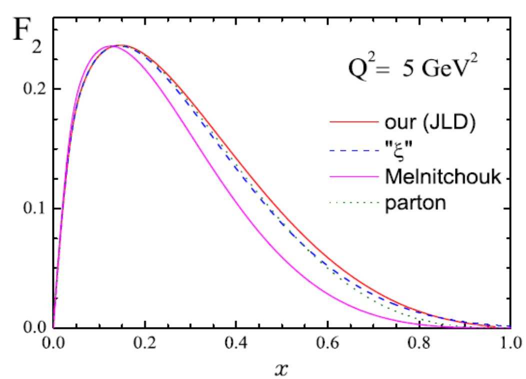
3. Steffens, Melnitchouk, Phys.Rev.C 73 (2006)

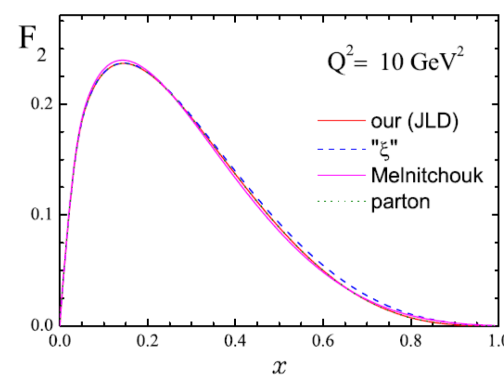
A Review of Target Mass Corrections,
J.Phys.G35:053101,2008.

$$\xi_{SM}(x) = x\frac{1+\sqrt{1+4\varepsilon}}{1+\sqrt{1+4\varepsilon x^2}}$$

$$A_n^{(\mathrm{SM})} \equiv \int_0^{\xi_0} d\xi\, \xi^n\, F(\xi,\xi_0)\,,$$

$$\xi_0 \equiv \xi(x=1) = 2/(1+\sqrt{1+4\epsilon}) < 1$$

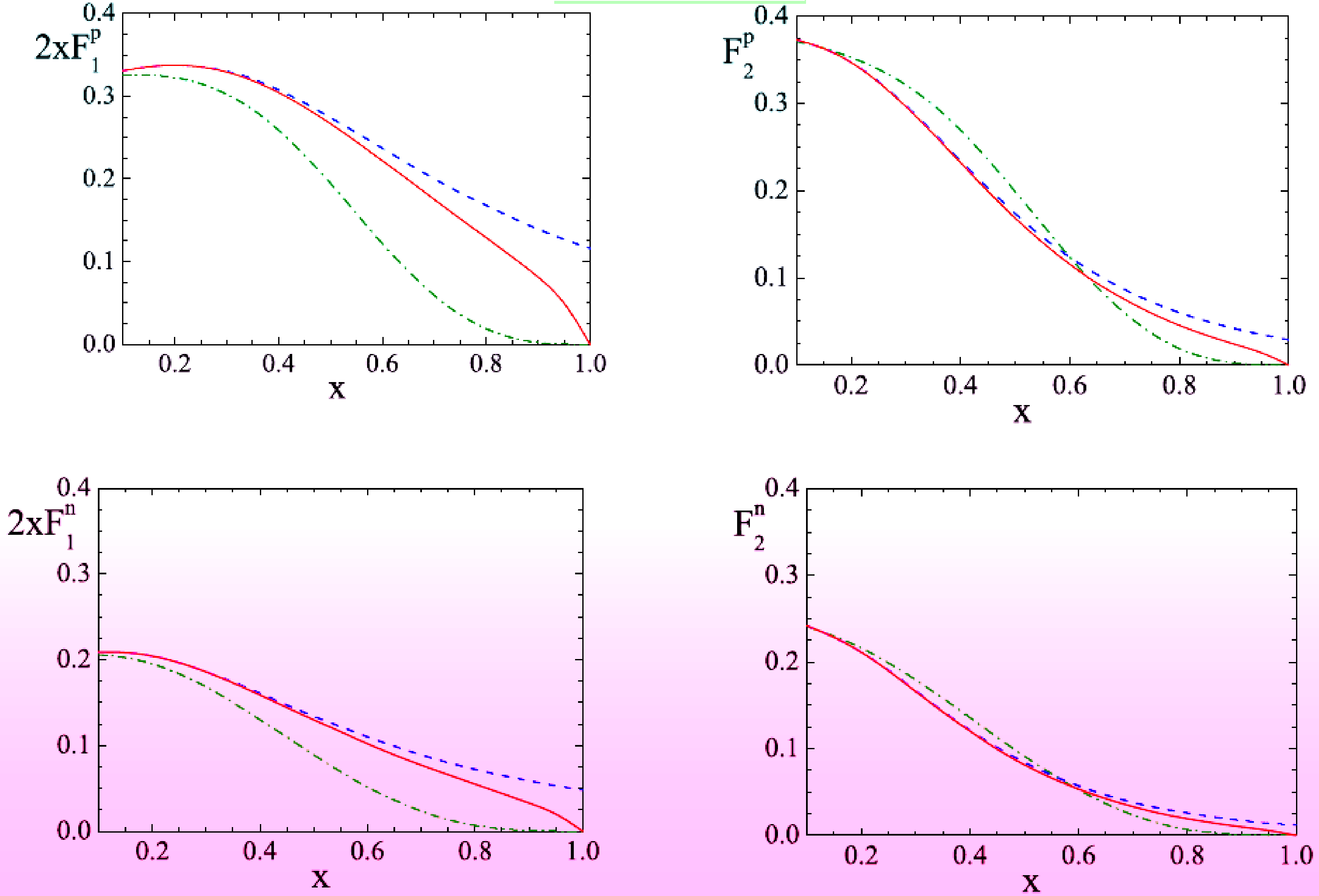




Figs. from **Lashkevich, O.S, Theor. Math. Phys., 160 (2009)**

Comparison of TMC for unpolarized nucleon structure functions

$Q^2 = 1\ GeV^2$



TMC calculated by using the JLD method (s-method) noticeably differ from the standard Georgi-Polizer method result (for x> 0.5) .

Summary

We performed the analyses some physical quantities by using the APT approach, which does not lead to any unphysical singularities. $\Lambda_{APT} > \Lambda_{PT}$

It was shown how works the idea of APT in DIS, and that at low Q^2, target mass corrections to structure functions have calculated by using JLD method noticeably differ from the standard Georgi-Polizer method result.

From the theoretical point of view, the remarkable properties of Shirkov-Solovtsov analytic approach in QCD create the basis for its further development and successful applications.

APT -> leads to correct analytic properties

APT -> improves convergence properties

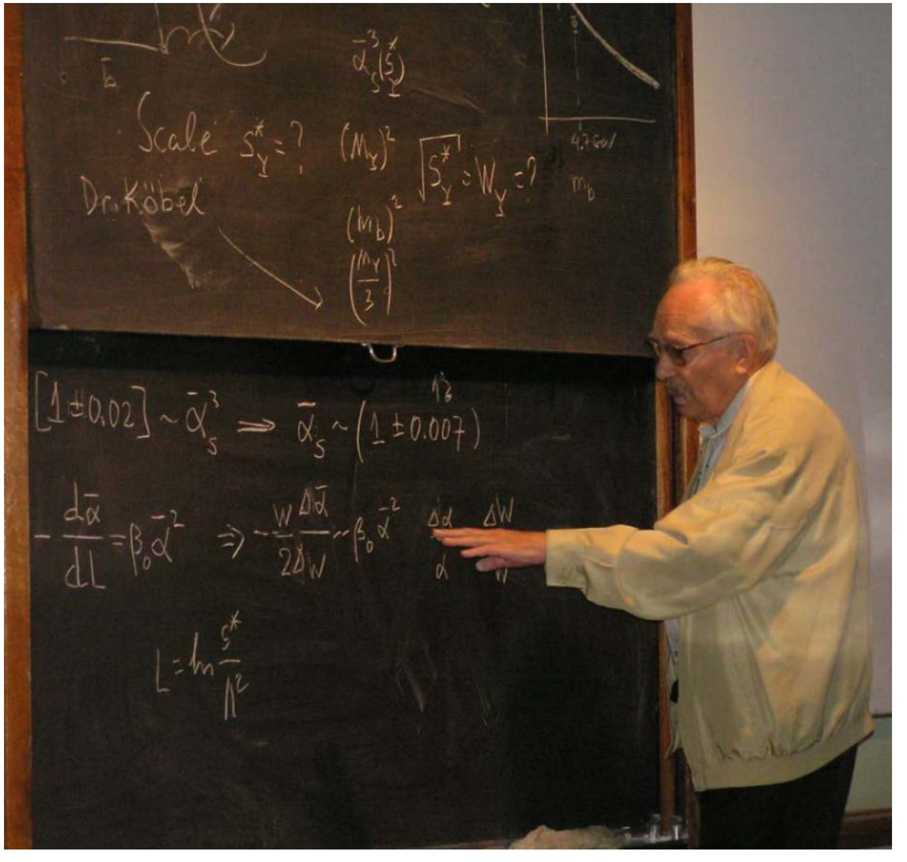
APT -> correct analytic continuation from Euclidean to Minkowskian region

APT -> gives stable results for the HT

and so on.

Thanks for your attention !

Postscript



Thanks for your attention again !