One-loop effective action with composite fields in gauge theories

O. Kolesnikov
Scientific supervisor: P. M. Lavrov

Tomsk State University
Tomsk, Russia

One of the most important objects in QFT is generating functional of vertex functions, effective action, which contains all information about given system.
One can obtain the effective action using the Legendre transformation applied to the generating functional of connected Green functions. Relation between those functionals has the form of functional Caliraut-type equation. In terms of standard QFT approach special properties of Clairaut-type equation cannot be used in perturbation theory. Those properties appear in composite fields approach. The aim of this talk is to present generalization of the results described in paper[1] for the case of gauge theories.

[1]Legendre transformations and Clairaut-type equations / P. Lavrov, B. Merzlikin // Phys.Lett. B756 (2016) 188-193.

Outline

We are going to discuss:

- Clairaut's equation on the Berezin algebra and its solution;
- Case of special function on the right-hand side of the equation;
- Functional Clairaut-type equation and its solution in special case;
- Effective action in terms of composite fields approach;
- One-loop effective action with composite fields in gauge theories as a solution of the Clairaut-type equation of special form.

Clairaut's equation

Clairaut's differential equation:

$$y - y'_A X^A = \psi(y'), \tag{1}$$

where

$$X = \{X^A, A = 1, \ldots, n\} \in \Lambda, \tag{2}$$
$$\varepsilon(X^A) = \varepsilon_A$$

$$y'_A = y(X) \frac{\overleftarrow{\partial}}{\partial X^A} \tag{3}$$

Using the notation

$$y'_A \equiv z_A(X), \tag{4}$$

one can rewrite the Clairaut's equation in following form

$$y - z_A X^A = \psi(Z). \tag{5}$$

Differentiating with respect to X^A one can get the system of equations

$$z_A \frac{\overleftarrow{\partial}}{\partial X^A} \left(\psi(X) \frac{\overleftarrow{\partial}}{\partial z_A} + (-1)^{\varepsilon_A} X^A \right) = 0, \quad A, B = 1, ..., n. \tag{6}$$

Solution

If

$$H_{AB} = z_A \frac{\overleftarrow{\partial}}{\partial X^B} = y \frac{\overleftarrow{\partial}}{\partial X^A} \frac{\overleftarrow{\partial}}{\partial X^B} = 0, \tag{7}$$

then

$$\begin{aligned} z_A = C_A = const \in \Lambda, \\ y(X) = C_A X^A + \psi(C), \quad C = \{C_1, \dots, C_n\} \end{aligned} \tag{8}$$

Solution

In case of

$$\mathrm{sDet}\, H_{AB} \neq 0 \tag{9}$$

and one can express z_A from the system algebraically

$$\psi(X)\frac{\overleftarrow{\partial}}{\partial z_A} + (-1)^{\varepsilon_A} X^A = 0, A = 1, \ldots, n \tag{10}$$
$$z_A = y'_A = \varphi_A(X),$$

then, if the conditions of integrability are satisfied, the solution can be presented in the form:

$$\varphi_A(X)\frac{\overleftarrow{\partial}}{\partial X^B} = (-1)^{\varepsilon_A \varepsilon_B}\varphi_B(X)\frac{\overleftarrow{\partial}}{\partial X^A} \tag{11}$$
$$y(X) = y(\xi) + \int\limits_{\xi}^{X} \varphi_A(X) dX^n \ldots dX^1, \quad \xi = \{\xi^1, \ldots, \xi^n\} \in \Lambda.$$

Special case

Let us consider function ψ of special form:

$$\psi(z) = \alpha \ln\left(1 - \beta(z_A a^A)\right), \tag{12}$$
$$\alpha, \beta \in \mathbb{R}, a = \{a^1, \ldots, a^n\} \in \Lambda, \varepsilon(a^A) = \varepsilon_A.$$

Then one can get the system of equations:

$$\left(X^A - \frac{\beta a^A}{1 - \beta(z_B a^B)}\right)\left(z_A(X)\frac{\overleftarrow{\partial}}{\partial X^B}\right) = 0. \tag{13}$$

Using the same method as before, in case of

$$H_{AB} = 0 \quad z_A = C_A = const \in \Lambda, \tag{14}$$

the solution takes the form:

$$y(X) = C_A X^A + \alpha \ln\left(1 - \beta(C_A a^A)\right), \quad C = \{C_1, \ldots, C_n\}. \tag{15}$$

Solution (special case)

If

$$\mathsf{sDet}\, H_{AB} \neq 0 \tag{16}$$

one can obtain the system of equations:

$$X^A - \frac{\beta a^A}{1 - \beta(z_B a^B)} = 0, A, B = 1, \ldots, n. \tag{17}$$

Introducing the set of supernumbers $b = b_1, \ldots, b_n \in \Lambda, b_A a^A = 1$, one can get the solution

$$y(X) = \frac{b_B X^B}{\beta} - \alpha \ln b_B X^B - \alpha - \alpha \ln \alpha\beta \tag{18}$$

Let us consider even functional $\Gamma = \Gamma[F], \quad \varepsilon(\Gamma) = 0$ where $F^i, \quad i = 1, \ldots, n$ are fields.

$$\Gamma - \Gamma \frac{\overleftarrow{\delta}}{\delta F^i(X)} F^i = \Psi\Big[\Gamma \frac{\overleftarrow{\delta}}{\delta F}\Big], \tag{19}$$

$\Psi[Z]$ - even functional of variables $Z_i(X) \equiv \Gamma \frac{\overleftarrow{\delta}}{\delta F^i}, i = 1, \ldots, N,$

$$\Gamma \frac{\overleftarrow{\delta}}{\delta F^i(X)} F^i = \int \Big(\Gamma \frac{\overleftarrow{\delta}}{\delta F^i(X)}\Big) F^i(X) dX^n \ldots dX^1, \tag{20}$$
$$F^i(X) \Gamma \frac{\overleftarrow{\delta}}{\delta F^j(Y)} = \delta^i_j \delta(X - Y).$$

Special case

We restrict ourselves with special form of the functional on the right-hand side of equation:

$$\Psi[Z] = \alpha \ln \left(1 - \beta(Z_i \mathcal{A}^i)\right) \tag{21}$$

$$\varepsilon(\mathcal{A}^i) = \varepsilon(Z_i), i = 1, \ldots, N, \alpha, \beta = const \in \mathbb{R}.$$

Using method similar to previous cases one can get the solution

$$\Gamma[F] = \frac{1}{\beta}(\mathcal{B}_i F^i) - \alpha \ln \left(\mathcal{B}_i F^i\right) - \alpha + \alpha \ln \beta, \tag{22}$$

where

$$\varepsilon(\mathcal{B}_i) = \varepsilon(\mathcal{A}^i), i = 1, \ldots, N, \int \mathcal{B}_i(X) \mathcal{A}^i(X) dX^n \ldots dX^1 = 1. \tag{23}$$

Theories with composite fields

Let us consider field model described by a set of fields $\phi^A(X), A = 1, \ldots, N, \varepsilon(\phi^A) = \varepsilon_A$ with non-degenerate action $S[\phi]$. Non-local composite fields are introduced as follows

$$L^i(\phi)(X,Y) = L^i(\phi)(Y,X) = \frac{1}{2}\mathcal{A}^i_{AB}\phi^A(X)\phi^B(Y), \tag{24}$$

where

$$\mathcal{A}^i_{AB} = (-1)^{\varepsilon_A\varepsilon_B}\mathcal{A}^i_{BA}, \quad \varepsilon(\mathcal{A}^i_{AB}) = \varepsilon_i + \varepsilon_A + \varepsilon_B. \tag{25}$$

Generating functional of Green functions

$$Z[J,K] = \int e^{i(S[\phi]+J_A\phi^A+K_iL^i(\phi))}\mathcal{D}\phi = e^{iW[J,K]}, \tag{26}$$

where $K_i(X,Y) = K_i(Y,X), i = 1, \ldots, M, \varepsilon(K_i) = \varepsilon_i$ are the sources to $L^i(\phi)(X,Y)$

Average fields $\Phi^A(X)$ and composite fields $F^i(X,Y) = F^i(Y,X)$ are defined by relations

$$\frac{\overrightarrow{\delta}}{\delta J_A(X)} W[J,K] = \Phi^A(X) \tag{27}$$
$$W[J,K] \frac{\overleftarrow{\delta}}{\delta K_i(X,Y)} = L^i(\phi)(X,Y) + \tfrac{1}{2} F^i(X,Y).$$

Then one can obtain effective action using the double Legendre transformation:

$$\Gamma[\Phi, F] = W[J,K] - J_A \Phi^A - K_i \Big(L^i(\phi) + \frac{1}{2} F^i \Big). \tag{28}$$

We introduce following notations:

$$S[\Phi] \frac{\overleftarrow{\delta}}{\delta \Phi^A} \frac{\overleftarrow{\delta}}{\delta \Phi^B} = S''. \tag{29}$$

Clairaut-type equation in gauge theories

It can be shown that the equation for one-loop effective action takes the form of the functional Clairaut-type equation

$$\Gamma^{(1)} - \Gamma^{(1)} \frac{\overleftarrow{\delta}}{\delta F^i} F^i = \frac{i}{2}\,\mathsf{sTr}\,\ln\left(S'' - 2\Gamma^{(1)} \frac{\overleftarrow{\delta}}{\delta F^i} \mathcal{A}^i_{AB}\right). \tag{30}$$

Introducing the notations

$$\Gamma^{(1)} \frac{\overleftarrow{\delta}}{\delta F^i} = Z_i(X, Y), \tag{31}$$

one can get following equation:

$$\Gamma^{(1)} = Z_i F^i + \frac{i}{2}\,\mathsf{sTr}\,\ln Q, \tag{32}$$

where

$$Q_{AB}(X, Y) = S'' - 2 Z_i \mathcal{A}^i_{AB}. \tag{33}$$

Solution

Making necessary transformations one can obtain the system

$$F^i(X,Y) - i(-1)^{\varepsilon_A}(Q^{-1})^{AB}(Y,X)\mathcal{A}^i_{AB} = 0. \tag{34}$$

In order to find solution let us introduce the set of matrices $\mathcal{B}^{AB}_i = (-1)^{\varepsilon_A \varepsilon_B}\mathcal{B}^{AB}_i$ which satisfy following equation

$$(-1)^{\varepsilon_B}\mathcal{A}^i_{AB}\mathcal{B}^{CD}_i = \frac{1}{2}(\delta^C_A \delta^D_B + \delta^C_A \delta^D_B). \tag{35}$$

Expressing $Z_i\mathcal{A}^i_{BA}$ and $Z_i F^i$ and substituting it in the equation one can find the solution for one-loop effective action

$$\Gamma^{(1)}[\Phi, F] = \frac{1}{2}\,\mathsf{sTr}\left((F^i\mathcal{B}_i)S''\right) - \frac{i}{2}\,\mathsf{sTr}\left(i(F^j\mathcal{B}_j)\right) - \frac{i}{2}\delta(0)N. \tag{36}$$

Discussions

- We considered and solved Clairaut's equation in case of anti-commuting variables.
- We studied functional Clairaut-type equation with special right-hand side, which appears in gauge field theory.
- We considered the composite fields approach which, in perturbation theory, leads to the functional Clairaut-type equation.
- One-loop effective action in gauge theories was found as a solution of the equation.