Lecture 1 — Introduction to diff. geometry.

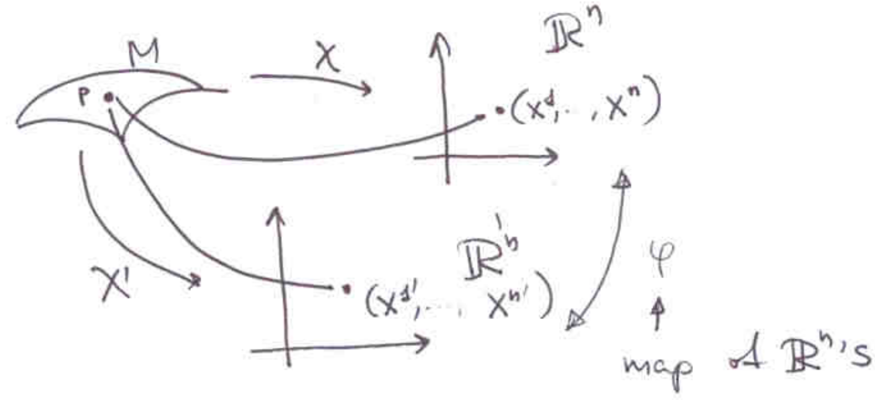
1. ~~Ger~~ Manifolds

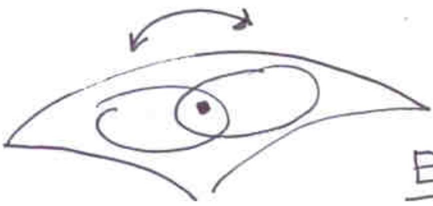
Manifold – set of points + coordinates.

$X : M \longrightarrow \mathbb{R}^n$

$\dim M = n$

defines coordinate patches





$$x^{\mu'} = x^{\mu'}(x^{\mu})$$

EG :

$$x^{1'} = x^1 \cdot \cos\theta + x^2 \sin\theta$$

$$x^{2'} = -x^1 \sin\theta + x^2 \cos\theta$$

2. Vector

def: $X^{\mu'}(x') = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \cdot X^{\nu}(x)$

for $\forall\ f : M \to \mathbb{R}$ – scalar function $f'(x') = f(x)$

define $X(f) = X^{\mu}(x)\partial_\mu f(x) = X^\mu \partial_\mu f$

↑ invariant $\quad X = X^\mu \partial_\mu$;

← tangent space $T_pM = \{ X \}$

$\text{bas}\, T_pM = \{\partial_\mu\}$.

3. Covector

$\partial_\mu f$ – component of covector space T_p^*M

def: $\omega'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu(x)$; $X(\omega) = X^\mu \omega_\mu$ – inv.

$$\omega = \omega_\mu(x) dx^\mu \leftarrow \text{1-form}; \quad df = \partial_\mu f\, dx^\mu$$

bas $T_p^*M = \{dx^\mu\}$;

3. de Rahm operator

$f(x)$ – 0-form.

df – 1-form, $df = \partial_\mu f\, dx^\mu$;

Integration: $\int df = \int \partial_\mu f\, dx^\mu = f(a) - f(b)$

$\int \omega = \int \omega_\mu dx^\mu$ – line integral

dx^μ – line element

$dx^\mu \wedge dx^\nu = \frac{1}{2}(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu)$ – surface element

$\vdots$

$dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \frac{1}{p!}(dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} + \ldots)$ – p-volume element.

p-form: $\omega^{(p)} = \frac{1}{p!}\omega^{(p)}_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$

$d: \omega^{(p)} \to \omega^{(p+1)}$

$\omega^{(p+1)} = \frac{1}{(p+1)!}\underbrace{\partial_{\mu_1}\omega^{(p)}_{\mu_2 \ldots \mu_{p+1}}}_{\omega^{(p+1)}_{\mu_1 \ldots \mu_{p+1}}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}}$

$\omega^{(p)} \in \wedge^p T^*M$

$d^2 = 0$ – check

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4. Metric and Hodge star

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ – interval

$g = g_{\mu\nu} dx^\mu \otimes dx^\nu$; $g: TM \otimes TM \to \mathbb{R}^+$

$g(X, Y) = X^\mu_{(x)} Y^\nu_{(x)} g_{\mu\nu}(x)$ – length.

$*: \wedge^p T^*M \to \wedge^{d-p} T^*M$ for $\dim M = d$

volume element $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_d} = \epsilon^{\mu_1 \ldots \mu_d} dx^1 \wedge \ldots \wedge dx^d$

$$(*\omega^{(p)})_{\mu_1\dots\mu_{d-p}} = \frac{1}{p!}\,\epsilon_{\mu_1\dots\mu_{d-p}\,\nu_1\dots\nu_p}\, g^{\nu_1\rho_1} g^{\nu_2\rho_2}\dots g^{\nu_p\rho_p}\,\omega^{(p)}_{\rho_1\dots\rho_p}$$

$$\shortparallel$$

$$\omega^{(d-p)}{}_{\mu_1\dots\mu_{d-p}} \qquad \epsilon_{\mu_1\dots\mu_d} = \sqrt{g}\,\underbrace{\varepsilon_{\mu_1\dots\mu_d}}_{1 \text{ or } 0}$$

check: $**\omega^{(p)} = (-1)^{p(d-p)}(-1)^{(s_+-s_-)}\omega_p$; (s_+, s_-) - number of $+$ and $-$ in g.

<u>EG</u>: $\dim \mathbb{R}^3 = 3$; $\omega^{(2)} = \frac{1}{2}dx^1\wedge dx^2$; the only non-zero coeff. $\omega^{(2)}_{12} \neq 0$

$\cancel{*(dx^1\wedge dx^2)} =$ $\quad *: \Lambda^2 TM \to \Lambda^1 TM$

$$= (*\omega^{(2)})_\mu = \frac{1}{1!}\,\epsilon_{\mu\,\mu_1\mu_2}\, g^{\mu_1\nu_1} g^{\mu_2\nu_2}\,\omega^{(2)}_{\nu_1\nu_2} = \epsilon_{\mu 12}\,\omega_{12} = \delta^3_\mu\,\omega_{12}$$

$$*\omega = dx^3$$

$$*(\tfrac{1}{2}dx^1\wedge dx^2) = dx^3 \;;$$

<u>Lagrangian</u> ; $d = 4$; $F = \frac{1}{2}F_{\mu\nu}\,dx^\mu\wedge dx^\nu$

$$*F = \frac{1}{2}\,\epsilon_{\rho\sigma\kappa\lambda}\,F^{\kappa\lambda}\,dx^\rho\wedge dx^\sigma$$

$$F\wedge *F = \frac{1}{4}\,F_{\mu\nu}\,\epsilon_{\rho\sigma\kappa\lambda}\,F^{\kappa\lambda}\,\underbrace{dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma}_{\varepsilon^{\mu\nu\rho\sigma}dx^1\wedge\dots\wedge dx^4} =$$

$$= \frac{\sqrt{g}}{4}\cdot\frac{2!\,2!}{\cancel{4}}\cdot F_{\mu\nu}F^{\mu\nu}\,dx^1\wedge\dots\wedge dx^4 = F_{\mu\nu}F^{\mu\nu}\,\cancel{g}\,d\mathrm{Vol}$$

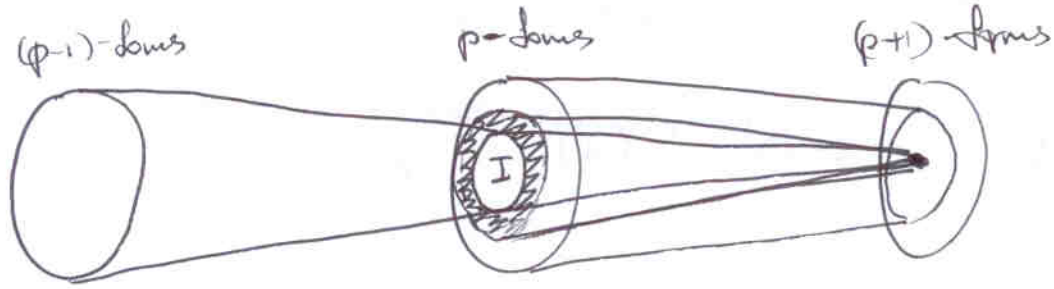
$$\mathcal{L}_{ED} = \int F\wedge *F \,;$$

<u>Check</u> $\quad \int F\wedge F = ?$

5. deRham complexes

exact forms: $\omega^{(p)} = d\omega^{(p-1)}$; $\in I_p = \{\omega^{(p)}; \omega^{(p)} = d\omega^{(p-1)}\}$

closed forms: $d\omega^{(p)} = 0$; $\in K_p = \{\omega^{(p)}; d\omega^{(p)} = 0\}$



$$\Lambda^{p-1} T^* M \xrightarrow{d} \Lambda^p T^* M \xrightarrow{d} \Lambda^{p+1} T^* M$$

$H_p(M) = K_p(M) / I_p(M)$ – such closed forms, which are not exact.

1) Poincaré theorem: for $\mathbb{R}^n$ all $H_p = 0$

2) EX: S^1 – circle: $\theta \sim \theta + 2\pi$; 1-form: $\omega(\theta) = d\theta$; $d\omega = 0$
$\omega(\theta+2\pi) = \omega(\theta)$

exact form: $\omega = d\varphi(\theta)$; $\varphi(\theta)$ – 0-form on S^1
$\varphi(\theta + 2\pi) = \varphi(\theta)$

however $\varphi(\theta) = \theta$ – is <u>not</u>

$\omega(\theta) = d\theta$ – Volume form (single)

$H_1(S^1) = 1$

3) ~~Show~~ find $H_p(T^2)$;

<u>H_p – cohomology groups.</u>