~~Sp.~~ Geometry of CY manifolds

1. Complex coord. and compl. structure.

$M \to \{x^\mu\}$ - coord. $\mu = \overline{1, 2n}$

$\hookrightarrow \text{bas}\, TM = \{\partial_\mu\}$

$\text{bas}\, T^*M = \{dx^\mu\}$

- Complex structure: $\mathcal{J}: V \to V$; $\mathcal{J}^2 = -\mathrm{id}$; V - lin. space

can choose such basis $\|\mathcal{J}_a{}^b\| = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix}$; $\mathcal{J}(e_a) = \mathcal{J}_a{}^b e_b$

$\text{bas}\, V = \{e_a\}$

$a = \overline{1, 2n}$

$\dim V = 2n$

natural basis: $\{e_1, \dots, e_n, \mathcal{J}(e_1), \dots, \mathcal{J}(e_n)\}$;

- Almost c.s. $\mathcal{J}: T_pM \to T_pM$; $\mathcal{J}^2 = -\mathrm{id}$, $\forall p \in M$;

choose $\{x^1, \dots, x^n\} \to \left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$; then define

$$\frac{\partial}{\partial y^a} = \mathcal{J}\left(\frac{\partial}{\partial x^a}\right); \Rightarrow \left\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^a}\right\} \coloneqq \text{bas}\, TM - \text{natural basis}$$

- Complex coord: $\frac{\partial}{\partial z^a} = \frac{\partial}{\partial x^a} - i\frac{\partial}{\partial y^a}$; $\frac{\partial}{\partial \bar{z}^a} = \frac{\partial}{\partial x^a} + i\frac{\partial}{\partial y^a}$;

$$z^a = x^a + i y^a; \quad \bar{z}^a = x^a - i y^a;$$

$\text{bas}\, T^h M = \left\{\frac{\partial}{\partial z^a}\right\}$; $\text{bas}\, T^{*h} M = \{dz^a\}$;

2. Differential forms

p-forms $\Rightarrow$ (p,q)-forms

$$\omega^{(p,q)} = \frac{1}{p!\,q!}\, \omega_{[a_1 \dots a_p][b_1 \dots b_q]}\, dz^{a_1} \wedge \dots \wedge dz^{a_p}\, d\bar{z}^{b_1} \wedge \dots \wedge d\bar{z}^{b_q}$$

$\partial \omega^{(p,q)} \in \Omega^{(p+1,q)}$ $\qquad d = \partial + \bar{\partial}$;

$\bar{\partial} \omega^{(p,q)} \in \Omega^{(p,q+1)}$

3. Hermitean structure and herm. metric

metric: $g = \frac{1}{2} g_{a\bar{b}} (dz^a \otimes d\bar{z}^b + d\bar{z}^b \otimes dz^a)$ $\boxed{g_{a\bar{b}} = g_{\bar{b}a}}$

1. $\overline{g(X,Y)} = g(X,Y)$, $X = X^a \partial_a + \bar{X}^a \bar{\partial}_a$

2. $g(X,Y) = g(Y,X)$ $g: T_pM \times T_pM \to \mathbb{R}$

Hermitean structure (analogue of Riemannian metric for complex manifolds)

$h: T_pM \times T_pM \longrightarrow \mathbb{C}$

1. $h(a_1 X_1 + a_2 X_2, Y) = a_1 h(X_1, Y) + a_2 h(X_2, Y)$; $a_1, a_2 \in \mathbb{R}$

2. $h(Y,X) = \overline{h(X,Y)}$

3. $h(JX, Y) = i\, h(X,Y)$

$h = h_{a\bar{b}}\, dz^a \otimes d\bar{z}^b$; $h_{a\bar{b}}$ - no sym. in indices

$h(X,Y) = h_{a\bar{b}} X^a \bar{Y}^b$

$g = \frac{1}{2}(h + \bar{h}) = \mathrm{Re}\, h$

$\omega = \frac{i}{2}(h - \bar{h}) = \mathrm{Im}\, h$; $\omega_{a\bar{b}} = h_{a\bar{b}}\, dz^a \wedge d\bar{z}^b$

$= \frac{i}{2} h_{a\bar{b}} (dz^a \otimes d\bar{z}^b - d\bar{z}^b \otimes dz^a)$ (no a/s in (ab)) no antisymm!

compl. herm. manifold: (g, J, ω)

$g(JX, JY) = g(X,Y)$

$\omega(X,Y) = g(JX, Y) \iff \omega_{\mu\nu} = J_\mu{}^\rho g_{\rho\nu}$

Kähler: $d\omega = 0 \Rightarrow \left.\begin{array}{l} \partial_c h_{a\bar{b}} = \partial_a h_{c\bar{b}} \\ \bar{\partial}_{\bar{c}} h_{a\bar{b}} = \bar{\partial}_{\bar{b}} h_{a\bar{c}} \end{array}\right\} h_{a\bar{b}} = \partial_{a\bar{b}} K$

$\partial\omega + \bar{\partial}\omega$

$K(z,\bar{z}) \sim K(z,\bar{z}) + f(z) + f'(\bar{z})$

4. <u>CY- manifold</u> $\quad \exists\, \Omega \in \Lambda^{(p,0)} T^* M \left(\Omega^{(p,0)}\right)$

$$\Omega = \Omega_{a_1 \dots a_n}\, dz^{a_1} \wedge \dots \wedge dz^{a_n}$$

$$d\Omega = 0$$

1) Ricci-flat : $R_{\mu\nu} = 0$

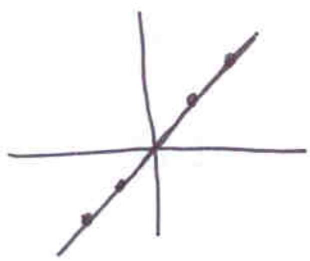
2) Holonomy : $SU(3)$ carries only 1 parallel spinor $\nabla \Psi = 0$
(comp Kähler) (preserves $N=2$ susy for II)

<u>CY-theorem</u> :

1) <u>Kähler manifold</u>:

0] $\mathbb{C}^m$: $\quad h = \sum_{j=1}^{m} dz^j\, d\bar{z}^j$; $\quad \omega = i \sum_{j=1}^{m} dz^j\, d\bar{z}^j$; $\quad d\omega = 0$

1] $\mathbb{CP}^n$: $\quad \mathbb{C}^{n+1} \to \{\zeta^1, \dots, \zeta^{n+1}\}$; $\quad \mathbb{CP}^n = \{\mathbb{C}^{n+1},\ z \sim \lambda z,\ \forall \lambda \in \mathbb{C}\}$
$= \mathbb{C}^{n+1}/\sim$;



$\mathbb{C}^2$

fix some $\zeta^j \neq 0$ (patch U_j)

define $z^A = \zeta^A / \zeta^j$; $\quad A = 1, \dots, j-1, j+1, \dots, n+1$

for $j = n+1$; $\quad z^a = \zeta^a / \zeta^{n+1}$; $\quad a = 1, \dots, n$

$$h = \frac{1}{(1+|z|^2)^2}\left(\delta_{ab}(1+|z|^2) - \bar{z}_a \bar{z}_b\right) dz^a \otimes d\bar{z}^b$$

$$\cancel{- \frac{i\, z_a \bar{z}_b}{(1+|z|^2)^2}\, dz^a\, d\bar{z}^b}$$

$$K(z, \bar{z}) = \ln(1 + |z|^2)$$

2) CY_n – subspace of $\mathbb{CP}^{n+1}$

CY_1: $F(z_0, z_1, z_2) = z_0^n + z_1^n + z_2^n = 0 \quad (n=3)$

patches: $z_0 \neq 0 \; U_0, \; \Omega_0 = \frac{dx}{y^2}$; $\quad x = \frac{z_1}{z_0}, \; y = \frac{z_2}{z_0}$

$z_1 \neq 0, \; U_1, \; \Omega_1 = \frac{d\tilde{x}}{\tilde{y}^2}$; $\quad \cancel{\tilde{x} = x^{-1}, \; \tilde{y} = y \cdot \tilde{x}}$

$\tilde{x} = \frac{z_0}{z_1}, \; \tilde{y} = \frac{z_2}{z_1}$;

$z_2 \neq 0 \quad \ldots$

CY_2: $F(z_0, z_1, z_2, z_3) = \sum a_{ijkl} z^i z^j z^k z^l$

$$\Omega_0 = \frac{dx_1 \wedge dx_2}{\partial f / \partial x_3}; \quad x_\mu = z_\mu / z_0$$

$$f(x_\mu) = \frac{F(z)}{z_0^4}$$

CY_3 (realistic example)

Y: $F = \sum_{i=1}^{5} z_i^5 = 0$; $\quad \chi(Y) = 200$; $\quad N_{gen} = 100$

$$\chi(Y/G) = \chi(Y)/n(G)$$

$G = \mathbb{Z}_5 \times \mathbb{Z}_5$

action of G: $\quad A: (z_1, z_2, z_3, z_4, z_5) \to (z_5, z_1, z_2, z_3, z_4)$

$B: (z_1, z_2, z_3, z_4, z_5) \to (\alpha z_1, \alpha^2 z_2, \alpha^3 z_3, \alpha^4 z_4, z_5)$

for some $\alpha^5 = 1$.

does not have fixed points on $\mathbb{CP}^4$