Quantum regime of laser-matter interactions at extreme intensities

1. **Volkov solutions**:

   (i) Solve equation
   $$(i\gamma^\mu \mathscr{D}_\mu - m)\Psi(x) = 0, \quad \mathscr{D}_\mu \equiv \partial_\mu - ie\mathscr{A}_\mu$$
   for plane wave field (PWF: $\mathscr{A}_\mu(x) = a_\mu f(nx)$, where $n^2 = 0$, $n^\mu a_\mu = 0$) and in particular for constant crossed field (CCF: $f(\xi) = \xi$) [hints: consider the squared equation and use the ansatz $\Psi_p(x) = e^{-ipx}\mathscr{F}_p(nx)$];

   (ii) Fourier-expand the obtained solution $\Psi_p(x)$ in plane waves e^{-ipx};

   (iii) Using the usual rules of diagram technique but replacing the free field modes with the obtained solutions, write down an explicit expression for the S-matrix element of the process of pair photoproduction $\gamma \to e^+e^-$ in the plane wave field and CCF[1]. Explain why (unlike the case of absence of external field) this $\mathcal{O}(\alpha)$-order process is not prohibited by energy-momentum conservation.

2. **Radiation correction for pair creation**: Think how one could try to easily estimate radiative correction for pair creation probability due to electron-positron Coulomb attraction.

3. **Estimation of formation scales in proper RF**: Estimate the characteristic times t_e, t_q for the elementary IFQED process $\gamma \to e^-e^+$ for the same conditions as assumed in lecture ($a_0 \gg 1$ and initial particle is ultrarelativistic and is moving across the field) in the proper RF of the initial particle[2]. Show how after return into the laboratory frame these estimates agree with those obtained in lectures.

4. **Acceleration in general time-varying field**:

[1] I suggest just to write it down to look at, as its further evaluation turns out to be rather tedious.

[2] Here by a 'proper' frame of a hard photon I denote any RF where the photon energy $k_0 \simeq m$.

(i) Show that the quantity $\chi = \frac{e}{m^3}\sqrt{-(F_{\mu\nu}p^\nu)^2}$ is conserved in a constant electromagnetic field and estimate its value for an initially slow particle;

(ii) How behaves χ in a field of a running plane wave?

(iii) Try to show that for initially slow particle *in general case of arbitrary time-varying field* (but different from the degenerate cases above, i.e. in case of focused laser field) $E \ll E_S$ with $a_0 \gg 1$

$$\chi(t) \simeq \frac{e^2E^2\omega t^2}{m^3}, \quad \frac{m}{eE} \ll t \ll \frac{1}{\omega}$$

[Hint: for $t \ll \frac{1}{\omega}$ use the expansion $F_{\mu\nu}(x) = F_{\mu\nu}(0) + F_{\mu\nu,\lambda}(0)x^\lambda + \frac{1}{2}F_{\mu\nu,\lambda\sigma}(0)x^\lambda x^\sigma + \ldots$]

5. **Classical limit of cascade equations**: consider the quantum cascade equation

$$\left\{\frac{\partial}{\partial t} + \frac{\boldsymbol{p}}{\varepsilon}\cdot\nabla - e\left(\boldsymbol{E} + \frac{\boldsymbol{p}}{\varepsilon}\times\boldsymbol{H}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}}\right\} f_{e^-}(\boldsymbol{p},t) = \int f_{e^-}(\boldsymbol{p}+\boldsymbol{k},t)w_{rad}(\boldsymbol{p}+\boldsymbol{k}\to\boldsymbol{k})d^3k$$
$$- f_{e^-}(\boldsymbol{p},t)\int w_{rad}(\boldsymbol{p}\to\boldsymbol{k})d^3k + \int f_\gamma(\boldsymbol{k},t)w_{cr}(\boldsymbol{k}\to\boldsymbol{p})d^3k$$

where $dW_{rad} = w_{rad}(\boldsymbol{p}\to\boldsymbol{k})d^3k$ and $dW_{cr} = w_{cr}(\boldsymbol{k}\to\boldsymbol{p})d^3p$ are the differential probabilities of hard photon emission and pair photoproduction, respectively. Try to explain how the classical equation of motion with radiation friction force included is recovered in classical limit $\chi \ll 1$.