Introduction to superconformal mechanics

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Plan

- Space-time symmetries. Conformal mechanics: examples and peculiarities.
- Superymmetries. Graded symmetry algebras. Brief sketch on supermatrix and supergroups.
- $1D$ super-Poincare and superconformal symmerties.
- Component formulation of $1D$ supersymmetric field theories.
- Superspace formulation of $1D$ supersymmetric field theories.
- $\mathcal{N}=2$ superconformal mechanics.
- Models of $\mathcal{N}=4$ superconformal mechanics.

Symmetries play fundamental role in the formulation of modern theories, specifying them. For example, systematization of the currently known particles

$$\text{Bosons}: \quad \underbrace{\overbrace{A_\mu \sim (\vec{E}, \vec{B})}^{\text{Maxwell Theory}, U(1)} \quad \oplus \quad W^\pm_\mu, W^0_\mu}_{\text{Electroweek Theory}, SU(2)\times U(1)} \quad \oplus \quad \underbrace{G^r_\mu, \; r=1,\dots,8}_{\text{Strong Interaction}, SU(3)} \quad \oplus \quad \underbrace{g_{\mu\nu}}_{\text{Gravity}} \quad \oplus \quad \underbrace{H}_{\text{Higgs}}$$

$$\text{Fermions}: \quad \psi^i_\alpha \,, \quad \bar{\psi}_{\dot{\alpha} i}$$

is defined by the symmetries:

- Maxwell theory: gauge group $U(1)$;
- Electro-week theory: gauge group $U(1) \times SU(2)$;
- Standard model: gauge group $U(1) \times SU(2) \times SU(3)$;
- Gravity: local diffeomorphism group of four-dimensional space-time;
- String theory: local diffeomorphism group of the worldsheet (two-dimensional space-time).

Symmetries are defined by concrete groups and corresponding algebras.

$$\psi \; \to \; g(\lambda)\, \psi \,, \text{ etc.}, \quad \text{where} \quad g(\lambda) = \exp\{\lambda^A B_A\} \,, \quad [B_A, B_B] = i c^C_{AB} B_C \,.$$

An important role is played by the space-time (relativistic) symmetries.

Space-time symmetries

$$\Big\{\text{Lorentz algebra } SO(D-1,1)\Big\} \quad\subset\quad \Big\{\text{Poincare algebra } T^D \in\!\!\!\!+ SO(D-1,1)\Big\} \quad\subset$$

$$\subset \quad \Big\{\text{ conformal algebra } SO(D,2)\Big\}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} {}[L_{\mu\nu},L_{\rho\lambda}] = i\,(\eta_{\nu\rho}L_{\mu\lambda} + \eta_{\mu\lambda}L_{\nu\rho} - (\mu\leftrightarrow\nu)) \\ {}[P_\mu,P_\nu] = 0\,, \quad [L_{\mu\nu},P_\lambda] = i\,(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu) \end{array}\right\}\text{Poincare algebra} \\ {}[D,P_\mu] = i\,P_\mu\,, \quad [D,K_\mu] = -i\,K_\mu\,, \quad [D,L_{\mu\nu}] = 0 \\ {}[K_\mu,K_\nu] = 0\,, \quad [P_\mu,K_\nu] = -2i\,(\eta_{\mu\nu}D + L_{\mu\nu})\,, \quad [L_{\mu\nu},K_\lambda] = i\,(\eta_{\nu\lambda}K_\mu - \eta_{\mu\lambda}K_\nu) \end{array}\right\}\begin{array}{l}\text{conformal}\\ \text{algebra}\end{array}$$

$$\eta_{\mu\nu} = \mathrm{diag}(+1,-1,\dots,-1), \quad \mu = 0,1,\dots,D-1$$

$D=1$ conformal algebra: $\quad so(2,1) \cong su(1,1) \cong sl(2,\mathbb{R})$

Internal symmetries

SU(n) algebra $\quad [T_j^i, T_l^k] = i\,(\delta_l^i T_j^k - \delta_j^k T_l^i)\,, \quad (T_j^i)^+ = -T_i^j\,, \quad T_i^i = 0\,, \qquad i = 1,\dots,n$

O(n) algebra $\quad [J_{ab}, J_{cd}] = i\,(\delta_{bc}J_{ad} + \delta_{ad}J_{bc} - (a\leftrightarrow b))\,, \quad (J_{ab})^+ = J_{ab} = -J_{ba}$

$$a = 1,\dots,n$$

Internal symmetries commute with space-time ones.

Example: $\quad O(3) \cong SU(2)$

$$[J_a,J_b] = i\epsilon_{abc}J_c\,, \quad J_{ab} = -\epsilon_{abc}J_c\,, \qquad J_a = \tfrac{1}{2}\sigma_a\,, \quad a = 1,2,3\,, \qquad T_i^j = J_a(\sigma_a)_i{}^j\,, \quad i = 1,2$$

Conformal mechanics and supersymmetric generalization of it have many applications now and have significant potential for use in the next.

In the near horizon limit the extreme ($M = Q$) Reissner-Nordström black hole solution of Einstein-Maxwell equations are (in the units with $G = 1$)

$$ds^2 = -\left(\frac{r}{M}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2 + M^2 d\Omega$$

But

$$-\left(\frac{r}{M}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

where

$$\eta_{\mu\nu} = diag(-,+,-), \qquad \eta_{\mu\nu} x^\mu x^\nu = -M^2,$$

$$x^0 = (2r)^{-1}[1 + r^2(M^2 - t^2)], \qquad x^1 = (2r)^{-1}[1 - r^2(M^2 + t^2)], \qquad x^2 = Mrt.$$

The near horizon limit the extreme Reissner-Nordström black hole possesses $AdS_2 \times S^2$ geometry.

[P. Claus, M. Derix, R. Kallosh, J. Kumar, P. Townsend, A. Van Proeyen; 1998]
[J.A. de Azcarraga, J.M. Izquierdo, J.C. Perez-Bueno, P.K. Townsend; 1998]
[S. Bellucci, A. Galajinsky, E. Ivanov, S. Krivonos, J. Niederle; 2002]

AdS_2 part, having $SO(2,1)$ symmetry, is described by conformal mechanics.

Superconformal mechanics models describe motion of the particle with angular momentum (spin) near horizon of the extreme Reissner-Nordström black hole.

In the large-n limit an n-particle generalization of the conformal (and superconformal) mechanics in the form of a (super)conformal Calogero model provides a microscopic description of multiple extremal Reissner-Nordström black holes in the near-horizon limit.

[G. Gibbons, P. Townsend; 1999]

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It was developed new class of inflation models with (spontaneously broken) conformal invariance [V. Rubakov; 2009];
[S. Ferrara, R. Kallosh, A. Linde, A. Marrani, A. Van Proeyen; 2010] (inspired by the superconformal approach to supergravity).
Observational consequences of a broad class of such models are stable with respect to strong deformations of the scalar potential. This universality is a critical phenomenon near the point of enhanced symmetry, SO(1,1), in case of conformal inflation
[R. Kallosh, A. Linde; 2013].

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In this regard, it should be emphasized the $d = 1 + 1$ and $d = 1 + 0$ dilaton gravity coupled to scalar matter fields proved to be a reliable model for higher dimensional black holes and string inspired cosmologies
[V. de Alfaro, A.T. Filippov, E.A. Davydov].

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Models of conformal and superconformal mechanics can be considered as some limits of higher dimensional systems and are the perfect arena for exploring the latest ones.

One-particle conformal mechanics

Conformal mechanics action: [V. de Alfaro, S. Fubini, G. Furlan; 1976]

$$S = \tfrac{1}{2} \int dt \, \left(\dot{x}^2 - \frac{g}{x^2} \right)$$

Conformal invariance:

$$\delta t = a + b\,t + c\,t^2 \equiv f(t), \qquad \delta x = \tfrac{1}{2}\,\dot{f}\,x\,, \qquad \delta S = \int dt\,\dot{\Lambda}\,, \quad \Lambda = \tfrac{1}{4}\,\ddot{f}\,x^2$$

Conserved charges $\left(p = \dot{x}; \qquad \frac{d}{dt}\,(H\,\delta t - p\,\delta x + \Lambda) = 0\right)$:

$$\begin{aligned} H &= \tfrac{1}{2}\left(p^2 + \tfrac{g}{x^2}\right) \\ D &= tH - \tfrac{1}{2}\,xp \\ K &= t^2 H - txp + \tfrac{1}{2}\,x^2 \end{aligned}$$

$$\frac{d}{dt}\,K = \frac{\partial}{\partial t}\,K + \{K, H\}_P = 0, \quad \frac{d}{dt}\,D = \frac{\partial}{\partial t}\,D + \{D, H\}_P = 0, \quad H \ - \ \text{the Hamiltonian}$$

$$\{H, D\}_P = H, \quad \{K, D\}_P = -K, \quad \{H, K\}_P = 2D \qquad - \qquad \text{dynamical symmetry}$$

$$[\mathbf{A}, \mathbf{B}] = i\{A, B\}_P\ : \quad [\mathbf{H}, \mathbf{D}] = i\,\mathbf{H}, \quad [\mathbf{K}, \mathbf{D}] = -i\,\mathbf{K}, \quad [\mathbf{H}, \mathbf{K}] = 2i\,\mathbf{D} \quad - \quad sl(2, \mathbb{R}) \ \text{algebra}$$

Properties of the conformal mechanics:

- If $\mathbf{H}\,|E>=E\,|E>$, then $\mathbf{H}\,e^{i\alpha D}\,|E>=e^{2\alpha}\,E\,|E>$ $\Rightarrow$ the spectrum of $\mathbf{H}$ is continuous;
- The eigenspectrum of $\mathbf{H}$ includes all $E>0$ values, for each of which there exists a plane wave narmalizable state;
- The spectrum of $\mathbf{H}$ does not have an endpoint (ground state), the state with $E=0$ is not even plane wave normalizible.

This is an obstacle to describe the conformal theory in terms of $\mathbf{H}$ eigenstates.

The $sl(2,\mathbb{R})$ algebra in the Virasoro form:

$$\mathbf{R}=\tfrac{1}{2}\,(a\,\mathbf{H}+\frac{1}{a}\,\mathbf{K}),\qquad \mathbf{L}_{\pm}=-\tfrac{1}{2}\,(a\,\mathbf{H}-\frac{1}{a}\,\mathbf{K}\mp i\,\mathbf{D});\qquad a\ \text{is a parameter}$$

$$[\mathbf{R},\mathbf{L}_{\pm}]=\pm\,\mathbf{L}_{\pm},\qquad [\mathbf{L}_{+},\mathbf{L}_{-}]=-2\,\mathbf{R}$$

$\mathbf{R}$ is the $u(1)$ generator in $sl(2,\mathbb{R})\sim o(1,2)$ algebra.

The eigenvalues of

$$\mathbf{R}|_{t=0,\,a=1}=\tfrac{1}{2}\,\left(p^2+\frac{g}{x^2}+x^2\right)$$

are given by a discrete series

$$r_n=r_0+n,\quad n=0,1,2,\ldots;\qquad r_0=\tfrac{1}{2}\left(1+\sqrt{g+\tfrac{1}{4}}\right)$$

Black hole interpretation of the Hamiltonian shift $\mathbf{H}\to\mathbf{R}$:

The time t, corresponding to the Hamiltonian $\mathbf{H}$, is ill defined near the black hole horizon. True time in this region is τ, corresponding to new Hamiltonian $\mathbf{R}$.

Multi-particle conformal mechanics

- the hermitian $n\times n$-matrix field $X_a^b(t)$, $(\overline{X_a^b})=X_b^a$,
- complex commuting U(n)-spinor field $Z_a(t)$, $\bar{Z}^a=(\overline{Z_a})$,
- n^2 non–propagating "gauge fields" $A_a^b(t)$, $(\overline{A_a^b})=A_b^a$.

$$S_0=\int dt\left[\tfrac{1}{2}\,\mathrm{Tr}\,(\nabla X\nabla X)+\tfrac{i}{2}\,(\bar{Z}\nabla Z-\nabla\bar{Z}Z)+c\,\mathrm{Tr}A\right],$$

$$\nabla X=\dot{X}+i[A,X],\qquad \nabla Z=\dot{Z}+iAZ\ .$$

The $1D$ conformal SO$(1,2)$ symmetry:

$$\delta t=f,\qquad \delta X_a^b=\tfrac{1}{2}\,\dot{f}X_a^b,\qquad \delta Z_a=0,\qquad \delta A_a^b=-\dot{f}A_a^b,\qquad \partial_t^3 f=0$$

The local U(n) symmetry, $g(\tau)\in$ U(n):

$$X\to gXg^\dagger,\quad Z\to gZ\,,\quad A\to gAg^\dagger+i\dot{g}g^\dagger.$$

The U(n) gauge fixing : $X_a^b=x_a\delta_a^b,\qquad \bar{Z}^a=Z_a.$

The algebraic equations of motion

$$(Z_a)^2=c\quad \text{(which implies } c>0);\qquad A_a^b=\frac{Z_aZ_b}{2(x_a-x_b)^2},\ {\scriptstyle a\neq b}$$

As result, we arrive at the standard Calogero action

$$S=\tfrac{1}{2}\int dt\left[\sum_a \dot{x}_a\dot{x}_a-\sum_{a\neq b}\frac{c^2}{(x_a-x_b)^2}\right],\qquad H=\tfrac{1}{2}\left[\sum_a p_ap_a+\sum_{a\neq b}\frac{c^2}{(x_a-x_b)^2}\right],$$

Supersymmetric generalization

Symmetry algebras of the supersymmetric models are graded Lie algebras or Lie superalgebras

$$[B_A, B_B] = i\, c_{AB}^{C} B_C\,, \qquad [B_A, Q_K] = i\, g_{AK}^{M} Q_M\,, \qquad \{Q_K\,, Q_M\} = i\, f_{KM}^{A} B_A$$

B_A are even (bosonic) elements; Q_K are odd (fermionic) elements

Graded Jacobi identities

$$[[G_1, G_2\}, G_3\} + \text{graded cyclic} = 0$$

(there is additional minus sign if two fermionic operators are interchanged)

Bosonic subalgebra B_A are defined by Coleman–Mandula theorem.

On the fermionic operators Q_M it is realized the representation of the bosonic subalgebra.

Q_M generate supersymmetric tansformations

$$Q\,|\text{boson}> = |\text{fermion}>\,, \qquad Q\,|\text{fermion}> = |\text{boson}>$$

Parity: $q(B) = 0, \quad q(Q) = 1, \quad q(|\text{boson}>) = 0, \quad q(|\text{fermion}>) = 1$

Simple example of SUSY algebra: BRST symmetry

$$[B_A, Q] = 0\,, \qquad \{Q, Q\} = 0$$

Q is BRST charge

Exponential representation of Lie supergroups are given by

$$X = \exp\left\{ i\left(\lambda^A B_A + \xi^M F_M\right)\right\}$$

where λ^A are c-number parameters whereas ξ^M are Grassmann parameters:

$$\xi^M \xi^N = -\xi^N \xi^M \qquad \Rightarrow \qquad (\xi^1)^2 = 0, \quad (\xi^2)^2 = 0, \quad \text{etc.}$$

$$X = \left(\begin{array}{c|c} B_1 & F_1 \\ \hline F_2 & B_2 \end{array}\right); \qquad \begin{array}{l} B_{1,2} \text{ are ordinary matrices,} \\ F_{1,2} \text{ are fermionic matrices} \end{array}$$

$$\mathrm{str}\, X = \mathrm{tr}\, B_1 - \mathrm{tr}\, B_2 , \qquad \mathrm{str}\, XY = \mathrm{str}\, YX$$

$$\mathrm{sdet}\begin{pmatrix} B_1 & F_1 \\ 0 & 1 \end{pmatrix} = \det B_1, \quad \mathrm{sdet}\begin{pmatrix} 1 & F_1 \\ 0 & B_2 \end{pmatrix} = \det B_2^{-1}; \qquad \mathrm{sdet}\, XY = \mathrm{sdet}\, X \cdot \mathrm{sdet}\, Y$$

$$\begin{pmatrix} B_1 & F_1 \\ F_2 & B_2 \end{pmatrix} = \begin{pmatrix} 1 & F_1 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} B_1 - F_1 B_2^{-1} F_2 & 0 \\ B_2^{-1} & 1 \end{pmatrix}, \quad \mathrm{sdet}\, X = \det\left(B_1 - F_1 B_2^{-1} F_2\right) \cdot \det B_2^{-1}$$

$$\mathrm{sdet}\, X = \exp\{\mathrm{str}\,(\ln X)\}$$

$$\mathrm{OSp}(m|n): \qquad G = e^X = \left(\begin{array}{c|c} \mathrm{Sp}(n) & F_1 \\ \hline F_2 & \mathrm{SO}(m) \end{array}\right)$$

$$\mathrm{U}(m,n|p): \qquad G = e^X = \left(\begin{array}{c|c} \mathrm{U}(m,n) & F_1 \\ \hline F_2 & \mathrm{U}(p) \end{array}\right)$$

$$\mathrm{SU}(m,n|p): \qquad G = e^X, \quad \mathrm{str}\, X = 0$$

For $m+n=p$ the identity matrix obeys $\mathrm{tr}\, B_1 = \mathrm{tr}\, B_2$ and generates $\mathrm{U}(1)$ subgroup.
The quotient $\mathrm{PSU}(m,n|p) = \mathrm{SU}(m,n|p)/\mathrm{U}(1)$ is simple and is often denoted just $\mathrm{SU}(m,n|p)$.

$4D$ $\mathcal{N}$-extended Poincare superalgebra

$$P_\mu,\, L_{\mu\nu},\, T^i_j \quad + \quad Q^i_\alpha,\, \bar{Q}_{\dot{\alpha}i} = (Q^i_\alpha)^+ \quad + \quad Z^{ij}, \bar{Z}_{ij} = (Z^{ij})^+$$

$$\{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2\delta^i_j(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu, \qquad \{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta}Z^{ij}, \qquad \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{Z}_{ij},$$

$$[P_\mu, Q^i_\alpha] = 0, \quad [P_\mu, \bar{Q}_{\dot{\alpha}i}] = 0, \qquad [L_{\mu\nu}, Q^i_\alpha] = -\tfrac{1}{2}(\sigma_{\mu\nu})^\beta_\alpha Q^i_\alpha, \quad [L_{\mu\nu}, \bar{Q}_{\dot{\alpha}i}] = \tfrac{1}{2}(\tilde{\sigma}_{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}} Q_{\dot{\alpha}i},$$

$$[T^i_j, Q^k_\alpha] = \delta^k_j Q^i_\alpha - \tfrac{1}{\mathcal{N}}\delta^i_j Q^k_\alpha, \qquad [T^i_j, \bar{Q}_{\dot{\alpha}k}] = -\delta^i_k \bar{Q}_{\dot{\alpha}j} + \tfrac{1}{\mathcal{N}}\delta^i_j \bar{Q}_{\dot{\alpha}k}$$

$Z^{ij}, \bar{Z}_{ij} = (Z^{ij})^+$ are central charges, $[Z, P] = [Z, L] = [Z, Q] = [Z, Z] = 0$

$4D$ $\mathcal{N}$-extended conformal superalgebra $SU(2,2|\mathcal{N})$

$$\underbrace{P_\mu,\, L_{\mu\nu},\, \overbrace{T^i_j,\; R,}^{U(\mathcal{N})} \qquad K_\mu,\, D,}_{even} \qquad \underbrace{Q^i_\alpha,\, \bar{Q}_{\dot{\alpha}i} = (Q^i_\alpha)^+, \qquad S_{\alpha i},\, \bar{S}^i_{\dot{\alpha}} = (S_{\alpha i})^+}_{odd}$$

$$\{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} = 2\delta^i_j(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu, \qquad \{S_{\alpha i}, \bar{S}^j_{\dot{\alpha}}\} = 2\delta^j_i(\sigma^\mu)_{\alpha\dot{\alpha}}K_\mu\,,$$

$$\{Q^i_\alpha, S^\beta_j\} = -\delta^i_j(\sigma^{\mu\nu})^\beta_\alpha L_{\mu\nu} - 4i\delta^\beta_\alpha T^i_j - 2i\delta^\beta_\alpha\delta^i_j D + \frac{2(4-N)}{N}\,\delta^\beta_\alpha\delta^i_j R\,,$$

$$[K_\mu, Q^i_\alpha] = (\sigma_\mu)_{\alpha\dot{\alpha}}\,\bar{S}^{\dot{\alpha}i}\,, \quad [K_\mu, \bar{Q}_{\dot{\alpha}i}] = -\,(\sigma_\mu)_{\alpha\dot{\alpha}}\,S^\alpha_i\,,$$

$$[P_\mu, S_{\alpha i}] = (\sigma_\mu)_{\alpha\dot{\alpha}}\,\bar{Q}^{\dot{\alpha}}_i\,, \quad [P_\mu, \bar{S}^i_{\dot{\alpha}}] = -\,(\sigma_\mu)_{\alpha\dot{\alpha}}\,Q^{\alpha i}\,,$$

$$[D, Q] = \tfrac{i}{2}\,Q\,, \qquad [D, \bar{Q}] = \tfrac{i}{2}\,\bar{Q}\,, \qquad [D, S] = -\tfrac{i}{2}\,S\,, \qquad [D, \bar{S}] = -\tfrac{i}{2}\,\bar{S}\,,$$

$$[R, Q] = -\tfrac{1}{2}\,Q\,, \qquad [R, \bar{Q}] = \tfrac{1}{2}\,\bar{Q}\,, \qquad [R, S] = \tfrac{1}{2}\,S\,, \qquad [R, \bar{S}] = -\tfrac{1}{2}\,\bar{S}\,.$$

One-dimensional superalgebras

$1D$ N-extended super-Poincare algebra

$$\{Q_a, Q_b\} = 2\delta_{ab}H\,, \qquad (Q_a)^+ = Q\,, \qquad a = 1, ..., N$$

$1D$ N-extended superconformal algebra

$1D$ superconformal symmetry $\supset$ $1D$ conformal symmetry
$\mathrm{SO}(1,2) \sim \mathrm{Sp}(2,\mathbb{R}) \sim \mathrm{SL}(2,\mathbb{R}) \sim \mathrm{SU}(1,1)$

$$\sim \left(\begin{array}{c|c} \mathrm{Sp}(2,\mathbb{R}) & Q+S \\ \hline Q-S & \mathrm{SO}(N) \end{array}\right), \qquad \sim \left(\begin{array}{c|c} \mathrm{SU}(1,1) & Q+S \\ \hline Q-S & \mathrm{SU}(M) \end{array}\right)$$

$$\{Q,Q\} \sim H, \quad \{S,S\} \sim K, \quad \{Q,S\} \sim D+J, \qquad (H,K,D) \subset su(1,1), \quad J \subset o(N) \text{ or } su(M)$$

$N{=}1$: $\mathrm{OSp}(1|2)$
$N{=}2$: $\mathrm{OSp}(2|2) \sim \mathrm{SU}(1,1|1)$
$N{=}4$: $D(2,1;\alpha)$

$\alpha = -1/2,\ \alpha = 1$: $D(2,1;\alpha) \sim \mathrm{OSp}(4|2)$
$\alpha = 0,\ \alpha = -1$: $D(2,1;\alpha) \sim \mathrm{SU}(1,1|2) \oplus_S \mathrm{SU}(2)$

$$D(2,1;\alpha): \quad \{Q^{ai'i}, Q^{bk'k}\} = 2\left(\epsilon^{ik}\epsilon^{i'k'}T^{ab} + \alpha\epsilon^{ab}\epsilon^{i'k'}J^{ik} - (1+\alpha)\epsilon^{ab}\epsilon^{ik}I^{i'k'}\right),$$

$$[T^{ab}, T^{cd}] = i(\epsilon^{ac}T^{bd} + \epsilon^{bd}T^{ac}), \quad \ldots, \quad [T^{ab}, Q^{ci'i}] = i\epsilon^{c(a}Q^{b)i'i}, \ldots$$

$$Q^{21'i} = -Q^i, \quad Q^{22'i} = -\bar{Q}^i, \quad Q^{11'i} = S^i, \quad Q^{12'i} = \bar{S}^i, \qquad T^{22} = H, \quad T^{11} = K, \quad T^{12} = -D.$$

Bosonic generators T^{ab}, J^{ik} and $I^{i'k'}$ form $su(1,1)$, $su(2)$ and $su'(2)$ algebras.

Component fields description

$$S = \int dt\, L, \qquad L = \frac{1}{2}\,\dot{\phi}^2 + \frac{i}{2}\,\psi\dot{\psi}$$

$$[\phi(t_1), \phi(t_2)] = \phi(t_1)\phi(t_2) - \phi(t_2)\phi(t_1) = 0, \qquad \{\psi(t_1), \psi(t_2)\} = \psi(t_1)\psi(t_2) + \psi(t_2)\psi(t_1) = 0$$

$$\phi^+ = \phi, \qquad \psi^+ = \psi; \qquad (AB)^+ = B^+A^+$$

$Q: \quad \phi \ \to \ \psi, \quad \psi \ \to \ \phi \qquad \Rightarrow \qquad$ the parameter $\varepsilon = \varepsilon^+$ must be anticommuting

$$[S/\hbar] = 0, \quad \hbar = 1 \qquad \Rightarrow \qquad [L] = +1, \quad [t] = -1 \qquad \Rightarrow \qquad [\phi] = -1/2, \quad [\psi] = 0$$

$$\delta\phi = i\varepsilon\psi \qquad \Rightarrow \qquad [\varepsilon] = -1/2 \qquad \Rightarrow \qquad \delta\psi \sim \varepsilon\dot{\phi}$$

$$\delta\phi = i\varepsilon\psi, \qquad \delta\psi = -\varepsilon\dot{\phi}$$

$$\delta L = \frac{i}{2}\,(\varepsilon\psi\dot{\phi})^{\cdot} + i\dot{\varepsilon}\psi\dot{\phi} = 0, \qquad \varepsilon = const, \quad \phi|_{t=\pm\infty} = \phi|_{t=\pm\infty} = 0$$

$$[\delta_1, \delta_2]\,\phi = 2i\varepsilon_1\varepsilon_2\dot{\phi}, \qquad [\delta_1, \delta_2]\,\psi = 2i\varepsilon_1\varepsilon_2\dot{\psi}$$

Note: In $N > 1$ $1D$ and $D > 1$ $\qquad [\delta_1, \delta_2]\,\psi = 2i\varepsilon_1\varepsilon_2\dot{\psi} +$ (eq.of motion)

Superfields

Superspace: Supersymmetry is realized by coordinate transformations

Q describes fermionic transformations → translations in odd direction of extended space

Usual $1D$ space: (t) $\Rightarrow$

$N{=}1$, $1D$ superspace: (t,θ), where $\theta=\bar\theta$ is Grassmann coordinate, $\theta\theta\equiv 0$

$$Q=Q^+=\partial_\theta+i\,\theta\,\partial_t\,,\qquad H=H^+=i\,\partial_t\,;\qquad \{Q,Q\}=2\,H\,,\qquad [H,Q]=0$$

$$\delta t=\varepsilon Q\cdot t,\quad \delta\theta=\varepsilon Q\cdot\theta\ :\qquad \delta t=i\,\varepsilon\theta,\qquad \delta\theta=\varepsilon$$

$N{=}1, 1D$ superfield : $\Phi(t,\theta)=\phi(t)+i\,\theta\psi(t)$

$$\Phi'(t',\theta')=\Phi(t,\theta),\quad \delta\Phi=\Phi(t',\theta')-\Phi(t,\theta)=\varepsilon Q\cdot\Phi=\delta\phi+i\,\theta\delta\psi\quad\Rightarrow$$

$$\delta\phi=i\,\varepsilon\psi,\quad \delta\psi=-\varepsilon\dot\phi$$

Integration over odd variable : $\int d\theta\, f(\theta)=\int d\theta\, f(\theta+\alpha)\quad\Rightarrow\quad \int d\theta\,\theta=1,\quad \int d\theta\,\alpha=0$

Covariant derivatives : $D_\theta=\partial_\theta-i\,\theta\,\partial_t\equiv D\,,\qquad D_t=\partial_t\,,\qquad \{Q,D\}=0\,,\quad [Q,\partial_t]=0$

$$S=\int dt\,d\theta\,\mathcal{L}(\Phi,\partial_t\Phi,D\Phi),\qquad \delta\,S=\int dt\,d\theta\,Q\,[\ldots]=\int\underbrace{dt\,d\theta\,\partial_\theta\,[\ldots]}_{=0,\,\partial_\theta[\ldots]\text{ contains no }\theta}+\int dt\,d\theta\;\underbrace{i\theta\partial_t\,[\ldots]}_{\text{the total derivative}}$$

Any action, built from superfields and covariant derivatives ∂_t and D, is always supersymmetric

Examples of the $N\!=\!1$ supermultiplets

$$\Phi(t,\theta) = \phi(t) + i\,\theta\psi(t) \qquad - \qquad \text{even superfield}$$

$$S = \tfrac{i}{2}\int dt\, d\theta\, \partial_t\Phi\, D\Phi = \tfrac{1}{2}\int dt\left(\dot{\phi}^2 + i\,\psi\dot{\psi}\right)$$

$(1,1,0)$ supermultiplet

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$$\Psi(t,\theta) = \psi(t) + \theta F(t) \qquad - \qquad \text{odd superfield}$$

$$S = \tfrac{1}{2}\int dt\, d\theta\, \Psi\, D\Psi = \tfrac{1}{2}\int dt\left(i\,\psi\dot{\psi} + F^2\right) \quad \xrightarrow{F=0} \quad \tfrac{i}{2}\int dt\, \psi\dot{\psi}$$

$(0,1,1)$ supermultiplet

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The supermultiplet $(m, n, n-m)$ contains

- m physical bosons
- n fermions
- $n-m$ auxiliary bosons

N-extended $1D$ superspace:

$$(t,\theta_i), \qquad \theta_k = (\overline{\theta_k}), \qquad \{\theta_i,\theta_k\} = 0, \qquad i,j,k = 1,\dots,N$$

Realization of super-Poincare algebra in superspace:

$$Q_k = Q_k^{+} = \frac{\partial}{\partial\theta_k} + i\,\theta_k\,\frac{\partial}{\partial t}\,, \qquad H = H^{+} = i\,\partial_t\,; \qquad \{Q_k,Q_j\} = 2\,\delta_{kj}\,H\,, \qquad [H,Q_k] = 0$$

$$\delta t = \varepsilon_k Q_k \cdot t, \quad \delta\theta_k = \varepsilon_j Q_j \cdot \theta_k \;\; : \qquad \delta t = i\,\varepsilon_k\theta_k\,, \qquad \delta\theta_k = \varepsilon_k$$

General supersfield:

$$\Phi(t,\theta_k) = \phi(t) + \theta_k\psi_k(t) + \theta_{k_1}\theta_{k_2}\phi_{k_1k_2}(t) + \theta_{k_1}\theta_{k_2}\theta_{k_3}\psi_{k_1k_2k_3}(t) + \cdots + \theta_{k_1}\ldots\theta_{k_N}\phi_{k_1\ldots k_N}(t)$$

Off-shell contents:

$$\left.\begin{array}{l} 2^{N-1}\ \text{bosonic (fermionic) component fields}\ \phi,\ \phi_{k_1k_2},\ldots \\ 2^{N-1}\ \text{fermionic (bosonic) component fields}\ \psi_{k_1},\ \psi_{k_1k_2k_3},\ldots \end{array}\right\} \quad \text{if}\ \Phi(t,\theta_k)\ \text{is bosonic (fermionic)}$$

Covariant derivatives:

$$D_k = \frac{\partial}{\partial\theta_k} - i\,\theta_k\,\frac{\partial}{\partial t}\,, \qquad \{Q_j,D_k\} = 0$$

$$F(D_k)\,\Phi = 0 \qquad - \qquad \text{covariant constraint}$$

On-shell (physical) contents of a model is defined by the action.

Real $N{=}2,\ 1D$ superspace: $(t,\theta_1,\theta_2),\qquad \theta_1=\theta_1^+,\quad \theta_2=\theta_2^+$

$$Q_1=\frac{\partial}{\partial\theta_1}+i\,\theta_1\,\partial_t\,,\quad Q_2=\frac{\partial}{\partial\theta_2}+i\,\theta_2\,\partial_t\,,\qquad H=i\,\partial_t\,;$$

$$\{Q_1,Q_1\}=2\,H,\qquad \{Q_2,Q_2\}=2\,H,\qquad \{Q_1,Q_2\}=0,\qquad [H,Q_1]=[H,Q_2]=0$$

$$\delta t=i\,(\varepsilon_1\theta_1+\varepsilon_2\theta_2),\qquad \delta\theta_1=\varepsilon_1,\qquad \delta\theta_2=\varepsilon_2$$

Complex $N{=}2,\ 1D$ superspace:

$$(t,\theta,\bar\theta),\qquad \theta=\tfrac{1}{\sqrt2}\,(\theta_1+i\,\theta_2),\ \bar\theta=\theta^+=\tfrac{1}{\sqrt2}\,(\theta_1-i\,\theta_2)$$

$$Q=\frac{\partial}{\partial\theta}+i\,\bar\theta\,\partial_t\,,\quad \bar Q=\frac{\partial}{\partial\bar\theta}+i\,\theta\,\partial_t\,,\qquad H=i\,\partial_t$$

$$\{Q,\bar Q\}=2\,H,\qquad \{Q,Q\}=\{\bar Q,\bar Q\}=0,\qquad [H,Q_{]}=[H,\bar Q]=0$$

$$\delta t=i\,(\varepsilon\bar\theta+\bar\varepsilon\theta),\qquad \delta\theta=\varepsilon,\qquad \delta\bar\theta=\bar\varepsilon,\qquad \bar\varepsilon=\varepsilon^+$$

General $N{=}2,\ 1D$ superfield:

$$\Phi(t,\theta)=\phi(t)+\theta\psi(t)+\bar\theta\chi(t)+\theta\bar\theta F(t)$$

$$\delta\phi=\varepsilon\psi+\bar\varepsilon\chi,\qquad \delta\psi=-i\,\bar\varepsilon\dot\phi+\bar\varepsilon F,\qquad \delta\chi=-i\,\varepsilon\dot\phi-\varepsilon F,\qquad \delta F=-i\,\varepsilon\dot\psi+i\,\bar\varepsilon\dot\chi$$

Covariant derivatives : $D=\dfrac{\partial}{\partial\theta}-i\,\bar\theta\,\partial_t\,,\quad \bar D=\dfrac{\partial}{\partial\bar\theta}-i\,\theta\,\partial_t\,,\qquad \{D,Q\}=\{D,\bar Q\}=0$

$\Phi^+=\Phi$ – the real superfield; $\bar D\,\Phi=0$ – the chiral superfield

Real superfield:

$$\Phi(t,\theta) = \Phi^{+} = \phi(t) + \theta\psi(t) - \bar{\theta}\bar{\psi}(t) + \theta\bar{\theta}F(t), \qquad \phi^{+} = \phi, \quad F^{+} = F, \quad \psi^{+} = \bar{\psi}$$

Off-shell SUSY transformations:

$$\delta\phi = \varepsilon\psi - \bar{\varepsilon}\bar{\psi}, \qquad \delta\psi = -i\,\bar{\varepsilon}\dot{\phi} + \bar{\varepsilon}F, \qquad \delta\bar{\psi} = i\,\varepsilon\dot{\phi} + \varepsilon F, \qquad \delta F = -i\,(\varepsilon\dot{\psi} + \bar{\varepsilon}\dot{\bar{\psi}})$$

$$S = \tfrac{i}{2}\int dt\,d\theta\,d\bar{\theta}\,\bar{D}\Phi\,D\Phi = \tfrac{1}{2}\int dt\left\{\dot{\phi}^{2} + i\,(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi}) + F^{2}\right\}$$

$$\mathrm{On-shell:} \qquad \ddot{\phi} = 0, \qquad \dot{\psi} = 0, \qquad \dot{\bar{\psi}} = 0, \qquad F = 0 \qquad (1,2,1)\ \mathrm{multiplet}$$

On-shell action:

$$S = \tfrac{1}{2}\int dt\left\{\dot{\phi}^{2} + i\,(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi})\right\}$$

On-shell SUSY transformations:

$$\delta\phi = \varepsilon\psi - \bar{\varepsilon}\bar{\psi}, \qquad \delta\psi = -i\,\bar{\varepsilon}\dot{\phi}, \qquad \delta\bar{\psi} = i\,\varepsilon\dot{\phi}$$

$$[\delta_1,\delta_2]\psi = i\,(\varepsilon_1\bar{\varepsilon}_2 - \varepsilon_2\bar{\varepsilon}_1)\,\dot{\psi} - \underbrace{2i\,\bar{\varepsilon}_1\bar{\varepsilon}_2\dot{\bar{\psi}}}_{=0\ \mathrm{on-shell}}$$

On-shell SUSY transformations are closed only on equations of motion.

Chiral superfield:

$$\bar{D}\,\Phi = 0 \quad \rightarrow \quad \Phi(t,\theta) = \phi(t) + \theta\psi(t) - i\,\theta\bar{\theta}\dot{\phi}(t), \qquad \phi,\ \psi \quad - \quad \text{complex fields}$$

$(2,2,0)$ multiplet

$$\Phi(t,\theta) = \phi(t) + \theta\psi(t) - i\,\theta\bar{\theta}\dot{\phi}(t) = \phi(t_L) + \theta\psi(t_L) = \Phi(t_L,\theta)$$

Chiral $N{=}2$, $1D$ subspace:

$$(t_L,\theta), \qquad t_L \equiv t - i\,\theta\bar{\theta}$$

$$\delta t = i\,(\varepsilon\bar{\theta} + \bar{\varepsilon}\theta), \quad \delta\theta = \varepsilon, \quad \delta\bar{\theta} = \bar{\varepsilon} \quad \Rightarrow \quad \delta t_L = 2i\,\bar{\varepsilon}\theta, \qquad \delta\theta = \varepsilon$$

Supercharges in superspace $(t_L,\theta,\bar{\theta})$:

$$Q = \frac{\partial}{\partial\theta}\,, \qquad \bar{Q} = \frac{\partial}{\partial\bar{\theta}} + 2i\,\theta\,\partial_{t_L}$$

SUSY transformations of component fields:

$$\delta\phi = \varepsilon\psi\,, \qquad \delta\psi = -2i\,\bar{\varepsilon}\dot{\phi}$$

SUSY invariant action:

$$S = -\tfrac{1}{2}\int dt\,d\theta\,d\bar{\theta}\,\bar{D}\Phi\,\bar{D}\bar{\Phi} = \tfrac{1}{2}\int dt\,\left\{4\dot{\phi}\dot{\bar{\phi}} - i\,(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi})\right\}$$

$\mathcal{N}=2$ superconformal mechanics

The $\mathcal{N}=2$ superconformal group OSp(2|2)∼SU(1,1|1)

$$\{Q,\bar{Q}\}=2H,\quad \{S,\bar{S}\}=2K,\quad \{Q,\bar{S}\}=2(D-U),\quad \{S,\bar{Q}\}=2(D+U),$$

$$i\left[P,\begin{pmatrix}S\\ \bar{S}\end{pmatrix}\right]=-\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix},\qquad i\left[K,\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix}\right]=\begin{pmatrix}S\\ \bar{S}\end{pmatrix},$$

$$i\left[D,\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix},\qquad i\left[D,\begin{pmatrix}S\\ \bar{S}\end{pmatrix}\right]=-\frac{1}{2}\begin{pmatrix}S\\ \bar{S}\end{pmatrix},$$

$$i\left[U,\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}Q\\ -\bar{Q}\end{pmatrix},\qquad i\left[U,\begin{pmatrix}S\\ \bar{S}\end{pmatrix}\right]=-\frac{1}{2}\begin{pmatrix}S\\ -\bar{S}\end{pmatrix}$$

The closure of S, $\bar{S}$ with Q, $\bar{Q}$ ⇒ the full OSp(2|2).

We obtain the superconformal transformations by nonlinear realization method.

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Coset realization of $N=2$ superspace:

$$\mathcal{G}=\{H,Q,\bar{Q},U\},\qquad \mathcal{H}=\{U\},\qquad \mathcal{K}=\{H,Q,\bar{Q}\}$$

$\mathcal{K}(t,\theta,\bar{\theta})=e^{itH+\theta Q+\bar{\theta}\bar{Q}}$, $\quad t,\ \theta,\ \bar{\theta}$ are the coordinates on the coset

$$e^{\varepsilon Q+\bar{\varepsilon}\bar{Q}}\,e^{itH+\theta Q+\bar{\theta}\bar{Q}}=e^{it'H+\theta' Q+\bar{\theta}'\bar{Q}}:\qquad \delta t=i(\varepsilon\bar{\theta}+\bar{\varepsilon}\theta),\quad \delta\theta=\varepsilon,\quad \delta\bar{\theta}=\bar{\varepsilon}$$

Note : $\quad e^{A}\,e^{B}=\exp\left\{A+B+\frac{1}{2}[A,B]+\frac{1}{12}\left([A,[A,B]]+[[A,B],B]\right)+\cdots\right\}$

Coset realization of SU(1,1|1):

$$\mathcal{G} = \{H, D, K, Q, \bar{Q}, S, \bar{S}, U\}, \qquad \mathcal{H} = \{U\}, \qquad \mathcal{K} = \{H, D, K, Q, \bar{Q}, S, \bar{S}\}$$

$$\mathcal{K} = e^{itH}\, e^{\theta Q + \bar{\theta}\bar{Q}}\, e^{iuD}\, e^{izK}\, e^{\zeta S + \bar{\zeta}\bar{S}}$$

$$e^{\varepsilon Q + \bar{\varepsilon}\bar{Q}}\, \mathcal{K} = \mathcal{K}'\mathcal{H}, \qquad e^{\eta S + \bar{\eta}\bar{S}}\, \mathcal{K} = \mathcal{K}'\mathcal{H}$$

Note :
$$e^{A}\, B\, e^{-A} = e^{A} \wedge B, \qquad 1 \wedge B \equiv B, \quad A \wedge B \equiv [A, B], \quad A^2 \wedge B \equiv [A, [A, B]], \quad \cdots$$

$$\delta t = i(\varepsilon\bar{\theta} + \bar{\varepsilon}\theta), \qquad \delta\theta = \varepsilon, \qquad \delta\bar{\theta} = \bar{\varepsilon};$$

$$\delta' t = i(\eta\bar{\theta} + \bar{\eta}\theta)t, \qquad \delta'\theta = \eta(t - i\theta\bar{\theta}), \qquad \delta'\bar{\theta} = \bar{\eta}(t + i\theta\bar{\theta})$$

$$\delta'(dtd^2\theta) = 0, \qquad \delta' D = -2i\, \eta\bar{\theta}\, D, \quad \delta'\bar{D} = -2i\, \bar{\eta}\theta\, \bar{D}$$

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$$\mathcal{X} = x(t) + \theta\psi - \bar{\theta}\bar{\psi}(t) + \theta\bar{\theta}F(t), \qquad \delta'\mathcal{X} = i(\eta\bar{\theta} + \bar{\eta}\theta)\,\mathcal{X}$$

$$S = \int dtd^2\theta \left(\tfrac{1}{2}\, D\mathcal{X}\bar{D}\mathcal{X} + \gamma\, \ln\, \mathcal{X} \right) = \tfrac{1}{2} \int dt \left\{ \dot{x}^2 + i\,(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi}) - \frac{\gamma^2 + \gamma\psi\bar{\psi}}{x^2} \right\}$$

Multi-particle generalization (N=2 superconformal Calogero):

$$S = \int dt\, d^2\theta \left(\tfrac{1}{2} \sum_{a} D\mathcal{X}_a \bar{D}\mathcal{X}_a + \gamma \sum_{a \neq b} \ln |\mathcal{X}_a - \mathcal{X}_b| \right)$$

$\mathcal{N}=4$ superconformal mechanics

The standard $\mathcal{N}=4$, $1D$ superspace:

$$\left\{ t, \theta_k, \bar\theta^k = (\theta_k)^+ \right\}, \qquad k = 1,2$$

Supersymmetry transformations from the $\mathcal{N}=4$, $1D$ superconformal group $D(2,1;\alpha)$:

$$\delta t = i(\theta_k \bar\varepsilon^k - \varepsilon_k \bar\theta^k), \qquad \delta\theta_k = \varepsilon_k, \qquad \delta\bar\theta^k = \bar\varepsilon^k;$$

$$\delta' t = -i(\eta_k \bar\theta^k - \bar\eta^k \theta_k)t + (1+2\alpha)\theta_j \bar\theta^j (\eta_k \bar\theta^k + \bar\eta^k \theta_k),$$

$$\delta'\theta_k = \eta_k t - 2i\alpha\theta_k\theta_j\bar\eta^j + 2i(1+\alpha)\theta_k\bar\theta^j\eta_j - i(1+2\alpha)\eta_k\theta_j\bar\theta^j$$

Covariant derivatives : $\quad D^k = \dfrac{\partial}{\partial\theta_k} + i\,\bar\theta^k\partial_t \qquad \bar D_k = \dfrac{\partial}{\partial\bar\theta^k} + i\,\bar\theta^k\partial_t$

Some types of the $\mathcal{N}=4$, $1D$ superfields:

- $D^k D_k \mathcal{X} = m,\ \bar D^k \bar D_k \mathcal{X} = m,\ [D^k, \bar D_k]\mathcal{X} = 0$ - scalar superfield, (1,4,3) multiplet
- $D^{(i}V^{jk)} = 0, \quad \bar D^{(i}V^{jk)} = 0$ - vector superfield, (3,4,1)

Superconformal models ($\mathcal{X} = (V^{ik}V_{ik})^{1/2}$ for vector superfield):

$$S \sim \int dt\, d^4\theta\ \mathcal{X}^{-1/2} \quad \text{for } \alpha \neq -1; \qquad S \sim \int dt\, d^4\theta\ \mathcal{X}\ln\mathcal{X} \quad \text{for } \alpha = -1$$

In components : $\quad S \sim \displaystyle\int dt \left[\dot x^2 + i\,(\psi_k \dot{\bar\psi}^k - \dot\psi_k \bar\psi^k) - \frac{g + F(\psi,\bar\psi)}{x^2}\right]$

More general formulations of $\mathcal{N}=4$, $1D$ models is achieved in harmonic superspace

Harmonic superspace for $\mathcal{N}=4$, $1D$ SUSY models

[A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev; 1984]
[E. Ivanov, O. Lechtenfeld; 2003]

$$\mathcal{N}=4,\ 1D\ \text{SUSY algebra}: \qquad \Big\{ H,\ Q^k,\ \bar{Q}_k = (Q^k)^+,\ \overbrace{\underbrace{J^{(ik)}}_{su_L(2)},\ \underbrace{I^{(i'k')}}_{su_R(2)}}^{R-\text{symmetry}} \Big\}, \quad i,k=1,2$$

$$\text{Standard } \mathcal{N}=4,\ 1D \text{ superspace}: \quad \Big\{ H,\ Q^k,\ \bar{Q}_k = (Q^k)^+,\ J^{ik},\ I^{i'k'} \Big\} \Big/ \Big\{ J^{ik},\ I^{i'k'} \Big\}$$

$$\text{Standard superspace coordinates}: \qquad \Big\{ t, \theta_k, \bar{\theta}^k = (\theta_k)^+ \Big\}$$

$$su_L(2)\ \text{algebra}: \qquad J^{(ik)} = \Big\{ J^{\pm},\ J^0 \Big\}, \qquad J^0 \ -\ u(1)\ \text{generator}$$

$$\mathcal{N}=4,\ 1D \text{ harmonic superspace}: \quad \Big\{ H,\ Q^k,\ \bar{Q}_k = (Q^k)^+,\ J^{ik},\ I^{i'k'} \Big\} \Big/ \Big\{ J^0,\ I^{i'k'} \Big\}$$

$$\text{Harmonic superspace coordinates}: \qquad \Big\{ t, \theta_k, \bar{\theta}^k,\ u_i^{\pm} \Big\}$$

Harmonic coordinates $u_i^{\pm}$ parametrize the sphere $S^2 \sim SU(2)/U(1)$

Parametrize $S^2 \sim SU(2)/U(1)$ by two $SU(2)$ spinors

$$u_i^{\pm}, \qquad u_i^- = (\overline{u^{+i}})$$

which subject to the constraint

$$u^{+i}u_i^- = 1 \quad \rightarrow \quad u_i^+u_k^- - u_k^+u_i^- = \epsilon_{ik}$$

and are defined up to a $U(1)$ phase transformations

$$u_i^+ \rightarrow e^{i\alpha}u_i^+, \qquad u_i^- \rightarrow e^{-i\alpha}u_i^-$$

$$||u|| = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} \in \ SU(2), \qquad ||u|| \rightarrow g\,||u||\,h\,, \quad g \in SU(2), \quad h \in U(1)$$

i, k are $SU(2)$ indices; ± are $U(1)$ charges

Any function on $S^2 \sim SU(2)/U(1)$ must have a definite $U(1)$ charge q

$$\Phi^{(q)}(u) = \sum_{n=0}^{\infty} \phi^{i_1\ldots i_{n+q}j_1\ldots j_n}u_{i_1}^+\ldots u_{i_{n+q}}^+u_{j_1}^-\ldots u_{j_n}^- \qquad \text{for} \quad n \geq 0$$

Harmonic functions are defined up to the transformations $\Phi^{(q)} \rightarrow e^{i\alpha q}\Phi^{(q)}$.
The use of such parametrization of S^2 has the advantage of manifest $SU(2)$ covariance

Covariant derivatives on the harmonic sphere S^2:

$$D^{\pm\pm} = u_i^{\pm} \frac{\partial}{\partial u_i^{\mp}} \equiv \partial^{\pm\pm}\,, \qquad D^0 = u_i^{+} \frac{\partial}{\partial u_i^{+}} - u_i^{-} \frac{\partial}{\partial u_i^{-}} \equiv \partial^0$$

$$[D^{++}, D^{--}] = D^0\,, \qquad [D^0, D^{\pm\pm}] = \pm 2\, D^{\pm\pm}$$

Harmonic fields satisfy

$$D^0\, \Phi^{(q)} = q\, \Phi^{(q)}$$

Harmonic integrals:

$$\int du\, u^+_{(i_1} \dots u^+_{i_m} u^-_{j_1} \dots u^-_{j_n)} = 0\,,$$

$$\int du\, 1 = 1\,,$$

$$\int du\, F^{(q)} = 0 \qquad \text{if} \quad q \neq 0$$

Central basis in harmonic superspace:

$$\left\{ t, \theta_k, \bar\theta^k, u_i^\pm \right\} \equiv \{ z, u \}$$

The $\mathcal{N}=4$, $1D$ Poincare supersymmetry:

$$\delta t = i(\theta_k \bar\varepsilon^k - \varepsilon_k \bar\theta^k), \qquad \delta\theta_k = \varepsilon_k, \qquad \delta\bar\theta^k = \bar\varepsilon^k, \qquad \delta u_i^\pm = 0$$

Analytic basis in harmonic superspace:

$$\left\{ t_A, \theta^\pm, \bar\theta^\pm, u_i^\pm \right\} \equiv \{ z_A, u \}, \qquad \theta^\pm = \theta^i u_i^\pm, \quad \bar\theta^\pm = \bar\theta^i u_i^\pm, \qquad t_A = t - i\left(\theta^+\bar\theta^- + \theta^-\bar\theta^+\right)$$

Analytic superspace

$$\left\{ t_A, \theta^+, \bar\theta^+, u_i^\pm \right\} \equiv \{ \zeta, u \}$$

is closed under $\mathcal{N}=4$ Poincare SUSY (and under $\mathcal{N}=4$ superconformal symmetry)

$$\delta t_A = -2i(\varepsilon^- \bar\theta^+ + \theta^+ \bar\varepsilon^-), \qquad \delta\theta^+ = \varepsilon^+ = \varepsilon^i u_i^+, \qquad \delta\bar\theta^+ = \bar\varepsilon^+ = \bar\varepsilon^i u_i^+, \qquad \delta u_i^\pm = 0$$

Covariant derivatives $D^\pm = D^i u_i^\pm$, $\bar D^\pm = \bar D^i u_i^\pm$ in analytic basis:

$$D^+ = \frac{\partial}{\partial\theta^-}, \qquad \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}, \qquad D^- = -\frac{\partial}{\partial\theta^+} + 2i\bar\theta^-\partial_A, \qquad \bar D^- = \frac{\partial}{\partial\bar\theta^-} + 2i\theta^-\partial_A$$

$$D^+\Psi(z,u) = \bar D^+\Psi(z,u) = 0 \qquad \Rightarrow \qquad \Psi = \Psi(\zeta, u)$$

Vector superfield (3,4,1) multiplet

$$D^+V^{++} = \bar{D}^+V^{++} = 0, \qquad D^{++}V^{++} = 0$$

Central basis:

$$D^{++}V^{++} = 0 \quad \Rightarrow \quad V^{++} = V^{ik}(z)u_i^+u_k^+$$
$$D^+V^{++} = \bar{D}^+V^{++} = 0 \quad \Rightarrow \quad D^{(i}V^{kl)} = \bar{D}^{(i}V^{kl)} = 0$$

Analytic basis:

$$D^+V^{++} = \bar{D}^+V^{++} = 0 \quad \Rightarrow \quad V^{++} = V^{++}(\zeta, u)$$
$$D^{++}V^{++} = 0 \quad \Rightarrow \quad V^{++} = v^{ik}u_i^+u_k^+ + \theta^+\psi^i u_i^+ + \bar{\theta}^+\bar{\psi}^i u_i^+ + i\theta^+\bar{\theta}^+(F + 2\dot{v}^{ik}u_i^+u_k^+)$$

$$S = \gamma \int dt\, d^4\theta\, du\, \mathcal{L}\left(V^{++}, D^{--}V^{++}, (D^{--})^2V^{++}, u\right)$$
$$+\gamma' \int dt\, d\theta^+ d\bar{\theta}^+\, du\, \mathcal{L}^{++}\left(V^{++}, u\right)$$

first term $\Rightarrow$ $\gamma \int dt\, \mathcal{H}(v)\left(\dot{v}^{ik}\dot{v}_{ik} + F^2\right)$

second term $\Rightarrow$ $\gamma' \int dt \left\{F\mathcal{V}(v) + \dot{v}^{ik}\mathcal{A}_{ik}(v)\right\}$

$\partial_{ik}\mathcal{A}_{lt} - \partial_{lt}\mathcal{A}_{ik} = (\epsilon_{il}\partial_{kt} - \epsilon_{kt}\partial_{il})\mathcal{V}$ – monopole – like potential

Hypermultiplet (4,4,0) multiplet

$$D^+ q^+_a = \bar{D}^+ q^+_a = 0, \qquad D^{++} q^+_a = 0, \qquad (\widetilde{q^+_a}) = \epsilon^{ab} q^+_b\,, \qquad a, b = 1, 2$$

Central basis:

$$D^{++} q^+_a = 0 \quad \Rightarrow \quad q^+_a = q^i_a(z) u^+_i$$

$$D^+ q^+_a = \bar{D}^+ q^+_a = 0 \quad \Rightarrow \quad D^{(i} q^{k)}_a = \bar{D}^{(i} q^{k)}_a = 0$$

Analytic basis:

$$D^+ q^+_a = \bar{D}^+ q^+_a = 0 \quad \Rightarrow \quad q^+_a = q^+_a(\zeta, u)$$

$$D^{++} q^+_a = 0 \quad \Rightarrow \quad q^+_a = f^i_a u^+_i + \theta^+ \chi_a + \bar{\theta}^+ \bar{\chi}_a + 2i \theta^+ \bar{\theta}^+ \dot{f}^i_a u^-_i$$

$$S = \gamma \int dt\, d^4\theta\, du\, \mathcal{L}\left(q^+_a, D^{--} q^+_a, u\right)$$

$$+\gamma' \int dt\, d\theta^+ d\bar{\theta}^+\, du\, \mathcal{L}^{++}\left(q^+_a, u\right)$$

first term $\Rightarrow$ $\gamma \int dt\, G^{ab}(f)\, \dot{f}^i_a \dot{f}_{ib}$

second term $\Rightarrow$ $\gamma' \int dt\, \dot{f}^{ia} \mathcal{A}_{ia}(f)$

$\mathcal{A}_{ia}$ – self – dual gauge potential

The $\mathcal{N}=4$ superconformal matrix model ($\mu_H = dudtd^4\theta,\ \mu_A^{(-2)} = dud\zeta^{(-2)}$):

$$S = -\frac{1}{2}\int \mu_H \mathrm{Tr}(\mathcal{X}^2) + \frac{1}{2}\int \mu_A^{(-2)} \mathcal{V}_0\, \widetilde{\mathcal{Z}}^+ \mathcal{Z}^+ + \frac{i}{2}\, c \int \mu_A^{(-2)}\, \mathrm{Tr}\, V^{++}\,,$$

Superfield contents:

- hermitian matrix superfields $\mathcal{X} = (\mathcal{X}_a^b)$:

$$\mathcal{D}^{++}\mathcal{X} = 0, \qquad \mathcal{D}^{+}\mathcal{D}^{-}\mathcal{X} = 0, \quad (\mathcal{D}^{+}\bar{\mathcal{D}}^{-} + \bar{\mathcal{D}}^{+}\mathcal{D}^{-})\,\mathcal{X} = 0;$$

- analytic superfields $\mathcal{Z}_a^+(\zeta, u)$: $\qquad \mathcal{D}^{++}\mathcal{Z}^+ = 0;$
- the gauge matrix connection $V^{++}(\zeta, u)$.

$$\mathcal{D}^{++} = D^{++} + i\, V^{++}, \quad \mathcal{D}^{++}\mathcal{X} = D^{++}\mathcal{X} + i\,[V^{++}, \mathcal{X}], \quad \text{etc.}$$

The superfield $\mathcal{V}_0(\zeta, u)$ is defined by the integral transform ($\mathcal{X}_0 \equiv \mathrm{Tr}\,(\mathcal{X})$)

$$\mathcal{X}_0(t, \theta_i, \bar{\theta}^i) = \int du\, \mathcal{V}_0\, (t_A, \theta^+, \bar{\theta}^+, u^\pm)\,\Big|_{\theta^\pm = \theta^i u_i^\pm,\ \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm}.$$

Symmetries

- The $\mathcal{N}=4$ superconformal symmetry $D(2,1;\alpha)$ with $\alpha=-\frac{1}{2}\simeq \mathrm{OSp}(4|2)$:

$$\delta'\mathcal{X}=-\Lambda_0\,\mathcal{X},\;\; \delta'\mathcal{Z}^+=\Lambda\,\mathcal{Z}^+,\;\; \delta' V^{++}=0,\quad \Lambda=2i\alpha(\bar\eta^-\theta^+-\eta^-\bar\theta^+),\; \Lambda_0=2\Lambda-D^{--}D^{++}\Lambda$$

It is important that just the field multiplier $\mathcal{V}_0$ in the action provides this invariance.

- The local $\mathrm{U}(n)$ invariance:

$$\mathcal{X}\,'=e^{i\lambda}\mathcal{X}e^{-i\lambda},\qquad \mathcal{Z}^{+\,\prime}=e^{i\lambda}\mathcal{Z}^+,\qquad V^{++\,\prime}=e^{i\lambda}\,V^{++}\,e^{-i\lambda}-i\,e^{i\lambda}(D^{++}e^{-i\lambda}),$$

where $\lambda_a^b(\zeta,u^\pm)\in u(n)$ is the 'hermitian' analytic matrix parameter, $\widetilde{\lambda}=\lambda$.

Using gauge freedom we choose the WZ gauge: $\quad V^{++}=-2i\,\theta^+\bar\theta^+A(t_A)$.
In the WZ gauge: $\quad S_4=S_b+S_f,$

$$S_b=\int dt\,\Big[\mathrm{Tr}\,(\nabla X\nabla X+c\,A)+\tfrac{i}{2}X_0\Big(\bar Z_k\nabla Z^k-\nabla\bar Z_k Z^k\Big)+\tfrac{n}{8}(\bar Z^{(i}Z^{k)})(\bar Z_i Z_k)\Big],$$

$$S_f\;=\;-i\mathrm{Tr}\int dt\,\Big(\bar\Psi_k\nabla\Psi^k-\nabla\bar\Psi_k\Psi^k\Big)-\int dt\,\frac{\Psi_0^{(i}\bar\Psi_0^{k)}(\bar Z_i Z_k)}{X_0}\,,$$

where $\quad \mathcal{X}=X(t_A)+\theta^-\Psi^i(t_A)u_i^+ +\bar\theta^-\bar\Psi^i(t_A)u_i^+ +\ldots,\qquad \mathcal{Z}^+=Z^i(t_A)u_i^+ +\ldots$

$X_0\equiv\mathrm{Tr}(X),\quad \Psi_0^i\equiv\mathrm{Tr}(\Psi^i),\quad \bar\Psi_0^i\equiv\mathrm{Tr}(\bar\Psi^i).$

- imposing the gauge $X_a^b = 0$, $a \neq b$,
- eliminating A_a^b, $a \neq b$, by the equations of motion,
- introducing the new fields $Z'^i_a = (X_0)^{1/2} Z_a^i$ (omit the primes):

$$S_b = \int dt \Big\{ \sum_a \dot{x}_a \dot{x}_a + \tfrac{i}{2} \sum_a (\bar{Z}^a_k \dot{Z}^k_a - \dot{\bar{Z}}^a_k Z^k_a) + \sum_{a \neq b} \frac{\mathrm{Tr}(S_a S_b)}{4(x_a - x_b)^2} - \frac{n\, \mathrm{Tr}(\hat{S}\hat{S})}{2(X_0)^2} \Big\},$$

where $\quad (S_a)_i{}^j \equiv \bar{Z}^a_i Z^j_a, \quad (\hat{S})_i{}^j \equiv \sum_a \left[(S_a)_i{}^j - \frac{1}{2} \delta_i^j (S_a)_k{}^k \right].$

The fields Z^k_a are subject to the constraints

$$\bar{Z}^a_i Z^i_a = c \qquad \forall\, a\,.$$

$$\tfrac{i}{2} \int dt \sum_a (\bar{Z}^a_k \dot{Z}^k_a - \dot{\bar{Z}}^a_k Z^k_a) \qquad \Rightarrow \qquad [\bar{Z}^a_i, Z^j_b]_D = i \delta^a_b \delta^j_i.$$

Thus the quantities S_a for each a form $u(2)$ algebras

$$[(S_a)_i{}^j, (S_b)_k{}^l]_D = i \delta_{ab} \left\{ \delta_i^l (S_a)_k{}^j - \delta_k^j (S_a)_i{}^l \right\}.$$

Modulo center-of-mass conformal potential, the bosonic limit

$$S'_b = \int dt \Big\{ \sum_a \dot{x}_a \dot{x}_a + \sum_{a \neq b} \frac{\mathrm{Tr}(S_a S_b)}{4(x_a - x_b)^2} \Big\}$$

is none other than the integrable U(2)-spin Calogero model

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THANK YOU !