Introduction to Entanglement Entropy and Holography

Helmholtz International Summer School "Cosmology, Strings, and New Physics"

Lecture 1

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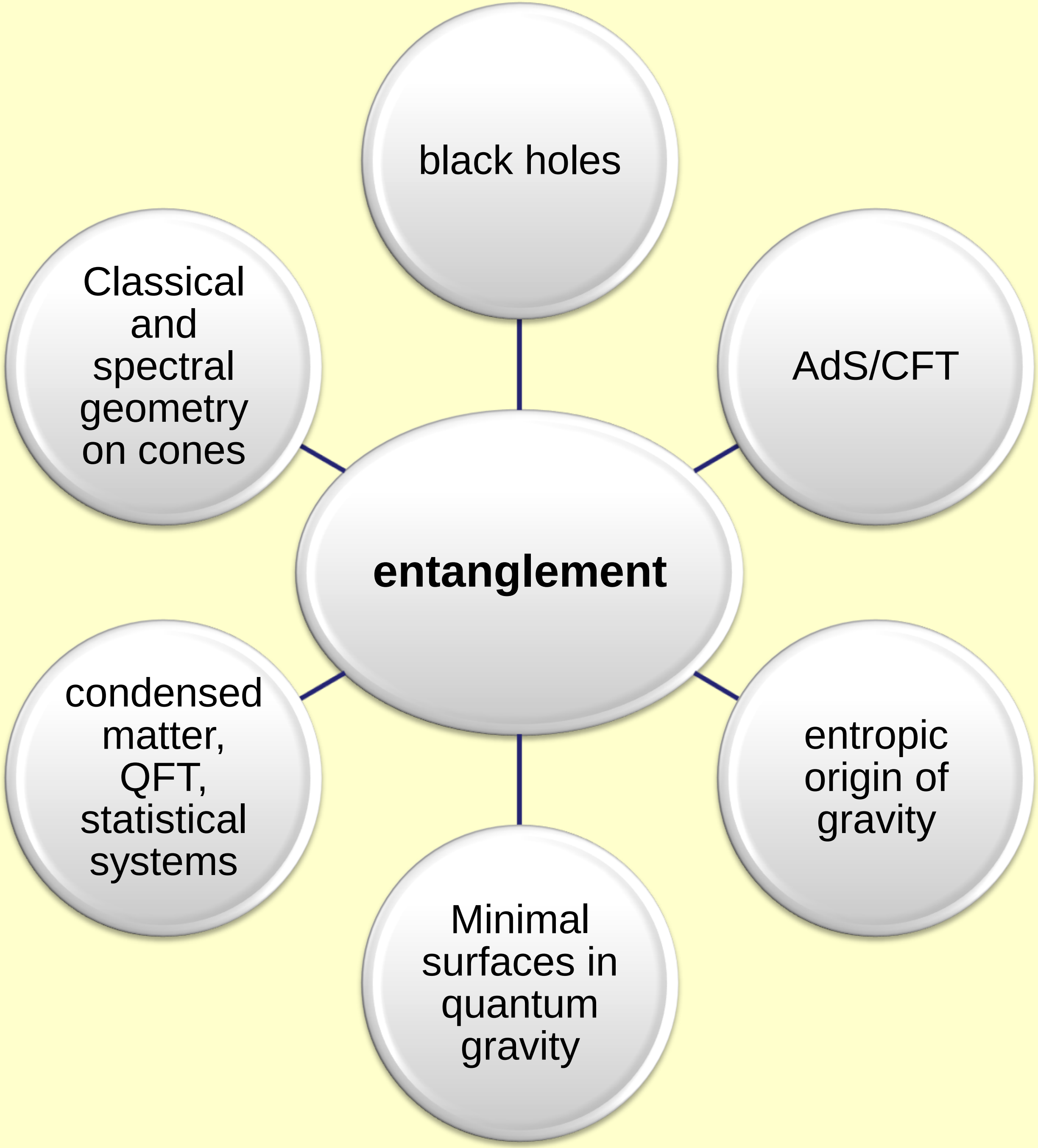

black holes
Classical and spectral geometry on cones
AdS/CFT
entanglement
condensed matter, QFT, statistical systems
entropic origin of gravity
Minimal surfaces in quantum gravity

Plan of lectures

Lecture 1: Entanglement entropy in QFT’s

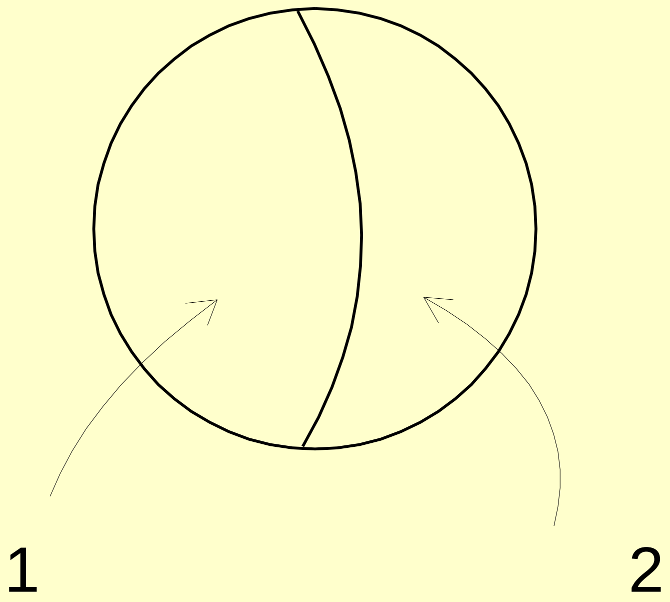
-Entanglement entropy (definitions and basic properties)
-EE and Renyi entropy (REE): methods of computations in free QFT's (spectral geometry and etc)
-logarithmic part of EE and conformal anomalies
- some developments

Lecture 2: Holographic approach to entanglement entropy

-Holographic EE (HEE)
- motivations for HEE
- HEE: how it works
- HEE and conformal anomalies
- Bekenstein-Hawking entropy of black holes
- Entanglement and Gravity

Quantum entanglement

<u>quantum mechanics</u>:
states of subsystems may not be described independently
= states are entangled



<u>importance:</u>
studying correlations of different systems (especially at strong couplings), critical phenomena and etc

entangled states (an example)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

Quantum state of particle «1» cannot be described independently from particle «2» (even for spatial separation at long distances)

measure of entanglement

$$S_2 = -\mathrm{Tr}_2 \rho_2 \ln \rho_2$$

- entropy of entanglement

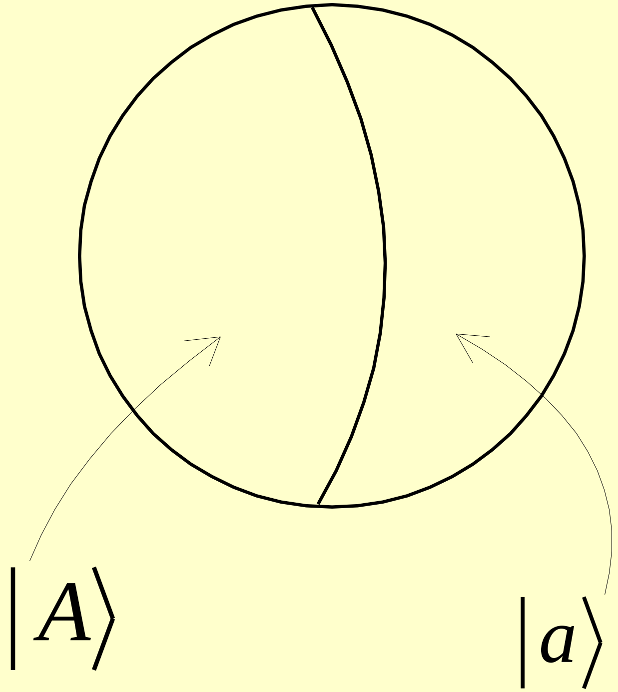
$$\rho_2 = \mathrm{Tr}_1 (|\psi\rangle\langle\psi|)$$

density matrix of particle «2» under integration over the states of «1»

«2» is in a mixed state when information about «1» is not available

S – measures the loss of information about "1" (or "2")

reduced density matrix- a general definition



$$\rho(A,a\,|\,B,b)$$

$$\rho_1(A\,|\,B) = \sum_a \rho(A,a\,|\,B,a),$$

$$\rho_2(a\,|\,b) = \sum_A \rho(A,a\,|\,A,b),$$

$$\rho_1 = Tr_2\,\rho, \quad \rho_2 = Tr_1\,\rho,$$

$$S_1 = -Tr_1\,\rho_1 \ln \rho_1, \quad S_2 = -Tr_2\,\rho_2 \ln \rho_2$$

Entanglement Renyi Entropy

$\rho_1 = \mathrm{Tr}_2\, \rho$ – reduced density matrix

$$S_1^{(\alpha)} = \frac{\ln \mathrm{Tr}_1\, \rho_1{}^{\alpha}}{1-\alpha}$$ – entanglement Renyi entropy

In general, $\alpha > 0$, and $\alpha \neq 1$

Next we consider integer values $\alpha = n = 2, 3, 4, \ldots.$

An example:

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_1 \left|\downarrow\right\rangle_2 + \left|\downarrow\right\rangle_1 \left|\uparrow\right\rangle_2\right) = \sum_{Aa} C_{Aa} \left|A\right\rangle_1 \left|a\right\rangle_2$$

$$A = "\uparrow,\downarrow", \quad a = "\uparrow,\downarrow",$$

$$C = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

$$(\rho_1)_{AB} = \sum_a C_{Aa} C^*{}_{Ba} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$\rho_1{}^{\alpha} = \begin{pmatrix} 2^{-\alpha} & 0 \\ 0 & 2^{-\alpha} \end{pmatrix}, \quad S^{(\alpha)} = S = 2\ln 2$$

I. **Basic properties**

$S_1^{(\alpha)} \geq 0, \quad (S_1^{(\alpha)} = 0,$ if and only if ρ_1 is pure state)

Different limits:

$$S_1^{(\alpha)} \to S_1 \quad , \quad \alpha \to 1$$

$$S_1 \equiv -\mathrm{Tr}_1\, \rho_1 \ln \rho_1$$ – entanglement entropy

$$\lim_{\alpha \to 0} S_1^{(\alpha)} = \ln D,$$

where D is the # of nonvanishing eigenvalues of reduced density matrix ρ_1

$$\lim_{\alpha \to \infty} S_1^{(\alpha)} = -\ln \lambda_1$$

where λ_1 is the largest eigenvalue of ρ_1

Basic properties

II. "Symmetry" in a pure state

$$S_1^{(\alpha)} = S_2^{(\alpha)}$$

sketch of the proof: let $\rho = |\psi\rangle\langle\psi|$, then $|\psi\rangle = \sum_{aA} C_{Aa} |A\rangle |a\rangle$

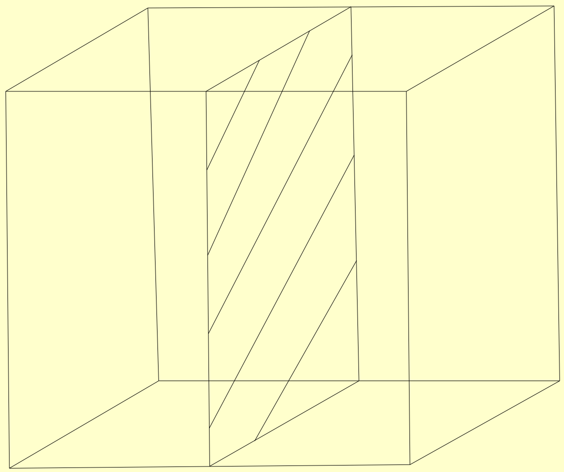
$$\rho_1(A|B) = \sum_a C_{Aa} C^*{}_{Ba} \quad \rightarrow \quad \rho_1 = CC^+$$

$$\rho_2(a|b) = \sum_A C_{Aa} C^*{}_{Ab}, \quad \rightarrow \quad \rho_2 = C^T C^*$$

non-vanishing eigenvalues of ρ_1 and ρ_2 coincide;

in general, $\rho = \dfrac{e^{-H/T}}{Tr\, e^{-H/T}} \quad \rightarrow S_1^{(\alpha)} \neq S_2^{(\alpha)}$

Basic properties: the entropy is a function of the characteristics of the separating surface



$$S_1^{(\alpha)} = S_2^{(\alpha)} = f(A)$$

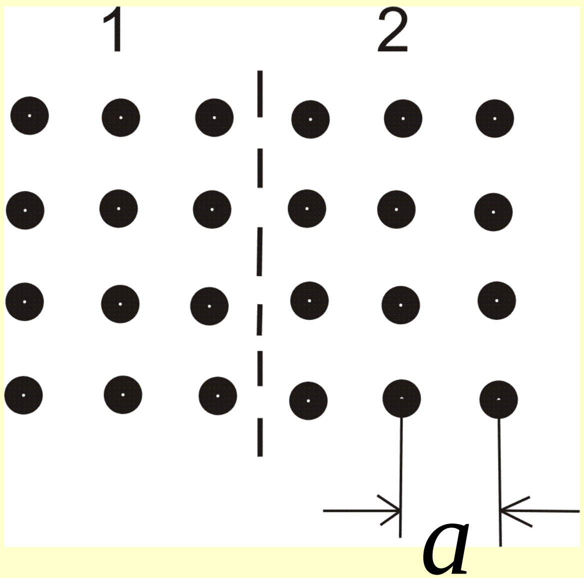
in a simple case the entropy
is a function of the area A of a separating
surface

$S^{(\alpha)} \sim A$ - in a relativistic QFT (for Srednicki 93, Bombelli et al, 86)

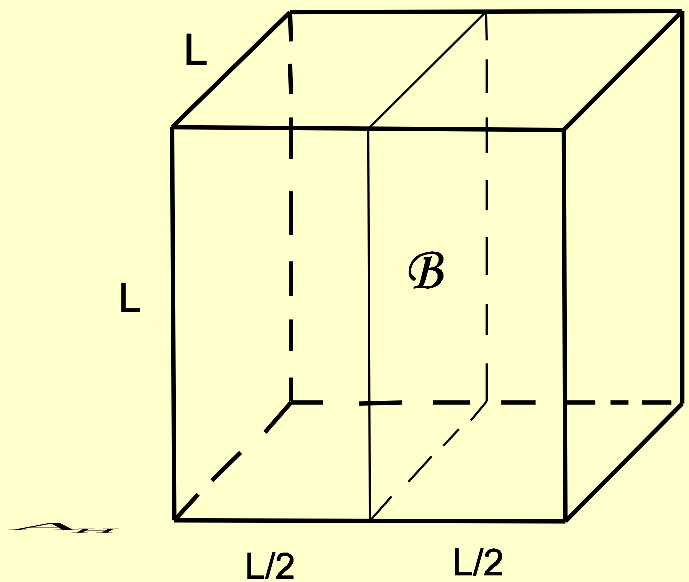
$S^{(\alpha)} \sim A^{D-2} \ln A$ - in some fermionic condensed matter systems (Gioev & Klich 06), e.g. for Fermi liquids (Swingle 1007.4825, Calabrese et al 1111.4836)

Basic properties: dependence on UV cutoff

spin lattice



continuum limit

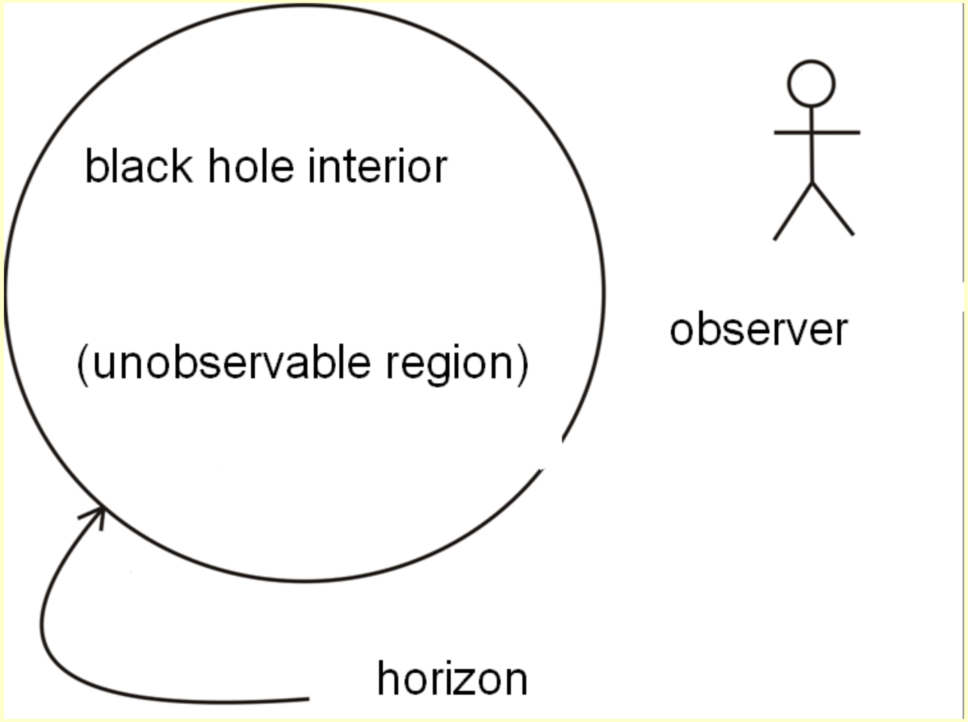


$$S \sim \frac{A}{a^{d-2}}$$

A – area of a flat separation surface $\mathcal{B}$ which divides the system into two parts (pure state)

entropy per unit area in a QFT is determined by a UV cutoff!

Entanglement as a possible source of black hole entropy



Bekenstein-Hawking entropy

$$S^{BH} = \frac{c^3 A_H}{4\hbar G_N}$$

A_H - area of the horizon

BH entropy may be a measure of the loss of information about States hidden under the horizon

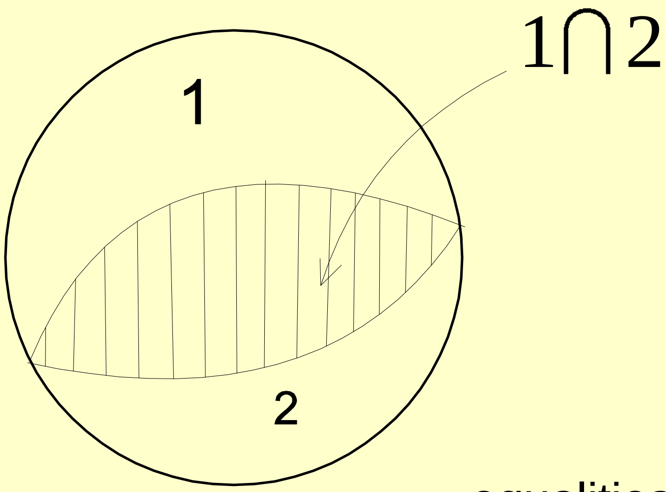
Basic properties :

subadditivity of the entanglement entropy (not Renyi!)

$$|S_1 - S_2| \leq S \leq S_1 + S_2, \quad S = -Tr\, \rho \ln \rho$$

$$S_1 + S_2 \geq S_{1\cup 2}$$

strong subadditivity



$$S_1 + S_2 \geq S_{1\cup 2} + S_{1\cap 2}$$

equalities are applied to the von Neumann entropy and are based on the concavity property

Basic properties : modular Hamiltonian and reduction to thermal states

$$\rho_1 \equiv e^{-H} = \rho$$

$H -$ is the "modular Hamiltonian" (non-local operator, in general)

$$U(s) = \rho^{is} = e^{-isH}$$

one-parameter modular group: $O(s) \equiv U(s) O U(-s)$

$\mathrm{Tr}(O(s)\rho) = \mathrm{Tr}(O\rho) -$ symmetry transformations

Kubo-Martin-Schwinger periodicity relation

$$\mathrm{Tr}(\rho O_1(i) O_2) = \mathrm{Tr}(\rho O_2 O_1)$$

$H -$ is a non-local operator, in general.

Some exceptions: planar and spherical entangling surfaces in flat space

Basic properties : planar entangling surface and Unruh effect

Minkowsky metric

$$ds^2 = -dt^2 + dx^2 + dl^2{}_{D-2} = -dx_+ dx_- + dl^2{}_{D-2}$$

position of entangling surface: $x = 0 \ (t = 0)$

one-parameter modular group is the group of boost transformations

$$x_\pm(s) = x_\pm e^{\pm 2\pi s},$$

Rindler coordinates $x_\pm = re^{\pm\tau/R} \to \tau(s) = \tau + 2\pi Rs$

$$ds^2 = -\frac{r^2}{R^2} d\tau^2 + dr^2 + dl^2{}_{D-2}$$

$$H = (2\pi R) H_R + \ln Z_R(T_R)$$

H_R - Rindler Hamiltonian, $Z_R(T) = \mathrm{Tr}\,(e^{-H_R/T})$ - Rindler partition function, $T_R = 1/(2\pi R)$

Basic properties : planar entangling surface and Unruh effect (continued)

Rindler thermal density matrix:

$$\rho_R = e^{-H_R/T_R} / \mathrm{Tr}\,(e^{-H_R/T_R}) -$$ thermal density matrix

$$T_R = (2\pi R)^{-1}$$ - Rindler temperature

$$\rho_R = \rho$$

$$S^{(\alpha)} = \frac{1}{\alpha - 1}\Big(F_R(T_R/\alpha) - \alpha F_R(T_R)\Big)$$

$$F_R(T) = -\ln Z_R(T)$$- Rindler free energy

Problem with path integral construction and geometrical representation for non-integer indexes

entanglement for a cylindrical entangling surface

$$ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 + dz^2$$

$r = a -$ position of a cylindrical entangling surface

$$r = a + x\cosh\eta, \qquad t = \sinh\eta$$

$$ds^2 = -x^2 d\eta^2 + dx^2 + (a + x\cosh\eta)^2 d\varphi^2 + dz^2$$

- observers at rest w.r.t. given coordinates are Rindler observers;
- metric is not static, constant η sections are conical surfaces (with different conical angles);
- there is a conical singularity if $\eta \to i\tau$ and $\tau \sim \tau + 2\pi\alpha$ with $\alpha \neq 1$, there is a jump of the curvature at $\tau = 2\pi\alpha$ (constant τ sections are different!)

the reduced density matrix is a time ordered opeartor

$$\hat{\rho} = \hat{U}(t = 2\pi i) = T \exp\left(-\int_0^{2\pi} d\tau \hat{H}(i\tau) \right), \quad \left[\hat{H}(\eta), \hat{H}(\eta') \right] \neq 0,$$

where $\hat{H}(\eta)$ is a time dependent generator of the $\eta-$ evolution;

consequences:

- in general, $\hat{\rho}^{\alpha}$ cannot be represented as an analogous evolution operator

(for non-integer α),

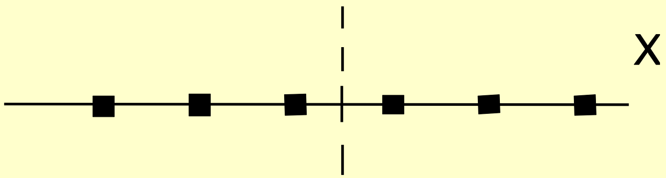
- there may not exist a geometrical construction of a

background manifold with conical singularities which corresponds to $\mathrm{Tr}\,\hat{\rho}^{\alpha}$

- modular Hamiltonian is a non-local operator

Computations of Entanglement Entropy

Ising spin chains



$$H=\sum_{K=1}^{N}(\sigma_K^X\sigma_{K+1}^X+\lambda\sigma_K^Z)$$

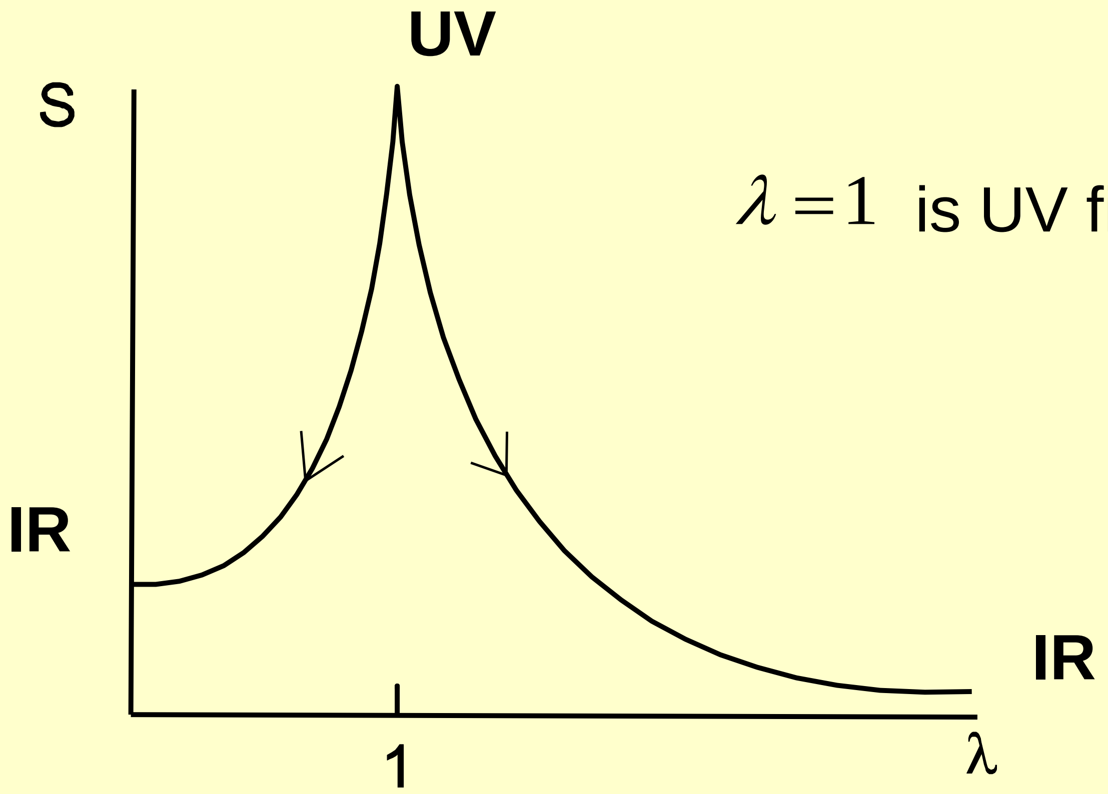
off-critical regime at large *N* $\lambda\neq 1$ $|\lambda-1|\ll 1$

$$S(N,\lambda)=-\frac{1}{6}\log_2|\lambda-1|$$

critical regime $\lambda=1$

$$S(N,\lambda)=\frac{1}{6}\log_2\frac{N}{2}$$

RG-evolution of the entropy



$\lambda = 1$ is UV fixed point

entropy does not increase under RG-flow (as a result of integration of high energy modes)

Explanation

Near the critical point the Ising model is equivalent to a 2D quantum field theory with mass *m* proportional to $|\lambda - 1|$

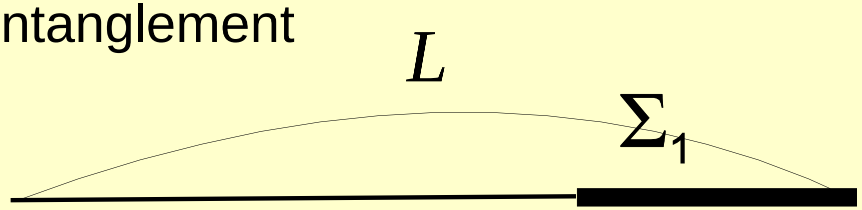
$$S = -\frac{c}{6}\ln ma$$

At the critical point it is equivalent to a 2D CFT with 2 massless fermions each having the central charge 1/2

$$S = \frac{c}{6}\ln\frac{L}{a}$$

entanglement in 2D models: analytical results

ground state entanglement on an interval



Calabrese, Cardy hep-th/0405152

L_1 is the length of Σ_1

massive case:

$$S = \frac{c}{6} \ln \frac{1}{ma}$$

a is a UV cutoff

$$1/m \ll L_1$$

massless case:

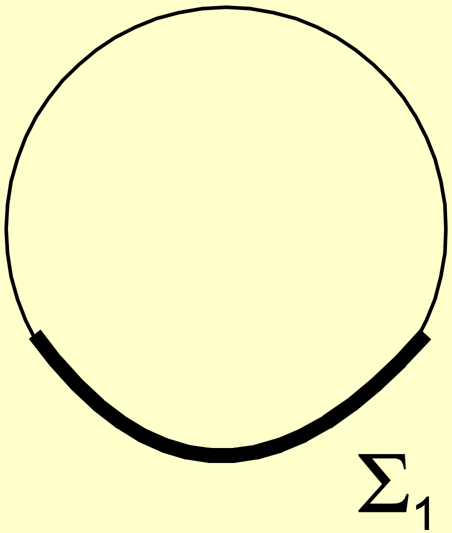
$$S = \frac{c}{6} \ln \left(\frac{L}{\pi a} \sin \frac{\pi L_1}{L} \right) + 2g$$

$$S = \frac{c}{6} \ln \frac{L_1}{a}$$

analytical results in 2D (continued)

$$S = \frac{c}{3}\ln\left(\frac{L}{\pi a}\sin\frac{\pi L_1}{L}\right)$$

L_1 is the length of Σ_1



ground state entanglement for a system on a circle

$$S = \frac{c}{3}\ln\left(\frac{\beta}{\pi a}\sinh\frac{\pi L_1}{\beta}\right)$$

system at a finite temperature $T = 1/\beta$

Computations of Entanglement Entropy in Higher Dimensions and Spectral Geometry

1st step: representation in terms of a ‘partition function’

we want to find an analog of Rindler relation

$$S^{(\alpha)} = \frac{1}{1-\alpha}\left(\ln Z_R(T_R/\alpha) - \alpha \ln Z_R(T_R)\right)$$

put $\rho = e^{-H/T} / \mathrm{Tr}\, e^{-H/T}$ – thermal density matrix

$Z(n,T) \equiv \mathrm{Tr}_1\, (\mathrm{Tr}_2\, e^{-H/T})^n$ – a partition function

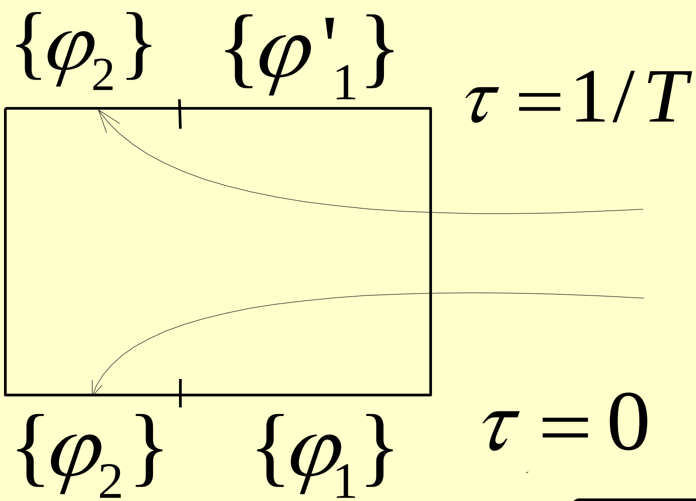
$\beta = 2\pi n -$ "inverse temperature"

$Z(T) = Z(1,T)$

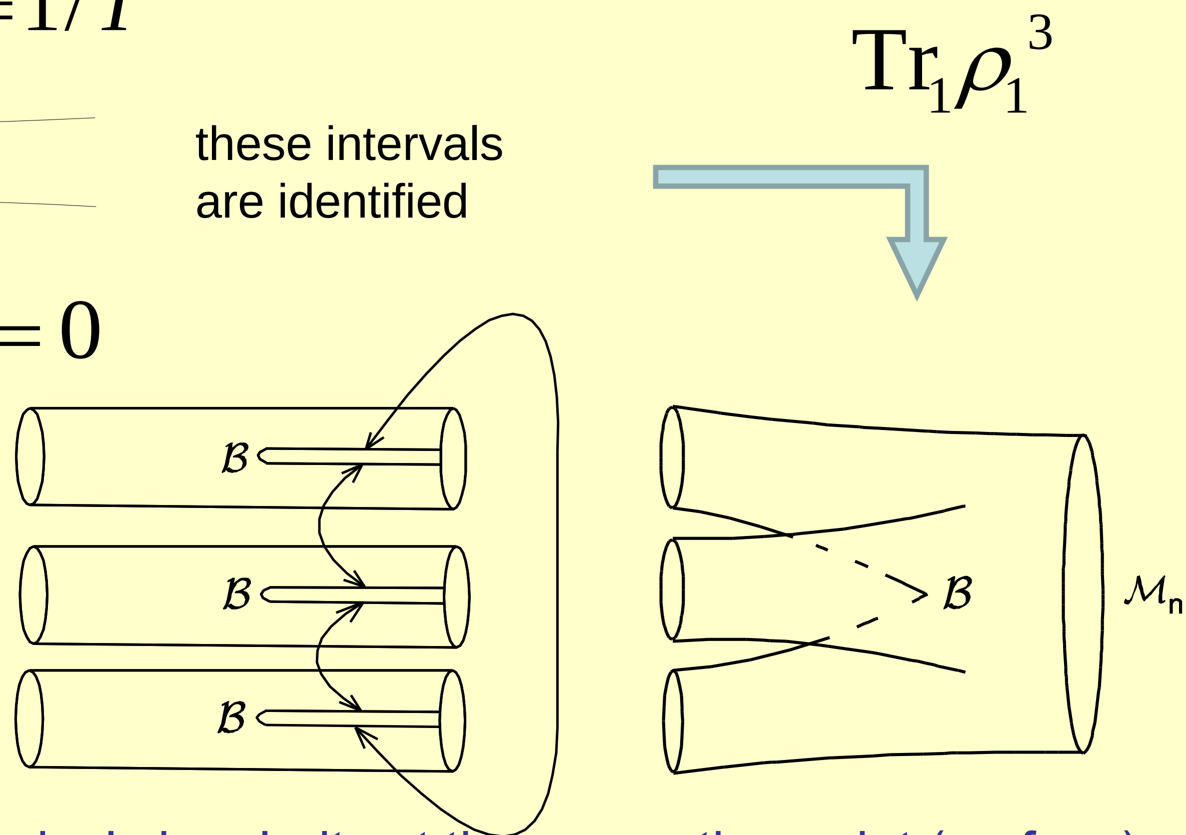
$$S_1^{(n)}(T) = \frac{\ln Z(n,T) - n \ln Z(T)}{1-n}$$

2d step: relation of a 'partition function' to an effective action on a 'curved space'

$$W(\beta,T) = -\ln Z(\beta,T) - \text{effective action}$$



these intervals are identified



one glues n (n=3) copies

a 'curved space' with conical singularity at the separating point (surface) effective action on a manifold with conical singularities is the gravity action (even if the manifold is locally flat)

curvature at the singularity is non-trivial:

$$R = 4\pi(1-n)\delta^{(2)}(B)$$

entanglement entropy in a flat space has to do with gravity effects!

3[d] step: use results of spectral geometry

$$W = \frac{1}{2}\sum_k \eta_k \ln\det L_k \quad , \quad \eta_k = \pm 1$$

$L_k -$ Laplace operators of different spin fields on M_n

$$W = \sum_{p=0}^{d-1} \Lambda^{d-p} \frac{A_p}{p-d} - A_d \ln(\Lambda/\mu) + \ldots$$ for dimension d even,

$$A_p = \sum_k \eta_k A_{k,p}$$, where $A_{k,p}$: $\mathrm{Tr}\, e^{-tL_k} \sim \sum_{p=0}^{\infty} t^{\frac{p-d}{2}} A_{k,p} \ldots;$

$\Lambda -$ is a UV cutoff; μ is a physical scale (mass, inverse syze etc)

an example: a scalar Laplacian $L_0 = -\nabla^2$:

$$A_0 = O(n)\,, \; A_2 = \frac{1}{24\pi}\int_{M_n} R + \frac{1}{12\gamma_n}({\gamma_n}^2 - 1)\int_B \quad , \quad \gamma_n = n^{-1}$$

There are non-trivial contributions from conical singularities located at the 'separating' surface B

computations

$$S = \sum_{p=2}^{d-2} \Lambda^{d-p} \frac{s_p}{d-p} + s_d \ln(\Lambda/\mu) + \dots,$$

$$S^{(n)} = \sum_{p=2}^{d-2} \Lambda^{d-p} \frac{s^{(n)}{}_p}{d-p} + s^{(n)}{}_d \ln(\Lambda/\mu) + \dots, - \text{Renyi entropies}$$

$$s_p \equiv -\lim_{n\to 1}(n\partial_n - 1)A_p(n) \quad , \quad s^{(n)}{}_p \equiv \frac{nA_p(1) - A_p(n)}{n-1}$$

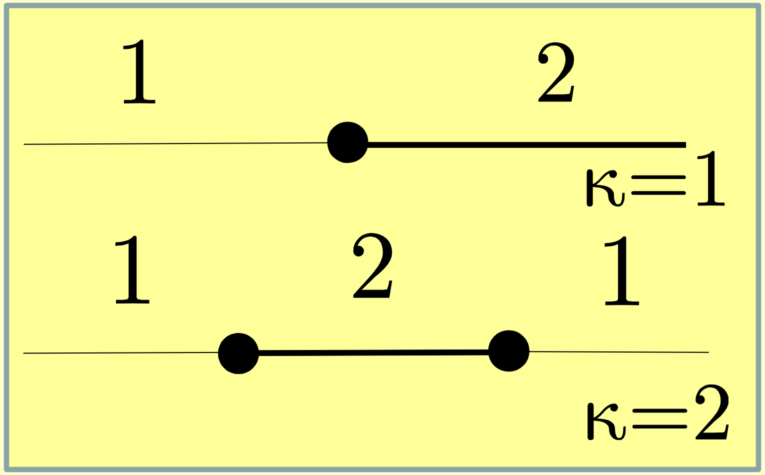
$$s_0 = s^{(n)}{}_0 = 0 \quad , \quad s_{2k+1} = s^{(n)}{}_{2k+1} = 0 - \text{(if boundaries are absent)}$$

2D CFT: "c" massless scalars and spinors

$$W = \frac{a}{2}\ln\det\nabla^2 - b\ln\det\gamma^\mu\nabla_\mu$$

$c = a + b$ $\;-$ CFT central charge

$$s_2 = \frac{c}{6}k \quad , \quad s^{(n)}{}_2 = \frac{c}{12}k(1+\gamma_n)$$



$$S = \frac{c}{6}k\ln(L/\varepsilon)\,,$$

$$S^{(n)} = \frac{c}{12}(1+\gamma_n)k\ln(L/\varepsilon),$$ – Renyi entropy

$\varepsilon \equiv \Lambda^{-1}$, $\quad L -$ a typical syze of the system,

the result holds for a system on an interval devided into 2 or 3 parts

$k = 1, 2$ - the number of separating points (which yield conical singularities)

4D N=4 super SU(N) Yang-Mills theory at weak coup.

6 scalar multiplets, 4 multiplets of Weyl spinors, 1 multiplet of gluon fields

$$S^{(n)} = \frac{1}{2}\Lambda^2 s^{(n)}{}_2 + s^{(n)}{}_4 \ln(\Lambda/\mu) + \ldots$$

$s^{(n)}{}_2 = \dfrac{d(N)}{4\pi}\gamma_n A(B)$ – area of the separating surface B

Conformal invariance of the heat coefficients

Let $L -$ be a Laplace type operator;

$$\mathrm{Tr}\, e^{-tL} \sim \sum_{p=0}^{\infty} t^{\frac{p-d}{2}} A_p, \quad t \to 0;$$

Let classical action $I[\phi, g] = \int d^d x \sqrt{g}\, \phi(x) L \phi(x) -$ be invariant w.r.t.

conformal transformations:

$$g_{\mu\nu}'(x) = e^{2\omega(x)} g_{\mu\nu}(x), \quad \phi'(x) = e^{k\omega(x)} \phi(x),$$

$$I[\phi, g] = I[\phi', g']$$

Then $A_{p=d}$ is conformal invariant: $A_{p=d}[g] = A_{p=d}[g']$

3 invariants on a smooth entangling surface *B* in d=4 (no boundaries)

$$F_a = -\frac{1}{2\pi}\int_B \sqrt{\sigma}\, d^2x\, R(B) \quad , \qquad R(B) - \text{scalar curvature of } B$$

$$F_c = \frac{1}{2\pi}\int_B \sqrt{\sigma}\, d^2x\, C_{\mu\nu\lambda\rho} n_i^\mu n_j^\nu n_i^\lambda n_j^\rho \quad , \quad C_{\mu\nu\lambda\rho} - \text{Weyl tensor of } M \text{ at } B,$$

$$F_b = \frac{1}{2\pi}\int_B \sqrt{\sigma}\, d^2x \left(\frac{1}{2}\mathrm{Tr}(k_i)\mathrm{Tr}(k_i) - \mathrm{Tr}(k_i k_i) \right) ,$$

$(k_i)_{\mu\nu}$ – extrinsic curvatures of B, n_i – normal vectors

F_a, F_b, F_c – are invariant with respect to the Weyl transformations $g_{\mu\nu}'(x) = e^{2\omega(x)} g_{\mu\nu}(x)$

EE and trace anomaly in d=4:

local conformal anomaly

$$\langle T^{\mu}_{\mu} \rangle = -2aE - cI - \frac{c'}{24\pi^2}\nabla^2 R$$

$$E = \frac{1}{16\pi^2}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$ – "density" of the Euler n.

$$I = -\frac{1}{16\pi^2}C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}, \quad C_{\mu\nu\lambda\rho}$$ – the Weyl tensor

"bulk charges" a, c

a- monotonically decreases under RG flow from UV to IR

suggested by J. Crardy, PLB 215, 749-752 (1988),

proved by Z.Komargodski and A.Schwimmer, JHEP 12 (2011)099

Logarithmic term in EE in d=4

$$s_{\log} = aF_a + cF_c + bF_b$$ (no boundaries)

- Ryu,Takayanagi, JHEP 0608, 045 (2006),

- Solodukhin, PLB 665, 305 (2008)
- Fursaev, Patrushev, Solodukhin, PRD 88, 044054 (2013)

$c = b$ for CFT's

conformal charges in the trace anomaly of a CFT uniquely fix the logarithmic term in EE (no boundaries) !

Why relation of the log part of EE and scale anomaly is interesting?

In supersymmetric CFT's 'charges' do not receive quantum corrections. This means computations in free theories can be used at strong couplings.

One can compare 'holographic' EE (strong couplings) and straightforward computations in free CFT's (to check they coincide)

Entanglement across a spatial surface is sensible to:

- the area of the surface (the leading terms);
- topology of the surface;
- extrinsic curvatures;
- geometrical properties of a spacetime

$$s^{(n)}{}_4 = -2\chi a(\gamma_n)F_a + b(\gamma_n)F_b \text{ - } \quad \text{in a flat space } \mathbb{R}^3$$

$\chi -$ Euler number of entangling surface

$$F_b = \frac{1}{2\pi}\int_B \sqrt{\sigma}\, d^2x \left(\frac{1}{2}\mathrm{Tr}(k)\mathrm{Tr}(k) - \mathrm{Tr}(k^2)\right),$$

k is an extrinsic curvature of B in $\mathbb{R}^3$

$F_b = 0$, - if $B = S^2$

$F_b = -\frac{L}{2R}$, - if B is a cylinder of radius R and length L

Computation of coefficient functions in SYM

$$s^{(n)}{}_4 = d(N)(a(\gamma_n)F_a + c(\gamma_n)F_c + b(\gamma_n)F_b)$$

contributions to heat kernel coefficients from conical singularities

$$\bar{A}_4(\Delta^{(i)}) = \bar{a}_i(\gamma_n)F_a + \bar{c}_i(\gamma_n)F_c + \bar{b}_i(\gamma_n)F_b$$

$$a(\gamma_n) = \frac{1}{n-1}(6\bar{a}_0(\gamma_n) - 4\bar{a}_{1/2}(\gamma_n) + \bar{a}_1(\gamma_n)) = \frac{1}{32}(\gamma_n{}^3 + \gamma_n{}^2 + 7\gamma_n + 15)$$

$$c(\gamma_n) = \frac{1}{n-1}(6\bar{c}_0(\gamma_n) - 4\bar{c}_{1/2}(\gamma_n) + \bar{c}_1(\gamma_n)) = \frac{1}{32}(\gamma_n{}^3 + \gamma_n{}^2 + 3\gamma_n + 3)$$

$\lim_{n\to 1} b(\gamma_n) = 1,$ ('holographic' arguments by S.N. Solodukhin, arXiv:0802.3117)

Some developments: EE in gauge theories

Definition of EE in gauge theories should be taken with care: the Hilbert space of physical degrees of freedom does not admit a tensor product description associated to spatial separation (the physical degrees of freedom are non-local: lines, loops,…)

to put it other way: spatial separation of physical states violates the Gauss law

(see Polikarpov and Buividovich, 2008)

A wayout is to embed the physical Hilbert space to a larger space which admits factorization, see

Donnelly (2012), Cassini, Huerta, Rosabal (2014), Ghosh, Soni, Trivedi (2015) and other

Some developments: EE and RG flow in d=3

The F-theorem (a 3D analog of C-theorem): finite part of the free energy on 3-sphere decrees along RG flow

Cassini, Huerta (2012): a monotonic RG behavior of EE in 3d (EE for a circle)

Some developments: EE and boundaries

Boundary effects in EE:

In d=4
Fursaev (2006), Wilczek and Hertzberg (2011), Fursaev (2013), Kuo-Wei Hung (2016)

In d=3 (and connection to boundary charges in the integrated scale anomaly)

Fursaev and Solodukhin (2016), Kuo-Wei Hung (2016)

thank you for attention