Soft-Collinear Effective Theory

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T. Becher, A. Broggio, A. Ferroglia,
Introduction to Soft-Collinear Effective Theory,
arXiv:1410.1892

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Problems with widely separated scales

- Method of regions
- Effective field theories

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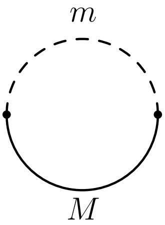
Problems with widely separated scales

- Method of regions
- Effective field theories

Plan

- Scalar φ^3
- QCD
- Some applications

Method of regions



$d = 2$

$M \gg m$

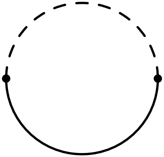
$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(M^2 - k^2 - i0)(m^2 - k^2 - i0)}$$

$$= \int \frac{d^d k}{\pi^{d/2}} \frac{1}{(k^2 + M^2)(k^2 + m^2)}$$

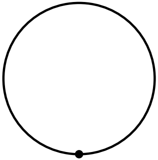
Exact solution

$$d = 2 - 2\varepsilon$$

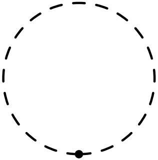
$$I = \frac{1}{M^2 - m^2} \int \frac{d^d k}{\pi^{d/2}} \left[-\frac{1}{k^2 + M^2} + \frac{1}{k^2 + m^2} \right]$$



$$= \frac{1}{M^2 - m^2} \Bigg[-$$



$$+$$



$$\Bigg]$$

Exact solution

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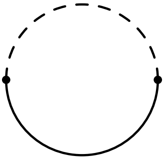
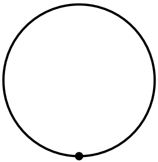
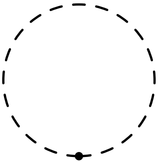


$$= \frac{1}{M^2 - m^2} \Bigg[-$$



$$+$$



$$\Bigg]$$

$$I = -\Gamma(\varepsilon) \frac{M^{-2\varepsilon} - m^{-2\varepsilon}}{M^2 - m^2} \to \frac{\log \frac{M^2}{m^2}}{M^2 - m^2}$$

$$= \frac{1}{M^2} \log \frac{M^2}{m^2} \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right]$$

Method of regions

$$I = I_h + I_s$$

hard $k \sim M$

soft $k \sim m$

Hard region

$k \sim M$

$$I_h = \int \frac{d^d k}{\pi^{d/2}} T_h \frac{1}{(k^2 + M^2)(k^2 + m^2)}$$

$$T_h \frac{1}{(k^2 + M^2)(k^2 + m^2)} = \frac{1}{k^2 + M^2} \frac{1}{k^2} \left[1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} - \cdots \right]$$

$$I_h = -\frac{M^{-2\varepsilon}}{M^2} \Gamma(\varepsilon) \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right]$$

Infrared divergence
Taylor series in m
Loop integrals with a single scale $M \Rightarrow M^{-2\varepsilon}$

Soft region

$k \sim m$

$$I_s = \int \frac{d^d k}{\pi^{d/2}} T_s \frac{1}{(k^2 + M^2)(k^2 + m^2)}$$

$$T_s \frac{1}{(k^2 + M^2)(k^2 + m^2)} = \frac{1}{M^2} \frac{1}{k^2 + m^2} \left[1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} - \cdots \right]$$

$$I_s = \frac{m^{-2\varepsilon}}{M^2} \Gamma(\varepsilon) \left[1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right]$$

Ultraviolet divergence
Taylor series in $1/M$
Loop integrals with a single scale $m \Rightarrow m^{-2\varepsilon}$

Proof

$m \ll \Lambda \ll M$

$$\begin{aligned} I &= \int_{k>\Lambda} \frac{d^d k}{\pi^{d/2}} \frac{1}{(k^2+M^2)(k^2+m^2)} \\ &\quad + \int_{k<\Lambda} \frac{d^d k}{\pi^{d/2}} \frac{1}{(k^2+M^2)(k^2+m^2)} \\ &= \int_{k>\Lambda} \frac{d^d k}{\pi^{d/2}} T_h \frac{1}{(k^2+M^2)(k^2+m^2)} \\ &\quad + \int_{k<\Lambda} \frac{d^d k}{\pi^{d/2}} T_s \frac{1}{(k^2+M^2)(k^2+m^2)} \\ &= I_h + I_s - \Delta I \end{aligned}$$

$$
\begin{aligned}
\Delta I = & \int_{k<\Lambda} \frac{d^d k}{\pi^{d/2}} T_h \frac{1}{(k^2+M^2)(k^2+m^2)} \\
& + \int_{k>\Lambda} \frac{d^d k}{\pi^{d/2}} T_s \frac{1}{(k^2+M^2)(k^2+m^2)} \\
= & \int_{k<\Lambda} \frac{d^d k}{\pi^{d/2}} T_s T_h \frac{1}{(k^2+M^2)(k^2+m^2)} \\
& + \int_{k>\Lambda} \frac{d^d k}{\pi^{d/2}} T_h T_s \frac{1}{(k^2+M^2)(k^2+m^2)} \\
= & \int \frac{d^d k}{\pi^{d/2}} T_s T_h \frac{1}{(k^2+M^2)(k^2+m^2)}
\end{aligned}
$$

$$T_s T_h \frac{1}{(k^2 + M^2)(k^2 + m^2)} = T_h T_s \frac{1}{(k^2 + M^2)(k^2 + m^2)}$$

$$= \frac{1}{M^2 k^2} \left[1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} - \cdots \right] \left[1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} - \cdots \right]$$

$$\Delta I = 0$$

No scale

Form factor

Massless φ^3 theory $d = 4$

$$L = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) - \frac{g}{3!}\varphi^3$$

$[L] = d$, $[\varphi] = (d-2)/2$, $[g] = (6-d)/2$

Form factor

Massless φ^3 theory $d = 4$

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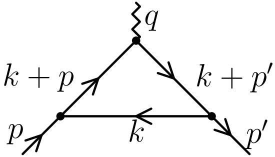


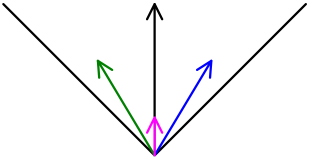
$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{[-k^2 - i0][-(k+p)^2 - i0][-(k+p')^2 - i0]}$$

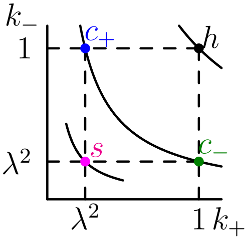
$-p^2, -p'^2 \ll -q^2$

Regions

$$\lambda^2 \sim \frac{-p^2}{-q^2} \sim \frac{-p'^2}{-q^2} \ll 1$$

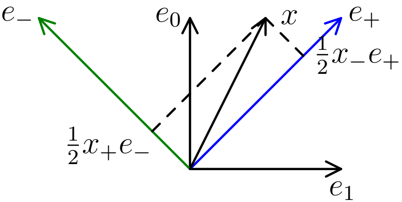
hard	h	$k \sim (1, 1, 1)Q$
collinear	c_+	$k \sim (1, \lambda^2, \lambda)Q$
collinear	c_-	$k \sim (\lambda^2, 1, \lambda)Q$
soft	s	$k \sim (\lambda^2, \lambda^2, \lambda^2)Q$





Light-front components

$$e_\pm = e_0 \pm e_1 \qquad e_\pm^2 = 0 \qquad e_+ \cdot e_- = 2$$
$$x_\pm = x \cdot e_\pm = x^0 \mp x^1$$
$$x = \frac{1}{2}\left(x_- e_+ + x_+ e_-\right) + x_\perp = (x_+, x_-, \vec{x}_\perp)$$



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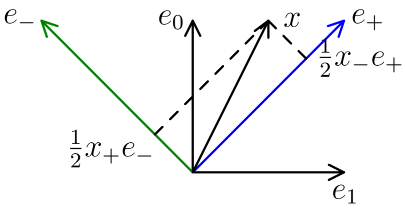
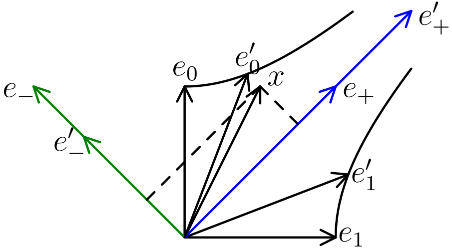


$$x \cdot y = \frac{1}{2}\left(x_+ y_- + x_- y_+\right) - \vec{x}_\perp \cdot \vec{y}_\perp$$

$$x^2 = x_+ x_- - \vec{x}_\perp^{\,2}$$

Light-front components

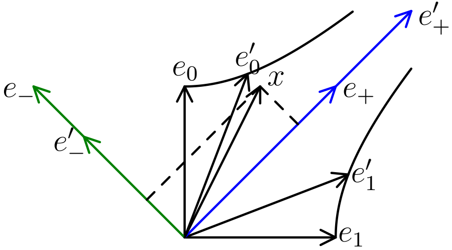
Lorentz transformation



$$x'_\pm = x_\pm e^{\pm\varphi}$$

Light-front components

Lorentz transformation



$$x'_\pm = x_\pm e^{\pm\varphi}$$

$$d^d x = \frac{1}{2} dx_+ \, dx_- \, d^{d-2} \vec{x}_\perp$$

Light-front components

$$
\begin{aligned}
df(x) &= \frac{\partial f}{\partial x^0} dx^0 + \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial \vec{x}_\perp} \cdot d\vec{x}_\perp \\
&= \partial^0 f \, dx^0 - \partial^1 f \, dx^1 - \vec{\partial}_\perp \cdot d\vec{x}_\perp
\end{aligned}
$$

$$
\partial^0 = \frac{\partial}{\partial x^0} \qquad \partial^1 = -\frac{\partial}{\partial x^1} \qquad \vec{\partial}_\perp = -\frac{\partial}{\partial \vec{x}_\perp}
$$

Light-front components

$$df(x) = \frac{\partial f}{\partial x^0} dx^0 + \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial \vec{x}_\perp} \cdot d\vec{x}_\perp$$

$$= \partial^0 f\, dx^0 - \partial^1 f\, dx^1 - \vec{\partial}_\perp \cdot d\vec{x}_\perp$$

$$\partial^0 = \frac{\partial}{\partial x^0} \qquad \partial^1 = -\frac{\partial}{\partial x^1} \qquad \vec{\partial}_\perp = -\frac{\partial}{\partial \vec{x}_\perp}$$

$$df(x) = \frac{\partial f}{\partial x_+} dx_+ + \frac{\partial f}{\partial x_-} dx_- + \frac{\partial f}{\partial \vec{x}_\perp} \cdot d\vec{x}_\perp$$

$$= \frac{1}{2}\partial_+ f\, dx_- + \frac{1}{2}\partial_- f\, dx_+ - \vec{\partial}_\perp \cdot d\vec{x}_\perp$$

$$\partial_+ = 2\frac{\partial}{\partial x_-} \qquad \partial_- = 2\frac{\partial}{\partial x_+}$$

Light-front components

$$df(x) = \frac{\partial f}{\partial x^0}dx^0 + \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial \vec{x}_\perp} \cdot d\vec{x}_\perp$$

$$= \partial^0 f \, dx^0 - \partial^1 f \, dx^1 - \vec{\partial}_\perp \cdot d\vec{x}_\perp$$

$$\partial^0 = \frac{\partial}{\partial x^0} \qquad \partial^1 = -\frac{\partial}{\partial x^1} \qquad \vec{\partial}_\perp = -\frac{\partial}{\partial \vec{x}_\perp}$$

$$df(x) = \frac{\partial f}{\partial x_+}dx_+ + \frac{\partial f}{\partial x_-}dx_- + \frac{\partial f}{\partial \vec{x}_\perp} \cdot d\vec{x}_\perp$$

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$$\partial_+ = 2\frac{\partial}{\partial x_-} \qquad \partial_- = 2\frac{\partial}{\partial x_+}$$

$$dx_+ = dx^0 - dx^1 \qquad dx_- = dx^0 + dx^1$$

$$\partial_+ = \partial^0 - \partial^1 \qquad \partial_- = \partial^0 + \partial^1$$

Kinematics

$$p = \frac{1}{2}\left(p_- e_+ - \frac{-p^2}{p_-} e_-\right)$$

$$p' = \frac{1}{2}\left(-\frac{-p'^2}{p'_+} e_+ + p'_+ e_-\right)$$

$$-q^2 = p_- p'_+ - p^2 - p'^2 + \frac{(-p^2)(-p'^2)}{p_- p'_+}$$

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$$I = I_h + I_{c_+} + I_{c_-} + I_s$$

Hard

$$k \sim (1,1,1)Q \quad p \sim (\lambda^2,1,0)Q \quad p' \sim (1,\lambda^2,0)Q \quad d^d k \sim \mathcal{O}(1)$$

$$-k^2 = -k_+k_- + \vec{k}_\perp^{\,2} = \mathcal{O}(1)$$

$$-(k+p)^2 = -(k_+ + p_+)(k_- + p_-) + \vec{k}_\perp^{\,2}$$
$$= \underbrace{-k^2 - p_-k_+}_{\mathcal{O}(1)} - \underbrace{p_+(k_- + p_-)}_{\mathcal{O}(\lambda^2)}$$

$$-(k+p')^2 = -(k_+ + p'_+)(k_- + p'_-) + \vec{k}_\perp^{\,2}$$
$$= \underbrace{-k^2 - p'_+k_-}_{\mathcal{O}(1)} - \underbrace{p'_-(k_+ + p'_+)}_{\mathcal{O}(\lambda^2)}$$

$$I_h = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)(-k^2 - p_-k_+ - i0)(-k^2 - p'_+k_- - i0)}$$

On-shell external legs

Hard

Single scale $-q^2 = p_- p'_+ + \mathcal{O}(\lambda^2)$

$$k_- = p_- k_- \qquad k_+ = p'_+ k_+ \qquad \vec{k}_\perp = \sqrt{p_- p'_+}\,\vec{k}_\perp$$

$$I_h = (-q^2)^{d/2-3}$$

$$\times \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)(-k^2 - k_+ - i0)(-k^2 - k_- - i0)}$$

Hard

Single scale $-q^2 = p_- p'_+ + \mathcal{O}(\lambda^2)$

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Feynman parametrization

$$\frac{1}{D_1 D_2 D_3} = \int d\alpha_1 d\alpha_2 d\alpha_3 e^{-\alpha_1 D_1 - \alpha_2 D_2 - \alpha_3 D_3}$$

Hard

Single scale $-q^2 = p_- p'_+ + \mathcal{O}(\lambda^2)$

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Feynman parametrization

$$\frac{1}{D_1 D_2 D_3} = \int d\alpha_1 d\alpha_2 d\alpha_3 e^{-\alpha_1 D_1 - \alpha_2 D_2 - \alpha_3 D_3} d\eta\, \delta(\alpha_1 + \alpha_2 + \alpha_3 - \eta)$$

Hard

Single scale $-q^2 = p_- p'_+ + \mathcal{O}(\lambda^2)$

$$k_- = p_- k_- \qquad k_+ = p'_+ k_+ \qquad \vec{k}_\perp = \sqrt{p_- p'_+}\,\vec{k}_\perp$$

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Feynman parametrization

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$$\alpha_i = \eta x_i$$

Hard

Single scale $-q^2 = p_- p'_+ + \mathcal{O}(\lambda^2)$

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Feynman parametrization

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$$\alpha_i = \eta x_i$$

$$= \int dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1) d\eta\, \eta^2 e^{-\eta(x_1 D_1 + x_2 D_2 + x_3 D_3)}$$

Hard

Single scale $-q^2 = p_- p'_+ + \mathcal{O}(\lambda^2)$

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Feynman parametrization

$$\frac{1}{D_1 D_2 D_3} = \int d\alpha_1 d\alpha_2 d\alpha_3 e^{-\alpha_1 D_1 - \alpha_2 D_2 - \alpha_3 D_3} d\eta\, \delta(\alpha_1 + \alpha_2 + \alpha_3 - \eta)$$

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$$= \Gamma(3) \int \frac{dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1)}{(x_1 D_1 + x_2 D_2 + x_3 D_3)^3}$$

Hard

Set $-q^2 = 1$

$$I_h = \Gamma(3) \int dx_1\, dx_2\, dx_3\, \delta(x_1 + x_2 + x_3 - 1) \frac{d^d k}{i\pi^{d/2}} \frac{1}{D^3}$$

$$D = -x_1 k^2 - x_2(k^2 + k_+) - x_3(k^2 + k_-) - i0$$

$$= -k^2 - (x_2 e_+ + x_3 e_-) \cdot k - i0 = -k'^2 + x_2 x_3 - i0$$

$$k' = k + \frac{1}{2}(x_2 e_+ + x_3 e_-)$$

Hard

Set $-q^2 = 1$

$$I_h = \Gamma(3) \int dx_1\, dx_2\, dx_3\, \delta(x_1 + x_2 + x_3 - 1) \frac{d^d k}{i\pi^{d/2}} \frac{1}{D^3}$$

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$$k' = k + \frac{1}{2}(x_2 e_+ + x_3 e_-)$$

$$I_h = \Gamma(3 - d/2) \int dx_2 dx_3 (x_2 x_3)^{d/2-3}$$

Hard

Set $-q^2 = 1$

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$$I_h = \Gamma(3 - d/2) \int dx_2 dx_3 (x_2 x_3)^{d/2-3}$$

$$x_2 = zx \qquad x_3 = z(1-x)$$

$$I_h = \Gamma(3 - d/2) \int_0^1 dz\, z^{d-5} \cdot \int_0^1 dx\, [x(1-x)]^{d/2-3}$$

Hard

Set $-q^2 = 1$

$$I_h = \Gamma(3) \int dx_1\, dx_2\, dx_3\, \delta(x_1 + x_2 + x_3 - 1) \frac{d^d k}{i\pi^{d/2}} \frac{1}{D^3}$$

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$$I_h = \Gamma(3 - d/2) \int dx_2 dx_3 (x_2 x_3)^{d/2-3}$$

$$x_2 = zx \qquad x_3 = z(1 - x)$$

$$I_h = \Gamma(3 - d/2) \int_0^1 dz\, z^{d-5} \cdot \int_0^1 dx\, [x(1 - x)]^{d/2-3}$$

$$= \frac{\Gamma(3 - d/2)\Gamma^2(d/2 - 2)}{\Gamma(d - 3)}$$

Hard

$$I_h = \frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} \Gamma^2(-\varepsilon)(-q^2)^{-1-\varepsilon}$$

Infrared and collinear divergences
Single scale $-q^2$

Collinear c_+

$$k \sim (\lambda^2, 1, \lambda)Q \quad p \sim (\lambda^2, 1, 0)Q \quad p' \sim (1, \lambda^2, 0)Q$$

$$d^d k \sim \mathcal{O}(\lambda^d)$$

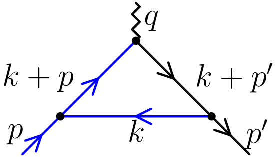
$$-k^2 = -k_+ k_- + \vec{k}_\perp^{\,2} = \mathcal{O}(\lambda^2)$$

$$-(k+p)^2 = -(k_+ + p_+)(k_- + p_-) + \vec{k}_\perp^{\,2} = \mathcal{O}(\lambda^2)$$

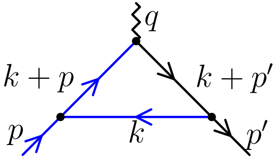
$$\begin{aligned} -(k+p')^2 &= -(k_+ + p'_+)(k_- + p'_-) + \vec{k}_\perp^{\,2} \\ &= -\underbrace{p'_+ k_-}_{\mathcal{O}(1)} - \underbrace{(k^2 + p'_+ p'_-)}_{\mathcal{O}(\lambda^2)} - \underbrace{p'_- k_+}_{\mathcal{O}(\lambda^4)} \end{aligned}$$

$$I_{c_+} = \frac{1}{p'_+} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)(-(k+p)^2 - i0)(-k_- - i0)}$$

Collinear c_+



Collinear c_+



Single scale $-p^2 = -p_+p_-$

$$k_+ = (-p_+)k_+ \quad k_- = p_-k_- \quad \vec{k}_\perp = \sqrt{-p_+p_-}\,\vec{k}_\perp \quad p = (-1, 1, \vec{0})$$

$$I_{c_+} = \frac{(p_+p_-)^{d/2-2}}{p_-p'_+} \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)(-(k+p)^2 - i0)(-k_- - i0)}$$

Collinear c_+

Feynman parametrization

$$I_{c_+} = \Gamma(3) \int dx_1 dx_2 \delta(x_1 + x_2 - 1) dy \frac{d^d k}{i\pi^{d/2}} \frac{1}{D^3}$$

$$D = -x_1 k^2 - x_2 (k+p)^2 - y k_- - i0$$

$$= -k^2 - (2x_2 p + y e_-) \cdot k + x_2 - i0$$

$$= -k'^2 + x_2(1 - x_2 + y) - i0$$

$$k' = k + x_2 p + \frac{1}{2} y e_-$$

$$I_{c_+} = \Gamma(3 - d/2) \int dx dy [x(1 - x + y)]^{d/2-3}$$

Collinear c_+

$$y = (1-x)z$$

$$\int = \int_0^\infty dz\,(1+z)^{d/2-3} \cdot \int_0^1 dx\, x^{d/2-3}(1-x)^{d/2-2}$$

$$I_{c_+} = \frac{\Gamma(2-d/2)\Gamma(d/2-2)\Gamma(d/2-1)}{\Gamma(d-3)}$$

Collinear c_+

$$y = (1-x)z$$

$$\int = \int_0^\infty dz\,(1+z)^{d/2-3} \cdot \int_0^1 dx\, x^{d/2-3}(1-x)^{d/2-2}$$

$$I_{c_+} = \frac{\Gamma(2-d/2)\Gamma(d/2-2)\Gamma(d/2-1)}{\Gamma(d-3)}$$

$$I_{c_+} = \frac{\Gamma(d/2-1)}{\Gamma(d-3)}\Gamma(2-d/2)\Gamma(d/2-2)\frac{(-p_+p_-)^{d/2-2}}{p_-p'_+}$$

$$I_{c_+} = \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\Gamma(\varepsilon)\Gamma(-\varepsilon)\frac{(-p^2)^{-\varepsilon}}{-q^2}$$

Soft

$$k \sim (\lambda^2, \lambda^2, \lambda^2)Q \quad p \sim (\lambda^2, 1, 0)Q \quad p' \sim (1, \lambda^2, 0)Q$$

$$d^d k \sim \mathcal{O}(\lambda^{2d})$$

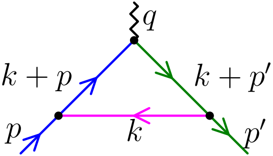
$$-k^2 = -k_+ k_- + \vec{k}_\perp^{\,2} = \mathcal{O}(\lambda^4)$$

$$\begin{aligned} -(k+p)^2 &= -(k_+ + p_+)(k_- + p_-) + \vec{k}_\perp^{\,2} \\ &= \underbrace{-p_-(k_+ + p_+)}_{\mathcal{O}(\lambda^2)} - \underbrace{(k^2 + p_+ k_-)}_{\mathcal{O}(\lambda^4)} \end{aligned}$$

$$\begin{aligned} -(k+p')^2 &= -(k_+ + p'_+)(k_- + p'_-) + \vec{k}_\perp^{\,2} \\ &= \underbrace{-p'_+(k_- + p'_-)}_{\mathcal{O}(\lambda^2)} - \underbrace{(k^2 + p'_- k_+)}_{\mathcal{O}(\lambda^4)} \end{aligned}$$

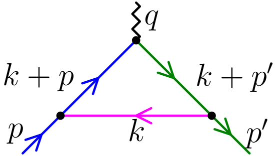
$$I_s = \frac{1}{p_- p'_+} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)(-k_+ - p_+ - i0)(-k_- - p'_- - i0)}$$

Soft



Off-shell collinear legs. Ultraviolet divergence.

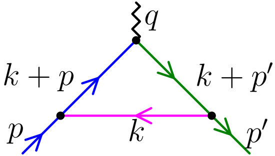
Soft



Off-shell collinear legs. Ultraviolet divergence.
Single scale

$$\frac{(-p^2)(-p'^2)}{-q^2} = \underbrace{p_+ p'_-}_{\mathcal{O}(\lambda^4)} + \mathcal{O}(\lambda^6)$$

Soft



Off-shell collinear legs. Ultraviolet divergence.
Single scale

$$\frac{(-p^2)(-p'^2)}{-q^2} = \underbrace{p_+ p'_-}_{\mathcal{O}(\lambda^4)} + \mathcal{O}(\lambda^6)$$

$$k_+ = (-p_+)k_+ \quad k_- = (-p'_-)k_- \quad \vec{k}_\perp = \sqrt{(-p_+)(-p'_-)}\vec{k}_\perp$$

$$I_s = \frac{(p_+ p'_-)^{d/2-2}}{p_- p'_+} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)(-k_+ + 1 - i0)(-k_- + 1 - i0)}$$

Soft

Feynman parametrization

$$I_s = \Gamma(3) \int dy_1 dy_2 \frac{d^d k}{i\pi^{d/2}} \frac{1}{D^3}$$

$$D = -k^2 + y_1(-k_+ + 1) + y_2(-k_- + 1) - i0$$

$$= -k^2 - (y_1 e_+ + y_2 e_-) \cdot k + y_1 + y_2 - i0$$

$$= -k'^2 + y_1 + y_2 + y_1 y_2 - i0$$

$$k' = k + \frac{1}{2}(y_1 e_+ + y_2 e_-)$$

$$I_s = \Gamma(3 - d/2) \int dy_1 dy_2 (y_1 + y_2 + y_1 y_2)^{d/2-3}$$

Soft

$$y_1 = yx \qquad y_2 = y(1-x)$$

$$\int = \int_0^\infty dy\, y^{d/2-2} \int_0^1 dx\, [1 + yx(1-x)]^{d/2-3}$$

Soft

$$y_1 = yx \qquad y_2 = y(1-x)$$

$$\int = \int_0^\infty dy\, y^{d/2-2} \int_0^1 dx\, [1 + yx(1-x)]^{d/2-3}$$

$$yx(1-x) = z$$

$$= \int_0^\infty dz\, z^{d/2-2}(1+z)^{d/2-3} \cdot \int_0^1 dx\, [x(1-x)]^{1-d/2}$$

Soft

$$y_1 = yx \qquad y_2 = y(1-x)$$

$$\int = \int_0^\infty dy\, y^{d/2-2} \int_0^1 dx\, [1 + yx(1-x)]^{d/2-3}$$

$$yx(1-x) = z$$

$$= \int_0^\infty dz\, z^{d/2-2}(1+z)^{d/2-3} \cdot \int_0^1 dx\, [x(1-x)]^{1-d/2}$$

$$1 + z = 1/u$$

$$= \int_0^1 du\, u^{3-d}(1-u)^{d/2-2} \cdot \int_0^1 dx\, [x(1-x)]^{1-d/2}$$

Soft

$$y_1 = yx \qquad y_2 = y(1-x)$$

$$\int = \int_0^\infty dy\, y^{d/2-2} \int_0^1 dx\, [1 + yx(1-x)]^{d/2-3}$$

$$yx(1-x) = z$$

$$= \int_0^\infty dz\, z^{d/2-2}(1+z)^{d/2-3} \cdot \int_0^1 dx\, [x(1-x)]^{1-d/2}$$

$$1 + z = 1/u$$

$$= \int_0^1 du\, u^{3-d}(1-u)^{d/2-2} \cdot \int_0^1 dx\, [x(1-x)]^{1-d/2}$$

$$\frac{\Gamma(d/2-q)}{\Gamma(3-d/2)}\Gamma^2(2-d/2)$$

Soft

$$
\begin{aligned}
I_s &= \Gamma(d/2-1)\Gamma^2(2-d/2)\frac{(p_+p'_-)^{d/2-2}}{p_-p'_+} \\
&= \Gamma(1-\varepsilon)\Gamma^2(\varepsilon)\frac{1}{-q^2}\left(\frac{(-p^2)(-p'^2)}{-q^2}\right)^{-\varepsilon}
\end{aligned}
$$

Result

$$
\begin{aligned}
I = {} & \frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\Gamma^2(-\varepsilon)(-q^2)^{-1-\varepsilon} \\
& + \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\Gamma(\varepsilon)\Gamma(-\varepsilon)(-q^2)^{-1}\left[(-p^2)^{-\varepsilon} + (-p'^2)^{-\varepsilon}\right] \\
& + \Gamma(1-\varepsilon)\Gamma^2(\varepsilon)(-q^2)^{-1}\left(\frac{(-p^2)(-p'^2)}{-q^2}\right)^{-\varepsilon} \\
= {} & \frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\Gamma^2(-\varepsilon)(-q^2)^{-1-\varepsilon}\left[1 - \left(\frac{-p^2}{-q^2}\right)^{-\varepsilon} - \left(\frac{-p'^2}{-q^2}\right)^{-\varepsilon}\right. \\
& \qquad \left. + \frac{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)}\left(\frac{-p^2}{-q^2}\right)^{-\varepsilon}\left(\frac{-p'^2}{-q^2}\right)^{-\varepsilon}\right]
\end{aligned}
$$

$$
I = \frac{1}{-q^2}\left(\log\frac{-p^2}{-q^2}\,\log\frac{-p'^2}{-q^2} + \frac{\pi^2}{3}\right)
$$

Divergences cancel

Other regions?

Let's try

$$k \sim (\lambda, \lambda, \lambda)Q \quad p \sim (\lambda^2, 1, 0)Q \quad p' \sim (1, \lambda^2, 0)Q$$

$$-k^2 = -k_+ k_- + \vec{k}_\perp^{\,2} = \mathcal{O}(\lambda^2)$$

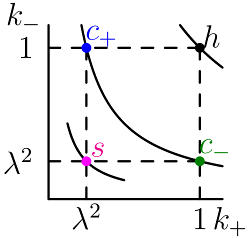
$$-(k+p)^2 = -(k_+ + p_+)(k_- + p_-) + \vec{k}_\perp^{\,2}$$
$$= -\underbrace{p_- k_+}_{\mathcal{O}(\lambda)} + \mathcal{O}(\lambda^2)$$

$$-(k+p')^2 = -(k_+ + p'_+)(k_- + p'_-) + \vec{k}_\perp^{\,2}$$
$$= -\underbrace{p'_+ k_-}_{\mathcal{O}(\lambda)} + \mathcal{O}(\lambda^2)$$

$$I_{sh} = \frac{1}{p_- p'_+} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)(-k_+ - i0)(-k_- - i0)} = 0$$

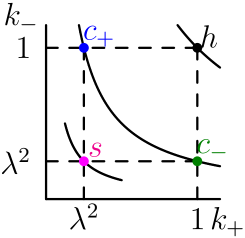
Effective theory

	k	k^2
Hard	$(1,1,1)Q$	Q^2
Collinear c_+	$(\lambda^2,1,\lambda)Q$	$Q^2\lambda^2$
Collinear c_-	$(1,\lambda^2,\lambda)Q$	$Q^2\lambda^2$
Soft	$(\lambda^2,\lambda^2,\lambda^2)Q$	$Q^2\lambda^4$



Effective theory

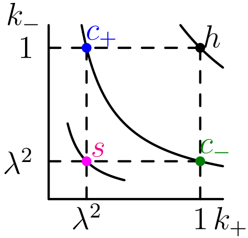
	k	k^2
Hard	$(1,1,1)Q$	Q^2
Collinear c_+	$(\lambda^2,1,\lambda)Q$	$Q^2\lambda^2$
Collinear c_-	$(1,\lambda^2,\lambda)Q$	$Q^2\lambda^2$
Soft	$(\lambda^2,\lambda^2,\lambda^2)Q$	$Q^2\lambda^4$



$$\varphi \to \varphi_{c_+} + \varphi_{c_-} + \varphi_s$$

Effective theory

	k	k^2
Hard	$(1,1,1)Q$	Q^2
Collinear c_+	$(\lambda^2,1,\lambda)Q$	$Q^2\lambda^2$
Collinear c_-	$(1,\lambda^2,\lambda)Q$	$Q^2\lambda^2$
Soft	$(\lambda^2,\lambda^2,\lambda^2)Q$	$Q^2\lambda^4$



$$\varphi \to \varphi_{c_+} + \varphi_{c_-} + \varphi_s$$

$$L = L_{c_+} + L_{c_-} + L_s + L_{cs}$$

$$L_{c_+} = \frac{1}{2}(\partial_\mu \varphi_{c_+})(\partial^\mu \varphi_{c_+}) - \frac{g}{3!}\varphi_{c_+}^3$$

$$L_{c_-} = \frac{1}{2}(\partial_\mu \varphi_{c_-})(\partial^\mu \varphi_{c_-}) - \frac{g}{3!}\varphi_{c_-}^3$$

$$L_s = \frac{1}{2}(\partial_\mu \varphi_s)(\partial^\mu \varphi_s) - \frac{g}{3!}\varphi_s^3$$

Power counting

Soft

$$k_s \sim \lambda^2 \qquad \partial_s \sim \lambda^2 \qquad x_s \sim \lambda^{-2}$$

Power counting

Soft

$$k_s \sim \lambda^2 \qquad \partial_s \sim \lambda^2 \qquad x_s \sim \lambda^{-2}$$

$$<T\varphi_s(x)\varphi_s(0)> = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \frac{i}{k^2 + i0}$$

$$\varphi_s \sim \lambda^{d-2}$$

Power counting

Soft

$$k_s \sim \lambda^2 \qquad \partial_s \sim \lambda^2 \qquad x_s \sim \lambda^{-2}$$

$$<T\varphi_s(x)\varphi_s(0)> = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \frac{i}{k^2 + i0}$$

$$\varphi_s \sim \lambda^{d-2}$$

$$L_s \sim (\partial_\mu \varphi_s)(\partial^\mu \varphi_s) \sim \lambda^{2d} \qquad S_s = \int d^d x \, L_s \sim 1$$

Power counting

Collinear c_+ ($k_+ x_- \sim 1$, $k_- x_+ \sim 1$, $k_\perp x_\perp \sim 1$)

$$k_{c_+} \sim (\lambda^2, 1, \lambda) \qquad \partial_{c_+} \sim (\lambda^2, 1, \lambda) \qquad x_{c_+} \sim (1, \lambda^{-2}, \lambda^{-1})$$

Power counting

Collinear c_+ $(k_+ x_- \sim 1,\ k_- x_+ \sim 1,\ k_\perp x_\perp \sim 1)$

$$k_{c_+} \sim (\lambda^2, 1, \lambda) \qquad \partial_{c_+} \sim (\lambda^2, 1, \lambda) \qquad x_{c_+} \sim (1, \lambda^{-2}, \lambda^{-1})$$

$$< T \varphi_{c_+}(x) \varphi_{c_+}(0) > = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \frac{i}{k^2 + i0}$$

$$\varphi_{c_+} \sim \lambda^{d/2-1}$$

Power counting

Collinear c_+ $(k_+ x_- \sim 1,\ k_- x_+ \sim 1,\ k_\perp x_\perp \sim 1)$

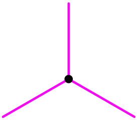
$$k_{c_+} \sim (\lambda^2, 1, \lambda) \qquad \partial_{c_+} \sim (\lambda^2, 1, \lambda) \qquad x_{c_+} \sim (1, \lambda^{-2}, \lambda^{-1})$$

$$<T\varphi_{c_+}(x)\varphi_{c_+}(0)> = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \frac{i}{k^2 + i0}$$

$$\varphi_{c_+} \sim \lambda^{d/2-1}$$

$$L_{c_+} \sim (\partial_+ \varphi_{c_+})(\partial_- \varphi_{c_+}) \sim \lambda^d \qquad S = \int d^d x\, L \sim 1$$

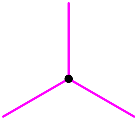
Interactions



$$-\frac{g}{3!}\varphi_s^3$$

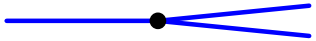
$$(\lambda^2, \lambda^2, \lambda^2) + (\lambda^2, \lambda^2, \lambda^2) = (\lambda^2, \lambda^2, \lambda^2)$$

Interactions



$$-\frac{g}{3!}\varphi_s^3$$

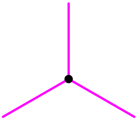
$$(\lambda^2, \lambda^2, \lambda^2) + (\lambda^2, \lambda^2, \lambda^2) = (\lambda^2, \lambda^2, \lambda^2)$$



$$-\frac{g}{3!}\varphi_{c_+}^3$$

$$(\lambda^2, 1, \lambda) + (\lambda^2, 1, \lambda) = (\lambda^2, 1, \lambda)$$

Interactions

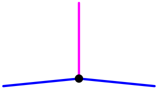


$$-\frac{g}{3!}\varphi_s^3$$

$$(\lambda^2, \lambda^2, \lambda^2) + (\lambda^2, \lambda^2, \lambda^2) = (\lambda^2, \lambda^2, \lambda^2)$$

$$-\frac{g}{3!}\varphi_{c_+}^3$$

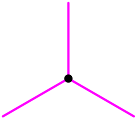
$$(\lambda^2, 1, \lambda) + (\lambda^2, 1, \lambda) = (\lambda^2, 1, \lambda)$$



$$L_{cs} = -\frac{g}{2}\varphi_{c_+}^2\varphi_s - \frac{g}{2}\varphi_{c_-}^2\varphi_s$$

$$(\lambda^2, 1, \lambda) + (\lambda^2, \lambda^2, \lambda^2) = (\lambda^2, 1, \lambda)$$

Interactions

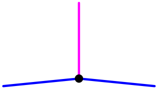


$$-\frac{g}{3!}\varphi_s^3$$

$$(\lambda^2,\lambda^2,\lambda^2)+(\lambda^2,\lambda^2,\lambda^2)=(\lambda^2,\lambda^2,\lambda^2)$$

$$-\frac{g}{3!}\varphi_{c_+}^3$$

$$(\lambda^2,1,\lambda)+(\lambda^2,1,\lambda)=(\lambda^2,1,\lambda)$$



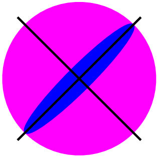
$$L_{cs}=-\frac{g}{2}\varphi_{c_+}^2\varphi_s-\frac{g}{2}\varphi_{c_-}^2\varphi_s$$

$$(\lambda^2,1,\lambda)+(\lambda^2,\lambda^2,\lambda^2)=(\lambda^2,1,\lambda)$$

Other interactions (${\varphi_{c_+}}^2\varphi_{c_-}$, $\varphi_{c_+}\varphi_{c_-}\varphi_s$, ${\varphi_{c_+}}^2\varphi_s$, $\dots$) are not allowed by momentum conservation

Multipole expansion

$$S_{cs} = \frac{g}{2}\int d^dx\, \varphi_{c_+}^2(x)\varphi_s(x)$$



$$x \sim (1, \lambda^{-2}, \lambda^{-1})Q^{-1}$$

$$x \sim (\lambda^{-2}, \lambda^{-2}, \lambda^{-2})Q^{-1}$$

$$\partial \sim (\lambda^2, \lambda^2, \lambda^2)Q$$

$$\varphi_s(x) = \varphi_s(\bar{x}_-) - \underbrace{\vec{x}_\perp \cdot \vec{\partial}_\perp}_{\mathcal{O}(\lambda)}\varphi_s(\bar{x}_-) + \frac{1}{2}\underbrace{x_+\partial_-}_{\mathcal{O}(\lambda^2)}\varphi_s(\bar{x}_-)$$

$$+\frac{1}{2}\underbrace{x_{\perp i}x_{\perp j}\partial_{\perp i}\partial_{\perp j}}_{\mathcal{O}(\lambda^2)}\varphi_s(\bar{x}_-) + \mathcal{O}(\lambda^3) \qquad \bar{x}_- \equiv \frac{1}{2}x_- e_+^\mu$$

$$S_{cs} = \frac{g}{2}\int d^dx\, \varphi_{c_+}^2(x)\varphi_s(\bar{x}_-) + \mathcal{O}(\lambda)$$

Multipole expansion

$$S_{cs} = \frac{g}{2}\int d^dx\, \varphi_{c_+}^2(x)\varphi_s(x)$$

$$= \frac{g}{2}\int d^dx \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_s}{(2\pi)^d} e^{-i(k_1+k_2+k_s)\cdot x}\tilde{\varphi}_{c_+}(k_1)\tilde{\varphi}_{c_+}(k_2)\tilde{\varphi}_s(k_s)$$

$$k_1 + k_2 + k_s = (\lambda^2, 1, \lambda)Q \quad x \sim (1, \lambda^{-2}, \lambda^{-1})Q^{-1}$$

$$k_s \sim (\lambda^2, \lambda^2, \lambda^2)Q$$

$$k_s \cdot x = \frac{1}{2}\underbrace{k_{s+}x_-}_{\mathcal{O}(1)} + \frac{1}{2}\underbrace{k_{s-}x_+}_{\mathcal{O}(\lambda^2)} - \underbrace{\vec{k}_{s\perp}\cdot\vec{x}_\perp}_{\mathcal{O}(\lambda)}$$

φ_s when interacting with φ_{c_+} carries $k_s = (k_{s+}, 0, \vec{0})$

$$\int \frac{d^dk_s}{(2\pi)^d} e^{ik_{s+}x_-}\tilde{\varphi}_s(k_s) = \varphi_s(\bar{x}_-)$$

SCET Lagrangian

$$
\begin{aligned}
L = & \frac{1}{2}(\partial_\mu \varphi_{c_+}(x))(\partial^\mu \varphi_{c_+}(x)) - \frac{g}{3!}\varphi_{c_+}^3(x) \\
& + \frac{1}{2}(\partial_\mu \varphi_{c_-}(x))(\partial^\mu \varphi_{c_-}(x)) - \frac{g}{3!}\varphi_{c_-}^3(x) \\
& + \frac{1}{2}(\partial_\mu \varphi_s(x))(\partial^\mu \varphi_s(x)) - \frac{g}{3!}\varphi_s^3(x) \\
& - \frac{g}{2}\varphi_{c_+}^2(x)\varphi_s(\bar{x}_-) - \frac{g}{2}\varphi_{c_-}^2(x)\varphi_s(\bar{x}_+) + \mathcal{O}(\lambda)
\end{aligned}
$$

Translation invariant up to $\mathcal{O}(\lambda)$

Matching

Effective theory

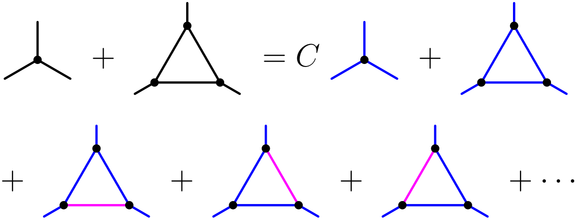
- Write the most general Lagrangian up to some order with unknown coefficients
- Calculate some scattering amplitudes in the full theory; if necessary, expand in small parameters up to some order
- Calculate the same amplitudes in the effective theory
- Equate, find the coefficients

Matching

For example, the c_+ collinear interaction

$$L_{c_+} = -\frac{g}{3!} C \varphi_{c_+}^3 \qquad C = 1 + C_1 g^2 + \cdots$$

All 3 external momenta c_+



All external momenta $\parallel e_+ \Rightarrow$ all loop diagrams $= 0$

$$C = 1$$

Current

$$J(x) = \frac{1}{2}\varphi^2(x) \to J_2(x) + J_3(x) + \cdots$$

Kinematics: $c_+ \to c_-$

$$J_2 = C_2 \varphi_{c_+} \varphi_{c_-} \qquad J_3 = \frac{1}{2} C_3 \left[\varphi_{c_+}^2 \varphi_{c_-} + \varphi_{c_+} \varphi_{c_-}^2 \right]$$

Current

$$J(x) = \frac{1}{2}\varphi^2(x) \to J_2(x) + J_3(x) + \cdots$$

Kinematics: $c_+ \to c_-$

$$J_2 = C_2 \varphi_{c_+} \varphi_{c_-} \qquad J_3 = \frac{1}{2} C_3 \left[\varphi_{c_+}^2 \varphi_{c_-} + \varphi_{c_+} \varphi_{c_-}^2 \right]$$

$t \sim 1/Q$

$t' \sim 1/Q$

$$\varphi_{c_+}(x + te_-) = \varphi_{c_+}(x) + \underbrace{t\partial_-}_{\mathcal{O}(1)} \varphi_{c_+}(x) + \frac{1}{2} \underbrace{t^2 \partial_-^2}_{\mathcal{O}(1)} \varphi_{c_+}(x) + \cdots$$

$$\varphi_{c_-}(x + t'e_+) = \varphi_{c_-}(x) + \underbrace{t'\partial_+}_{\mathcal{O}(1)} \varphi_{c_-}(x) + \frac{1}{2} \underbrace{t'^2 \partial_+^2}_{\mathcal{O}(1)} \varphi_{c_-}(x) + \cdots$$

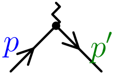
J_2: tree level

$$J_2(x) = \int dt\, dt'\, C_2(t, t') \varphi_{c_+}(x + te_-) \varphi_{c_-}(x + t'e_+)$$

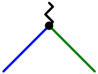
J_2: tree level

$$J_2(x) = \int dt\, dt'\, C_2(t,t')\varphi_{c_+}(x + te_-)\varphi_{c_-}(x + t'e_+)$$

On-shell $p = (0, p_-, \vec{0})$, $p' = (p'_+, 0, \vec{0})$



$$= \tilde{C}_2(p_-, p'_+)$$

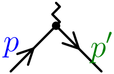


$$\tilde{C}_2(p_-, p'_+) = \int dt\, dt'\, e^{ip_- t - ip'_+ t'} C_2(t,t')$$

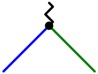
J_2: tree level

$$J_2(x) = \int dt\, dt'\, C_2(t,t')\varphi_{c_+}(x + te_-)\varphi_{c_-}(x + t'e_+)$$

On-shell $p = (0, p_-, \vec{0})$, $p' = (p'_+, 0, \vec{0})$



$$= \tilde{C}_2(p_-, p'_+)$$

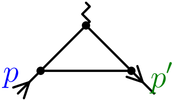


$$\tilde{C}_2(p_-, p'_+) = \int dt\, dt'\, e^{ip_- t - ip'_+ t'} C_2(t,t')$$

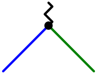
$$\tilde{C}_2^{(0)}(p_-, p'_+) = 1 \qquad C_2^{(0)}(t,t') = \delta(t)\delta(t')$$

J_2: 1 loop

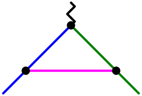
On-shell $p = (0, p_-, \vec{0})$, $p' = (p'_+, 0, \vec{0})$



$= \tilde{C}_2^{(1)}(p_-, p'_+)$



$+$



SCET loop vanishes

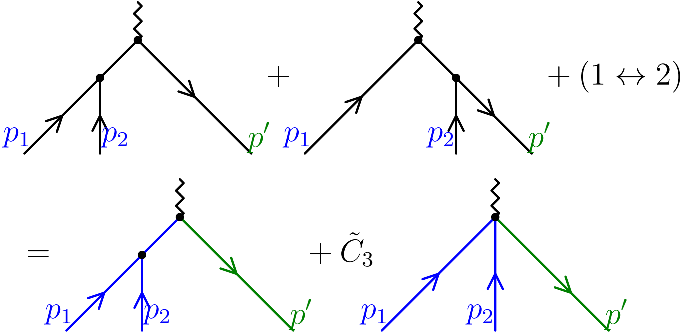
$$\tilde{C}_2^{(1)}(p_-, p'_+) = \frac{g^2}{(4\pi)^{d/2}} I_h(p_- p'_+)$$

J_3: tree level

$$J_3(x) = \frac{1}{2} \int dt_1\, dt_2\, dt'\, C_3(t_1, t_2, t')$$
$$\varphi_{c_+}(x + t_1 e_-)\varphi_{c_+}(x + t_2 e_-)\varphi_{c_-}(x + t' e_+) + (+ \leftrightarrow -)$$

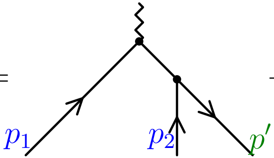
J_3: tree level

$$J_3(x) = \frac{1}{2}\int dt_1\, dt_2\, dt'\, C_3(t_1, t_2, t')$$
$$\varphi_{c_+}(x + t_1 e_-)\varphi_{c_+}(x + t_2 e_-)\varphi_{c_-}(x + t' e_+) + (+ \leftrightarrow -)$$



J_3: tree level

On-shell $p_1 = (0, p_{1-}, \vec{0})$, $p_2 = (0, p_{2-}, \vec{0})$, $p' = (p'_+, 0, \vec{0})$

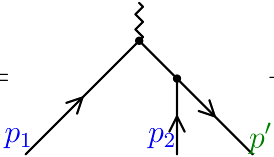
$\tilde{C}_3(p_{1-}, p_{2-}, p'_+) =$ 

$+ (1 \leftrightarrow 2)$

J_3: tree level

On-shell $p_1 = (0, p_{1-}, \vec{0})$, $p_2 = (0, p_{2-}, \vec{0})$, $p' = (p'_+, 0, \vec{0})$

$$\tilde{C}_3(p_{1-}, p_{2-}, p'_+) =$$



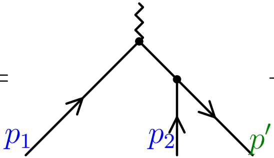
$$+ (1 \leftrightarrow 2)$$

$$= \frac{g}{p_{2-}p'_+ - i0} + (1 \leftrightarrow 2)$$

J_3: tree level

On-shell $p_1 = (0, p_{1-}, \vec{0})$, $p_2 = (0, p_{2-}, \vec{0})$, $p' = (p'_+, 0, \vec{0})$

$$\tilde{C}_3(p_{1-}, p_{2-}, p'_+) =$$



$$+ (1 \leftrightarrow 2)$$

$$= \frac{g}{p_{2-}p'_+ - i0} + (1 \leftrightarrow 2)$$

$$C_3(t_1, t_2, t') = \int \frac{dp_{1-}}{2\pi} \frac{dp_{2-}}{2\pi} \frac{dp'_+}{2\pi} e^{-ip_{1-}t_1 - ip_{2-}t2 + ip'_+t'}$$

$$\times \tilde{C}_3(p_{1-}, p_{2-}, p'_+) = g\delta(t_1)\theta(-t_2)\theta(t') + (1 \leftrightarrow 2)$$

J_3: coordinate space

$$J_3(x) = g\,\varphi_{c_+}(x) \int_{-\infty}^{0} dt\,\varphi_{c_+}(x + te_-) \int_0^{\infty} dt'\,\varphi_{c_-}(x + t'e_+)$$

$$= g\,\varphi_{c_+}(x)\,\frac{1}{i\partial_- - i0}\varphi_{c_+}(x)\,\frac{1}{-i\partial_+ - i0}\varphi_{c_-}(x)$$

Incoming $p_- \to i\partial_-$, outgoing $p'_+ \to -i\partial_+$

J_3: coordinate space

$$J_3(x) = g\,\varphi_{c_+}(x)\int_{-\infty}^{0} dt\,\varphi_{c_+}(x+te_-)\int_0^\infty dt'\,\varphi_{c_-}(x+t'e_+)$$

$$= g\,\varphi_{c_+}(x)\,\frac{1}{i\partial_- - i0}\varphi_{c_+}(x)\,\frac{1}{-i\partial_+ - i0}\varphi_{c_-}(x)$$

Incoming $p_- \to i\partial_-$, outgoing $p'_+ \to -i\partial_+$

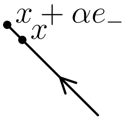
$$\int_{-\infty}^{0} dt\,\varphi(x+te_-) = \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot x}\tilde{\varphi(k)}\int_{-\infty}^{0} dt\,e^{-ik_-t+0t}$$

$$= \int \frac{d^dk}{(2\pi)^d}\frac{i}{k_- + i0}e^{-ik\cdot x}\tilde{\varphi(k)} = \frac{i}{i\partial_- + i0}\int \frac{d^dk}{(2\pi)^d}e^{-ik\cdot x}\tilde{\varphi(k)}$$

$$= \frac{i}{i\partial_- + i0}\varphi(x)$$

$$\int_0^\infty dt\,\varphi(x+te_+) = \frac{-i}{i\partial_+ - i0}\varphi(x)$$

J_3: coordinate space



$$F(x) = \int_{-\infty}^{0} dt\, \varphi(x + te_-)$$

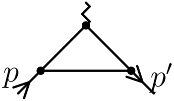
$$F(x + \alpha e_-) = F(x) + \alpha \partial_- F(x)$$

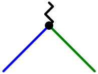
$$= \int_{-\infty}^{\alpha} dt\, \varphi(x + te_-) = F(x) + \alpha \varphi(x)$$

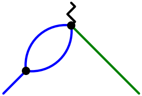
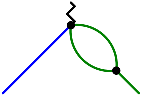
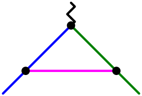
$$\partial_- F(x) = \varphi(x)$$

Form factor

Off-shell $p = (p_+, p_-, \vec{0})$, $p' = (p'_+, p'_-, \vec{0})$,

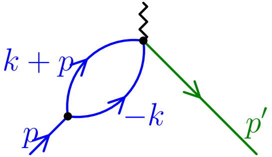


$= \tilde{C}_2^{(1)}$ 

$+ \tilde{C}_3^{(0)}$  $+ \tilde{C}_3^{(0)}$  $+$ 

$$\tilde{C}_2^{(1)} = \frac{g^2}{(4\pi)^{d/2}} I_h$$

Form factor



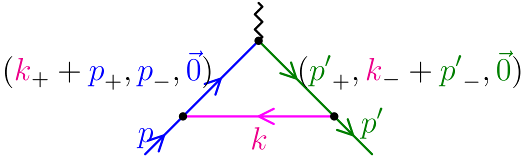
$$\frac{1}{2}\frac{g^2}{(4\pi)^{d/2}}\int\frac{d^d k}{i\pi^{d/2}}\frac{\tilde{C}_3(k_- + p_-, -k_-, p'_+)}{(-k^2 - i0)(-(k+p)^2 - i0)}$$

$$= \frac{g^2}{(4\pi)^{d/2}}\int\frac{d^d k}{i\pi^{d/2}}\frac{1}{(-k^2 - i0)(-(k+p)^2 - i0)(-p'_+ k_- - i0)}$$

$$= \frac{g^2}{(4\pi)^{d/2}} I_{c_+}$$

Form factor

$$p = (p_+, p_-, \vec{0}),\ p' = (p'_+, p'_-, \vec{0}),\ k = (k_+, k_-, \vec{k}_\perp)$$



Momentum conservation up to $\mathcal{O}(\lambda)$

$$\frac{g^2}{(4\pi)^{d/2}} \int \frac{d^d k}{i\pi^{d/2}}$$

$$\frac{1}{(-k^2 - i0)(-p_-(k_+ + p_+) - i0)(-p'_+(k_- + p'_-) - i0)}$$

$$= \frac{g^2}{(4\pi)^{d/2}} I_s$$

QCD

$k_c \sim (\lambda^2, 1, \lambda)Q$, $k_s \sim (\lambda^2, \lambda^2, \lambda^2)Q$

$$\psi \to \psi_c + \psi_s \qquad A^\mu \to A_c^\mu + A_s^\mu$$

QCD

$k_c \sim (\lambda^2, 1, \lambda)Q$, $k_s \sim (\lambda^2, \lambda^2, \lambda^2)Q$

$$\psi \to \psi_c + \psi_s \qquad A^\mu \to A_c^\mu + A_s^\mu$$

Soft

$$<T\psi_s(x)\bar{\psi}_s(0)> = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i\not{k}}{k^2 + i0} \sim \lambda^6$$

$$<TA_s^\mu(x)A_s^\nu(0)> = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 + i0} \left[-g^{\mu\nu} + \xi \frac{k^\mu k^\nu}{k^2}\right] \sim \lambda^4$$

$$\psi_s \sim \lambda^3 \qquad A_s^\mu \sim \lambda^2 \qquad iD_s^\mu = i\partial_s^\mu + gA_s^\mu \sim \lambda^2$$

Collinear fields

$$\psi_c = \xi + \eta \qquad \xi = \frac{\gamma_+\gamma_-}{4}\psi_c \qquad \eta = \frac{\gamma_-\gamma_+}{4}\psi_c$$

$$<T\xi(x)\xi(0)> = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 + i0} \frac{\gamma_+\gamma_-}{4} \underbrace{\not{k}}_{\frac{1}{2}k_-\gamma_+} \frac{\gamma_-\gamma_+}{4} \sim \lambda^2$$

$$<T\eta(x)\eta(0)> = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 + i0} \frac{\gamma_-\gamma_+}{4} \underbrace{\not{k}}_{\frac{1}{2}k_+\gamma_-} \frac{\gamma_+\gamma_-}{4} \sim \lambda^4$$

$$\xi \sim \lambda \qquad \eta \sim \lambda^2$$

Collinear fields

$$\psi_c = \xi + \eta \qquad \xi = \frac{\gamma_+\gamma_-}{4}\psi_c \qquad \eta = \frac{\gamma_-\gamma_+}{4}\psi_c$$

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$$<T\eta(x)\eta(0)> = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2+i0} \frac{\gamma_-\gamma_+}{4} \underbrace{\not{k}}_{\frac{1}{2}k_+\gamma_-} \frac{\gamma_+\gamma_-}{4} \sim \lambda^4$$

$$\xi \sim \lambda \qquad \eta \sim \lambda^2$$

$$<TA_c^\mu(x)A_c^\nu(0)> = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2+i0} \left[-g^{\mu\nu} + \xi\frac{k^\mu k^\nu}{k^2}\right]$$

$$<A_{c+}A_{c+}> \sim \lambda^4 \qquad <A_{c-}A_{c-}> \sim 1 \qquad <A_{c\perp}A_{c\perp}> \sim \lambda^2$$

$$A_c \sim (\lambda^2, 1, \lambda) \qquad iD_c^\mu = i\partial_c^\mu + gA_c^\mu \sim (\lambda^2, 1, \lambda)$$

Soft Lagrangian

$$L_s = \bar{\psi}_s i \not{D}_s \psi_s - \frac{1}{4} F_s^{a\mu\nu} F_{s\mu\nu}^a \qquad igF_s^{\mu\nu} = [iD_s^\mu, iD_s^\nu]$$

Collinear quark Lagrangian

$$L = i(\bar{\xi} + \bar{\eta}) \left(\frac{1}{2} D_+ \gamma_- + \frac{1}{2} D_- \gamma_+ + \not{D}_\perp \right) (\xi + \eta)$$
$$= \frac{i}{2} \bar{\xi} D_+ \gamma_- \xi + \frac{i}{2} \bar{\eta} D_- \gamma_+ \eta + i\bar{\xi} \not{D}_\perp \eta + i\bar{\eta} \not{D}_\perp \xi$$

Collinear quark Lagrangian

$$L = i(\bar{\xi} + \bar{\eta}) \left(\frac{1}{2} D_+ \gamma_- + \frac{1}{2} D_- \gamma_+ + \not{D}_\perp \right) (\xi + \eta)$$
$$= \frac{i}{2} \bar{\xi} D_+ \gamma_- \xi + \frac{i}{2} \bar{\eta} D_- \gamma_+ \eta + i \bar{\xi} \not{D}_\perp \eta + i \bar{\eta} \not{D}_\perp \xi$$

Equations of motion

$$\not{D} \psi_c = \left(\frac{1}{2} D_+ \gamma_- + \frac{1}{2} D_- \gamma_+ + \not{D}_\perp \right) (\xi + \eta) = 0$$
$$\gamma_+ \not{D} \psi_c = 2 D_+ \xi + \not{D}_\perp \eta = 0$$
$$\gamma_- \not{D} \psi_c = 2 D_- \eta + \not{D}_\perp \xi = 0$$

Collinear quark Lagrangian

$$L = i(\bar{\xi} + \bar{\eta}) \left(\frac{1}{2} D_+ \gamma_- + \frac{1}{2} D_- \gamma_+ + \not{D}_\perp \right) (\xi + \eta)$$
$$= \frac{i}{2} \bar{\xi} D_+ \gamma_- \xi + \frac{i}{2} \bar{\eta} D_- \gamma_+ \eta + i \bar{\xi} \not{D}_\perp \eta + i \bar{\eta} \not{D}_\perp \xi$$

Equations of motion

$$\not{D} \psi_c = \left(\frac{1}{2} D_+ \gamma_- + \frac{1}{2} D_- \gamma_+ + \not{D}_\perp \right) (\xi + \eta) = 0$$
$$\gamma_+ \not{D} \psi_c = 2 D_+ \xi + \not{D}_\perp \eta = 0$$
$$\gamma_- \not{D} \psi_c = 2 D_- \eta + \not{D}_\perp \xi = 0$$
$$\eta = -\frac{1}{2} \gamma_- \frac{1}{i D_- + i0} i \not{D}_\perp \xi \qquad \bar{\eta} = -\frac{1}{2} \bar{\xi} i \overleftarrow{\not{D}}_\perp \frac{1}{i \overleftarrow{D}_- + i0} \gamma_-$$

Collinear quark Lagrangian

Substituting η via ξ = integrating out η

$$L = \frac{i}{2}\bar{\xi} D_+ \gamma_- \xi - \frac{1}{2}\bar{\xi} i \not{D}_\perp \frac{\gamma_-}{iD_- + i0} i\not{D}_\perp \xi - \frac{1}{2}\bar{\xi} i \overleftarrow{\not{D}}_\perp \frac{\gamma_-}{i\overleftarrow{D}_- + i0} i\not{D}_\perp \xi$$

$$+ \frac{1}{8}\bar{\xi} i \overleftarrow{\not{D}}_\perp \frac{\gamma_-}{i\overleftarrow{D}_- + i0} iD_- \gamma_+ \frac{\gamma_-}{D_-} \not{D}_\perp \xi$$

$$= \frac{1}{2}\bar{\xi} iD_+ \gamma_- \xi + \frac{1}{2}\bar{\xi} i\not{D}_\perp \frac{1}{iD_- + i0} i\not{D}_\perp \gamma_- \xi$$

Leading-order Lagrangian

$$iD_- \to iD_{c-} = i\partial_- + gA_{c-},\ iD_\perp \to iD_{c\perp} = i\partial_\perp + gA_{c\perp}$$

$$\begin{aligned} L &= \bar{\psi}_s i\not{D}_s \psi_s - \frac{1}{4} F_s^{a\mu\nu} F^a_{s\mu\nu} \\ &+ \frac{1}{2}\bar{\xi}\left[i \underbrace{D_+}_{i\partial_+ + gA_{c+} + gA_{s+}(\bar{x}_-)} + i\not{D}_{c\perp} \frac{1}{iD_{c-} + i0} i\not{D}_{c\perp} \right] \gamma_- \xi \\ &- \frac{1}{4} F_c^{a\mu\nu} F^a_{c\mu\nu} \end{aligned}$$

$$\bar{x}_- \equiv \frac{1}{2} x_- e_+^\mu$$

Leading-order Lagrangian

$$iD_- \to iD_{c-} = i\partial_- + gA_{c-},\; iD_\perp \to iD_{c\perp} = i\partial_\perp + gA_{c\perp}$$

$$L = \bar{\psi}_s i\not{D}_s \psi_s - \frac{1}{4} F_s^{a\mu\nu} F^a_{s\mu\nu}$$
$$+ \frac{1}{2}\bar{\xi}\left[i \underbrace{D_+}_{i\partial_+ + gA_{c+} + gA_{s+}(\bar{x}_-)} + i\not{D}_{c\perp} \frac{1}{iD_{c-} + i0} i\not{D}_{c\perp} \right] \gamma_- \xi$$
$$- \frac{1}{4} F_c^{a\mu\nu} F^a_{c\mu\nu} \qquad \bar{x}_- \equiv \frac{1}{2} x_- e_+^\mu$$
$$igF_c^{\mu\nu} = [iD^\mu, iD^\nu]$$
$$iD^\mu = \frac{1}{2} iD_+ e_-^\mu + \frac{1}{2} iD_{c-} e_+^\mu + iD_{c\perp}^\mu$$

Soft gauge transformations

$$U_s(x) = e^{if_s^a(x)t^a} \qquad x \sim (\lambda^{-2}, \lambda^{-2}, \lambda^{-2}) \qquad \partial \sim (\lambda^2, \lambda^2, \lambda^2)$$

$$\psi_s(x) \to U_s(x)\psi_s(x)$$

$$A_s^\mu(x) \to U_s(x)A_s^\mu(x)U_s^+(x) - \frac{i}{g}\big(\partial^\mu U_s(x)\big)U_s^{-1}(x)$$

Soft gauge transformations

$$U_s(x) = e^{if_s^a(x)t^a} \qquad x \sim (\lambda^{-2}, \lambda^{-2}, \lambda^{-2}) \qquad \partial \sim (\lambda^2, \lambda^2, \lambda^2)$$

$$\psi_s(x) \to U_s(x)\psi_s(x)$$

$$A_s^\mu(x) \to U_s(x)A_s^\mu(x)U_s^+(x) - \frac{i}{g}\big(\partial^\mu U_s(x)\big)U_s^{-1}(x)$$

$$\xi(x) \to U_s(\bar{x}_-)\xi(x)$$

$$A_c^\mu(x) \to U_s(\bar{x}_-)A_c^\mu(x)U_s^{-1}(\bar{x}_-)$$

Soft gauge transformations

$$U_s(x) = U_s(\bar{x}_-) - \underbrace{\vec{x}_\perp \cdot \vec{\partial}_\perp U_s(\bar{x}_-)}_{\mathcal{O}(\lambda)} + \mathcal{O}(\lambda^2)$$

Soft gauge transformations

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$\left(\partial^\mu U_s\right) U_s^{-1} \sim \lambda^2$: not needed for A_{c-} and $A_{c\perp}$
A_{c+} only appears in D_+

$$\begin{aligned} & A_{c+}(x) + A_{s+}(\bar{x}_-) \to \\ & U_s(\bar{x}_-)\left[A_{c+}(x) + A_{s+}(\bar{x}_-)\right] U_s^{-1}(\bar{x}_-) \\ & + \frac{i}{g} U_s(\bar{x}_-)\left(\partial_+ U_s^{-1}(\bar{x}_-)\right) \end{aligned}$$

Soft gauge transformations

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Collinear gauge transformations

$$U_c(x) = e^{if_c^a(x)t^a} \qquad x \sim (1, \lambda^{-2}, \lambda^{-1}) \qquad \partial \sim (\lambda^2, 1, \lambda)$$

$$\psi_s(x) \to \psi_s(x) \qquad A_s^\mu(x) \to A_s^\mu(x)$$

Collinear gauge transformations

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$$\psi_s(x) \to \psi_s(x) \qquad A_s^\mu(x) \to A_s^\mu(x)$$

$$\xi(x) \to U_c(x)\xi(x)$$

$$A_c^\mu(x) \to U_c(x) A_c^\mu(x) U_c^{-1}(x)$$

$$+ \frac{1}{g} U_c(x) \left[i\partial^\mu + \tfrac{1}{2} g A_{s+}(\bar{x}_-) e_-^\mu, U_c^{-1}(x) \right]$$

Collinear gauge transformations

$$A_{c\perp}(x) \to U_c(x) A_{c\perp}(x) U_c^{-1}(x) + \frac{i}{g} U_c(x) \left(\partial_\perp U_c^{-1}(x) \right)$$

$$A_{c-}(x) \to U_c(x) A_{c-}(x) U_c^{-1}(x) + \frac{i}{g} U_c(x) \left(\partial_- U_c^{-1}(x) \right)$$

$$A_{c+}(x) \to U_c(x) A_{c+}(x) U_c^{-1}(x) + \frac{i}{g} U_c(x) \left[D_{s+}(\bar{x}_-), U_c^{-1}(x) \right]$$

Collinear gauge transformations

$$A_{c\perp}(x) \to U_c(x) A_{c\perp}(x) U_c^{-1}(x) + \frac{i}{g} U_c(x) \left(\partial_\perp U_c^{-1}(x)\right)$$

$$A_{c-}(x) \to U_c(x) A_{c-}(x) U_c^{-1}(x) + \frac{i}{g} U_c(x) \left(\partial_- U_c^{-1}(x)\right)$$

$$A_{c+}(x) \to U_c(x) A_{c+}(x) U_c^{-1}(x) + \frac{i}{g} U_c(x) \left[D_{s+}(\bar{x}_-), U_c^{-1}(x)\right]$$

$$iD_+ \to U_c(x) iD_+ U_c^{-1}(x)$$

Reparametrization invariance

$$e_+ \to (1+\varepsilon)e_+ \qquad e_- \to (1-\varepsilon)e_-$$
$$k_+ \to (1+\varepsilon)k_+ \qquad k_- \to (1-\varepsilon)k_-$$

Lorentz boost

Reparametrization invariance

$$e_+ \to (1+\varepsilon)e_+ \qquad e_- \to (1-\varepsilon)e_-$$
$$k_+ \to (1+\varepsilon)k_+ \qquad k_- \to (1-\varepsilon)k_-$$

Lorentz boost

$$e_+ \to e_+ \qquad e_- \to e_- + \varepsilon_\perp$$
$$k_- \to \underbrace{k_-}_{\mathcal{O}(1)} + \underbrace{k\cdot\varepsilon_\perp}_{\mathcal{O}(\lambda)}$$
$$k_\perp = k - \frac{1}{2}(k\cdot e_+)e_- - \frac{1}{2}(k\cdot e_-)e_+$$
$$\to \underbrace{k_\perp}_{\mathcal{O}(\lambda)} - \underbrace{\frac{1}{2}k_+\varepsilon_\perp}_{\mathcal{O}(\lambda^2)} - \underbrace{\frac{1}{2}(k\cdot\varepsilon_\perp)e_+}_{\mathcal{O}(\lambda)}$$

Reparametrization invariance

$$e_+ \to e_+ + \lambda \varepsilon_\perp \qquad e_- \to e_-$$

$$k_+ \to \underbrace{k_+}_{\mathcal{O}(\lambda^2)} + \underbrace{\lambda k \cdot \varepsilon_\perp}_{\mathcal{O}(\lambda^2)}$$

$$k_\perp \to \underbrace{k_\perp}_{\mathcal{O}(\lambda)} - \underbrace{\frac{\lambda}{2}(k \cdot \varepsilon_\perp) e_-}_{\mathcal{O}(\lambda^2)} - \underbrace{\frac{\lambda}{2} k_- \varepsilon_\perp}_{\mathcal{O}(\lambda)}$$

Mixes defferent orders in λ

Wilson lines

$$[x', x] = P \exp ig \int_x^{x'} dy_\mu \, A^\mu(y)$$

Wilson lines

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Gauge transformation

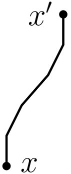
$$1 + igA^\mu(x)\, dx_\mu \to$$

$$1 + ig \left[U(x)A^\mu(x)U^{-1}(x) - \frac{i}{g} \left(\partial^\mu U(x)\right) U^{-1}(x) \right] dx_\mu$$

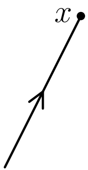
$$= U(x + dx)U^{-1}(x) + igU(x)A^\mu(x)u^{-1}(x)\, dx_\mu$$

$$= U(x + dx) \left[1 + igA^\mu(x)\, dx_\mu\right] U^{-1}(x)$$

$[x', x] \to U(x')[x', x]U^{-1}(x)$ $\qquad$ $\bar{\psi}(x')[x', x]\psi(x) =$ gauge invariant



Wilson lines

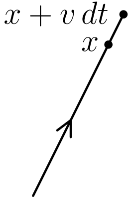


$$W(x) = [x, x - \infty v] = P \exp ig \int_{-\infty}^{0} dt\, A^{\mu}(x + vt)$$

$$U(x - \infty v) = 1 \qquad W(x) \to U(x)W(x)$$

$$W^{-1}(x)\psi(x),\ \bar{\psi}(x)W(x) = \text{gauge invariant}$$

Wilson lines



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$$W^{-1}(x)\psi(x),\ \bar\psi(x)W(x) = \text{gauge invariant}$$

$$W(x + v\,dt) = [1 + igv \cdot A(x)\,dt]\, W(x)$$

$$v \cdot \partial W(x) = igv \cdot A(x)\, W(x) \qquad \color{red}{v \cdot DW(x) = 0}$$

Collinear–soft interaction: quarks

$$L_q = \frac{1}{2}\bar{\xi}(x) iD_+ \gamma_- \xi(x)$$

$$iD_+ = i\partial_+ + gA_{c+}(x) + gA_{s+}(\bar{x}_-) = iD_{c+} + gA_{s+}(\bar{x}_-)$$

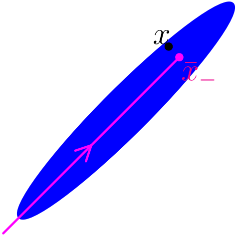
Collinear–soft interaction: quarks

$$L_q = \frac{1}{2}\bar{\xi}(x) iD_+ \gamma_- \xi(x)$$

$$iD_+ = i\partial_+ + gA_{c+}(x) + gA_{s+}(\bar{x}_-) = iD_{c+} + gA_{s+}(\bar{x}_-)$$

$$S(x) = P \exp ig \int_{-\infty}^{0} dt\, A_{s+}(x + e_+ t) \qquad D_{s+}S(x) = 0$$

$$\xi(x) = S(\bar{x}_-)\xi_0(x) \qquad A_c^\mu(x) = S(\bar{x}_-) A_{c0}^\mu(x) S^{-1}(\bar{x}_-)$$



Collinear–soft interaction: quarks

$$
\begin{aligned}
& iD_+\xi(x) \\
& = \Big[S(\bar{x}_-)i\partial_+ + \Big(\big(i\partial_+ + gA_s(\bar{x}_-)\big)S(\bar{x}_-)\Big) + gS(\bar{x}_-)A_{c0+}(x)\Big]\xi_0(x) \\
& = S(\bar{x}_-)iD_{c0+}\xi_0(x)
\end{aligned}
$$

$$
L_q = \frac{1}{2}\bar{\xi}_0(x)iD_{c0+}\xi_0(x)
$$

Collinear–soft interaction: gluons

$$L_g = -\frac{1}{2}\,\mathrm{Tr}\,F_{c\mu\nu}F_c^{\mu\nu} \qquad igF_c^{\mu\nu} = [iD^\mu, iD^\nu]$$

$$iD^\mu = \frac{1}{2}iD_+e_-^\mu + \frac{1}{2}iD_{c-}e_+^\mu + iD_{c\perp}^\mu$$

$$iD_+ = i\partial_+ + gA_{c+}(x) + gA_{s+}(\bar{x}_-)$$

$$A_c(x) = S(\bar{x}_-)A_{c0}(x)S^{-1}(\bar{x}_-)$$

Collinear–soft interaction: gluons

$$L_g = -\frac{1}{2}\,\mathrm{Tr}\,F_{c\mu\nu}F_c^{\mu\nu} \qquad igF_c^{\mu\nu} = [iD^\mu, iD^\nu]$$

$$iD^\mu = \frac{1}{2}iD_+e_-^\mu + \frac{1}{2}iD_{c-}e_+^\mu + iD_{c\perp}^\mu$$

$$iD_+ = i\partial_+ + gA_{c+}(x) + gA_{s+}(\bar{x}_-)$$

$$A_c(x) = S(\bar{x}_-)A_{c0}(x)S^{-1}(\bar{x}_-)$$

$$iD_{c-} = i\partial_- + gS(\bar{x}_-)A_{c0-}(x)S^{-1}(\bar{x}_-) = S(\bar{x}_-)iD_{c0-}S^{-1}(\bar{x}_-)$$

because $\partial_- S(\bar{x}_-) = 2\frac{\partial S(\bar{x}_-)}{\partial x_+} = 0$

$$iD_{c\perp} = i\partial_\perp + gS(\bar{x}_-)A_{c0\perp}(x)S^{-1}(\bar{x}_-)$$

Collinear–soft interaction: gluons

$$
\begin{aligned}
iD_+ &= S(\bar{x}_-) i\partial_+ S^{-1}(\bar{x}_-) - S(\bar{x}_-)\big(i\partial_+ S^{-1}(\bar{x}_-)\big) \\
&+ gA_{s+}(\bar{x}_-) + gS(\bar{x}_-) A_{c0+}(x) S^{-1}(\bar{x}_-) \\
&= S(\bar{x}_-) iD_{c0+} S^{-1}(\bar{x}_-)
\end{aligned}
$$

because

$S(\bar{x}_-)\big(i\partial_+ S^{-1}(\bar{x}_-)\big) = -\big(i\partial_+ S(\bar{x}_-)\big) S^{-1}(\bar{x}_-) = gA_{s+}(\bar{x}_-)$,

$iD_{s+} S = (i\partial_+ + gA_{s+}) S = 0$

Collinear–soft interaction: gluons

$$\begin{aligned} iD_+ &= S(\bar{x}_-) i\partial_+ S^{-1}(\bar{x}_-) - S(\bar{x}_-)\big(i\partial_+ S^{-1}(\bar{x}_-)\big) \\ &+ gA_{s+}(\bar{x}_-) + gS(\bar{x}_-)A_{c0+}(x)S^{-1}(\bar{x}_-) \\ &= S(\bar{x}_-) iD_{c0+} S^{-1}(\bar{x}_-) \end{aligned}$$

because

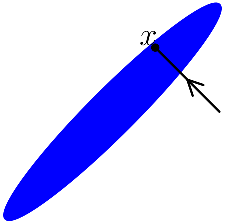
$S(\bar{x}_-)\big(i\partial_+ S^{-1}(\bar{x}_-)\big) = -\big(i\partial_+ S(\bar{x}_-)\big)S^{-1}(\bar{x}_-) = gA_{s+}(\bar{x}_-)$,

$iD_{s+}S = (i\partial_+ + gA_{s+})S = 0$

$$F_c^{\mu\nu} = S(\bar{x}_-)F_{c0}^{\mu\nu}S^{-1}(\bar{x}_-) \qquad L_g = -\frac{1}{2}\,\mathrm{Tr}\, F_{c0\mu\nu}F_{c0}^{\mu\nu}$$

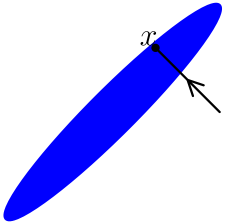
Collinear fields

$$W(x) = P \exp ig \int_{-\infty}^{0} dt\, A_{c-}(x + te_-)$$



Collinear fields

$$W(x) = P \exp ig \int_{-\infty}^{0} dt\, A_{c-}(x + te_-)$$



$$\xi(x) = W(x)\chi(x) \qquad \mathcal{A}^\mu(x) = W^{-1}(x)\big(iD_c^\mu W(x)\big)$$

$$\mathcal{A}_- = 0 \qquad \mathcal{A}_+ \sim \lambda^2 \qquad \mathcal{A}_\perp \sim \lambda$$

$$i\mathcal{D}^\mu = W^{-1}(x) iD_c^\mu W(x) = i\partial^\mu + \mathcal{A}^\mu$$

Collinear quark Lagrangian

$$L_q = \frac{1}{2}\bar{\xi} i D_{c+}\gamma_- \xi + \frac{1}{2}\bar{\xi} i \not{D}_{c\perp} \frac{1}{iD_{c-} + i0} i\not{D}_{c\perp}\gamma_- \xi$$

$$= \frac{1}{2}\bar{\chi} i \mathcal{D}_{c+}\gamma_- \chi + \frac{1}{2}\bar{\chi} i \not{\mathcal{D}}_{c\perp} \frac{1}{i\partial_- + i0} i\not{\mathcal{D}}_{c\perp}\gamma_- \chi$$

$$= \frac{1}{2}\bar{\chi} i \mathcal{D}_{c+}\gamma_- \chi - \frac{i}{2}\bar{\chi}\gamma_- i\not{\mathcal{D}}_{c\perp} \int_{-\infty}^{0} dt\ (i\not{\mathcal{D}}_{c\perp}\chi)_{x+te_-}$$

Collinear quark Lagrangian

$$L_q = \frac{1}{2}\bar{\xi} iD_{c+}\gamma_-\xi + \frac{1}{2}\bar{\xi} i\not{D}_{c\perp}\frac{1}{iD_{c-}+i0} i\not{D}_{c\perp}\gamma_-\xi$$
$$= \frac{1}{2}\bar{\chi} i\mathcal{D}_{c+}\gamma_-\chi + \frac{1}{2}\bar{\chi} i\not{\mathcal{D}}_{c\perp}\frac{1}{i\partial_- + i0} i\not{\mathcal{D}}_{c\perp}\gamma_-\chi$$
$$= \frac{1}{2}\bar{\chi} i\mathcal{D}_{c+}\gamma_-\chi - \frac{i}{2}\bar{\chi}\gamma_- i\not{\mathcal{D}}_{c\perp}\int_{-\infty}^{0} dt\ (i\not{\mathcal{D}}_{c\perp}\chi)_{x+te_-}$$

Nonlocality in e_- direction $\sim Q^{-1}$

$$W^{-1}(x) iD_{c-} W(x) = i\partial_-$$
$$\Rightarrow W^{-1}(x)\frac{1}{iD_{c-}+i0}W(x) = \frac{1}{i\partial_- + i0}$$

Collinear gluon Lagrangian

$$W^{-1} F_{c\mu\nu} W = \mathcal{F}_{\mu\nu} = \frac{1}{g} \left(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu] \right)$$

$$L_g = -\frac{1}{2} \operatorname{Tr} F_{c\mu\nu} F_c^{\mu\nu} = -\frac{1}{2g^2} \operatorname{Tr} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}$$

Quark vector current

$$J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$$

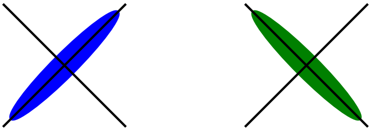
Kinematics: $c_- \to c_+$

$\psi(x) \to S_-(\bar{x}_+)\chi_-(x)$ $(\bar{x}_+ = \frac{1}{2}x_+e_-,\ \gamma_-\chi_- = 0)$

$\bar{\psi}(x) \to S_+^\dagger(\bar{x}_-)\bar{\chi}_+(x)$ $(\bar{x}_- = \frac{1}{2}x_-e_+,\ \gamma_+\chi_+ = 0)$

$$J^\mu(x) = \int dt\, dt'\, C_V(t,t')\bar{\chi}_+(x+te_-)S_+^\dagger(\bar{x}_-)S_-(\bar{x}_+) \gamma_\perp^\mu\chi_-(x+t'e_+) + \cdots$$

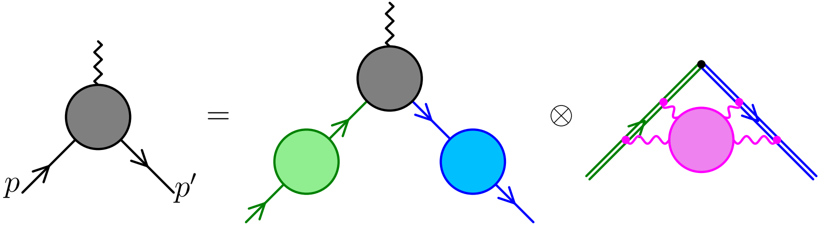
Quark vector current



$$x^\mu \sim (1, 1, \lambda^{-1})Q^{-1}$$

$$J^\mu(x) = \int dt\, dt'\, C_V(t, t')\bar{\chi}_+(\bar{x}_+ + x_\perp + te_-)S_+^\dagger(0)S_-(0)$$
$$\gamma_\perp^\mu \chi_-(\bar{x}_- + x_\perp + t'e_+) + \cdots$$

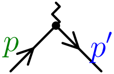
Factorization

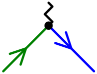


$$F(-q^2, -p^2, -p'^2) = \tilde{C}_V(-q^2) J_-(-p^2) J_+(-p'^2) S\left(\frac{(-p^2)(-p'^2)}{-q^2}\right)$$

Matching coefficient

On-shell $p = (p_+, 0, \vec{0})$, $p' = (0, p'_-, \vec{0})$
Tree level



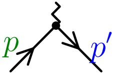
$= \tilde{C}_V(p_+, p'_-)$


$$\tilde{C}_V^{(0)}(p_+, p'_-) = 1 \qquad C_V^{(0)}(t, t') = \delta(t)\delta(t')$$

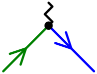
Matching coefficient

On-shell $p = (p_+, 0, \vec{0})$, $p' = (0, p'_-, \vec{0})$

Tree level

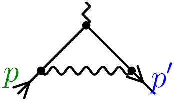


$$= \tilde{C}_V(p_+, p'_-)$$

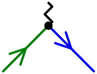


$$\tilde{C}_V^{(0)}(p_+, p'_-) = 1 \qquad C_V^{(0)}(t, t') = \delta(t)\delta(t')$$

1 loop



$$= \tilde{C}_V^{(1)}(p_+, p'_-)$$



SCET loops vanish

Matching coefficient: 1 loop

$$-C_F \frac{g_0^2}{(4\pi)^{d/2}} \int \frac{d^d k}{i\pi^{d/2}} \frac{\gamma^\nu (\not{k} + \not{p}') \gamma_\perp^\mu (\not{k} + \not{p}) \gamma_\nu}{(-k^2 - i0)[-(k+p)^2 - i0][-(k+p')^2 - i0]}$$

Matching coefficient: 1 loop

$$-C_F \frac{g_0^2}{(4\pi)^{d/2}} \int \frac{d^d k}{i\pi^{d/2}} \frac{\gamma^\nu (\not{k} + \not{p}') \gamma_\perp^\mu (\not{k} + \not{p}) \gamma_\nu}{(-k^2 - i0)[-(k+p)^2 - i0][-(k+p')^2 - i0]}$$

Numerator

$$\begin{aligned}
&\gamma^\nu \not{k} \gamma_\perp^\mu \not{k} \gamma_\nu + 2\gamma_\perp^\mu \not{k} \not{p}' + 2\not{p} \not{k} \gamma_\perp^\mu + 2p_+ p'_- \gamma_\perp^\mu \\
&= -(d-2) \not{k} \gamma_\perp^\mu \not{k} + 2p'_- k_+ \gamma_\perp^\mu + 2p_+ k_- \gamma_\perp^\mu + 2p_+ p'_- \gamma_\perp^\mu \\
&= \left[(d-2) k^2 - 2k_\perp^2 + 2p_+ k_- + 2p'_- k_+ + 2p_+ p'_- \right] \gamma_\perp^\mu
\end{aligned}$$

Matching coefficient: 1 loop

$$-C_F \frac{g_0^2}{(4\pi)^{d/2}} \int \frac{d^d k}{i\pi^{d/2}} \frac{\gamma^\nu (\not{k} + \not{p}') \gamma_\perp^\mu (\not{k} + \not{p}) \gamma_\nu}{(-k^2 - i0)[-(k+p)^2 - i0][-(k+p')^2 - i0]}$$

Numerator

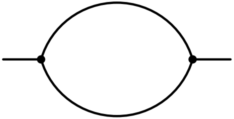
$$\begin{aligned}
&\gamma^\nu \not{k} \gamma_\perp^\mu \not{k} \gamma_\nu + 2\gamma_\perp^\mu \not{k} \not{p}' + 2\not{p} \not{k} \gamma_\perp^\mu + 2p_+ p'_- \gamma_\perp^\mu \\
&= -(d-2) \not{k} \gamma_\perp^\mu \not{k} + 2p'_- k_+ \gamma_\perp^\mu + 2p_+ k_- \gamma_\perp^\mu + 2p_+ p'_- \gamma_\perp^\mu \\
&= \left[(d-2)k^2 - 2k_\perp^2 + 2p_+ k_- + 2p'_- k_+ + 2p_+ p'_-\right] \gamma_\perp^\mu
\end{aligned}$$

$$C_V(-q^2) = -C_F \frac{g_0^2}{(4\pi)^{d/2}}$$

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{2(-q^2) - (d-8)(-k^2) + 2(-k^2)^2/(-q^2)}{(-k^2 - i0)(-k^2 - p_+ k_- - i0)(-k^2 - p'_- k_+ - i0)}$$

$$C_V(-q^2) = -C_F \frac{g_0^2}{(4\pi)^{d/2}} \Bigg[2(-q^2) I_h - (d-8) \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{[-(k+p)^2 - i0][-(k+p')^2 - i0]} + \frac{2}{-q^2} \int \frac{d^d k}{i\pi^{d/2}} \frac{-k^2}{[-(k+p)^2 - i0][-(k+p')^2 - i0]} \Bigg]$$

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{[-(k+p)^2 - i0][-(k+p')^2 - i0]}$$
$$= \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)[-(k+q)^2 - i0]}$$
$$= \frac{\Gamma(\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)} (-q^2)^{-\varepsilon}$$



$$\int \frac{d^dk}{i\pi^{d/2}} \frac{-k^2}{[-(k+p)^2 - i0][-(k+p')^2 - i0]}$$
$$= \int \frac{d^dk}{i\pi^{d/2}} \frac{-(k-p)^2}{(-k^2 - i0)[-(k+q)^2 - i0]}$$
$$= 2p \cdot \int \frac{d^dk}{i\pi^{d/2}} \frac{k}{(-k^2 - i0)[-(k+q)^2 - i0]}$$
$$= \frac{2p \cdot q}{q^2} \int \frac{d^dk}{i\pi^{d/2}} \frac{k \cdot q}{(-k^2 - i0)[-(k+q)^2 - i0]}$$
$$= -\frac{-q^2}{2} \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{(-k^2 - i0)[-(k+q)^2 - i0]}$$

Result

$$C_V(-q^2) = 1 - C_F \frac{g_0^2(-q^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left[\frac{2}{\varepsilon^2} + \frac{3+2\varepsilon}{\varepsilon(1-2\varepsilon)}\right]$$

$$= 1 - C_F \frac{g_0^2(-q^2)^{-\varepsilon}}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - \frac{\pi^2}{6} + 8\right)$$

Renormalized matching coefficient

$$\overline{\text{MS}} \qquad \frac{g_0^2}{(4\pi)^{d/2}} = \frac{\alpha_s(\mu)}{4\pi}\mu^{-2\varepsilon}e^{\gamma\varepsilon}$$

$$C_V(-q^2) = 1 - C_F\frac{\alpha_s(\mu)}{4\pi}\left(\frac{-q^2}{\mu^2}\right)^{-\varepsilon}\left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - \frac{\pi^2}{6} + 8\right)$$

Renormalized matching coefficient

$$\overline{\text{MS}} \qquad \frac{g_0^2}{(4\pi)^{d/2}} = \frac{\alpha_s(\mu)}{4\pi}\mu^{-2\varepsilon}e^{\gamma\varepsilon}$$

$$C_V(-q^2) = 1 - C_F\frac{\alpha_s(\mu)}{4\pi}\left(\frac{-q^2}{\mu^2}\right)^{-\varepsilon}\left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - \frac{\pi^2}{6} + 8\right)$$

Renormalization

$$C_V(-q^2) = Z^{-1}C_V(-q^2,\mu)$$

$$Z = 1 + C_F\left(\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon}\log\frac{-q^2}{\mu^2} + \frac{3}{\varepsilon}\right)\frac{\alpha_s(\mu)}{4\pi}$$

$$C_V(-q^2,\mu) = 1 + C_F\left(-\log^2\frac{-q^2}{\mu^2} + 3\log\frac{-q^2}{\mu^2} + \frac{\pi^2}{6} - 8\right)\frac{\alpha_s(\mu)}{4\pi}$$

RG equation

$$\frac{d\log C_V(-q^2)}{d\log\mu} = 0 = -\frac{d\log Z}{d\log\mu} + \frac{d\log C_V(-q^2,\mu)}{d\log\mu}$$

$$\frac{d\log \alpha_s(\mu)}{d\log\mu} = -2\varepsilon + \mathcal{O}(\alpha_s)$$

$$\frac{d\log Z}{d\log\mu} = \Gamma(\alpha_s(\mu))\log\frac{-q^2}{\mu^2} + \gamma_V(\alpha_s(\mu))$$

$$\frac{d\log C_V(-q^2,\mu)}{d\log\mu} = \Gamma(\alpha_s(\mu))\log\frac{-q^2}{\mu^2} + \gamma_V(\alpha_s(\mu))$$

$$\Gamma(\alpha_s) = 4C_F\frac{\alpha_s}{4\pi} \qquad \gamma_V(\alpha_s) = -6C_F\frac{\alpha_s}{4\pi}$$

Solution

$$C_V(-q^2,\mu) = U(\mu_0,\mu) C_V(-q^2,\mu_0)$$

$$\frac{d\log U(\mu_0,\mu)}{d\log\mu} = \Gamma(\alpha_s(\mu))\log\frac{-q^2}{\mu^2} + \gamma_V(\alpha_s(\mu))$$

$$\Gamma(\alpha_s) = \Gamma_0\frac{\alpha_s}{4\pi} + \Gamma_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$\gamma_V(\alpha_s) = \gamma_{V0}\frac{\alpha_s}{4\pi} + \gamma_{V1}\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

Solution

$$C_V(-q^2,\mu) = U(\mu_0,\mu) C_V(-q^2,\mu_0)$$

$$\frac{d\log U(\mu_0,\mu)}{d\log\mu} = \Gamma(\alpha_s(\mu))\log\frac{-q^2}{\mu^2} + \gamma_V(\alpha_s(\mu))$$

$$\Gamma(\alpha_s) = \Gamma_0\frac{\alpha_s}{4\pi} + \Gamma_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$\gamma_V(\alpha_s) = \gamma_{V0}\frac{\alpha_s}{4\pi} + \gamma_{V1}\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$\frac{d\log\alpha_s(\mu)}{d\log\mu} = -2\beta(\alpha_s(\mu)) \qquad \beta(\alpha_s) = \beta_0\frac{\alpha_s}{4\pi} + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

Solution

$$C_V(-q^2,\mu) = U(\mu_0,\mu)C_V(-q^2,\mu_0)$$
$$\frac{d\log U(\mu_0,\mu)}{d\log\mu} = \Gamma(\alpha_s(\mu))\log\frac{-q^2}{\mu^2} + \gamma_V(\alpha_s(\mu))$$
$$\Gamma(\alpha_s) = \Gamma_0\frac{\alpha_s}{4\pi} + \Gamma_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$
$$\gamma_V(\alpha_s) = \gamma_{V0}\frac{\alpha_s}{4\pi} + \gamma_{V1}\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$\frac{d\log\alpha_s(\mu)}{d\log\mu} = -2\beta(\alpha_s(\mu)) \qquad \beta(\alpha_s) = \beta_0\frac{\alpha_s}{4\pi} + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$\frac{d\log U(\mu_0,\mu)}{d\log\alpha_s} = -\frac{1}{2\beta(\alpha_s)}\left[\Gamma(\alpha_s)\left(\log\frac{-q^2}{\mu_0^2} - 2\log\frac{\mu}{\mu_0}\right) + \gamma_V(\alpha_s)\right]$$

$$\log\frac{\mu}{\mu_0} = -\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha_s}{\alpha_s}\frac{1}{2\beta(\alpha_s)}$$

Solution

$$U(\mu_0,\mu) = \exp\left[S(\mu_0,\mu) - A_{\gamma_V}(\mu_0,\mu)\right]\left(\frac{-q^2}{\mu_0^2}\right)^{-A_\Gamma(\mu_0,\mu)}$$

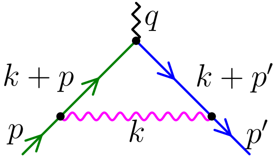
$$A_\gamma(\mu_0,\mu) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\alpha_s}\frac{\gamma(\alpha_s)}{2\beta(\alpha_s)} = \frac{\gamma_0}{2\beta_0}\log r + \mathcal{O}(\alpha_s)$$

$$S(\mu_0,\mu) = -\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\alpha_s}\frac{\Gamma(\alpha_s)}{2\beta(\alpha_s)}\int_{\alpha_s(\mu_0)}^{\alpha_s}\frac{d\alpha_s'}{\alpha_s'}\frac{1}{\beta(\alpha_s')}$$

$$= \frac{\Gamma_0}{2\beta_0^2}\left[\frac{4\pi}{\alpha_s(\mu_0)}\left(\frac{r-1}{r} - \log r\right) + \left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0}\right)(1 - r + \log r) + \frac{\beta_1}{2\beta_0}\log^2 r\right] + \mathcal{O}(\alpha_s)$$

$$r = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

Soft function



$$S_1 = C_F \int \frac{d^d k}{(2\pi)^d} i g_0 e_+^\mu \frac{i}{e_+ \cdot (k+p') + i0} \frac{i}{e_- \cdot (k+p) + i0} i g_0 e_{-\mu} \frac{-i}{k^2 + i0}$$

$$= -2 C_F (-q^2) I_s$$

Soft function

$$S(\Lambda_s^2) = 1 - 2C_F \frac{g_0^2(\Lambda_s^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(1-\varepsilon)\Gamma^2(\varepsilon)$$

$$= 1 - 2C_F \frac{\alpha_s(\mu)}{4\pi} \left(\frac{\Lambda_s^2}{\mu^2}\right)^{-\varepsilon} \left(\frac{1}{\varepsilon^2} + \frac{\pi^2}{4}\right)$$

$$= Z_S^{-1} S(\Lambda_s^2, \mu)$$

$$Z_S = 1 + 2C_F \left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \log \frac{\Lambda_s^2}{\mu^2}\right) \frac{\alpha_s(\mu)}{4\pi}$$

$$S(\Lambda_s^2, \mu) = 1 - C_F \left(\log^2 \frac{\Lambda_s^2}{\mu^2} + \frac{\pi^2}{2}\right) \frac{\alpha_s(\mu)}{4\pi}$$

$$\frac{d \log S(\Lambda_s^2, \mu)}{d \log \mu} = \Gamma(\alpha_s(\mu)) \log \frac{\Lambda_s^2}{\mu^2} + \gamma_S(\alpha_s(\mu))$$

$$\gamma_S(\alpha_s) = 0 + \mathcal{O}(\alpha_s^2)$$

Resummed form factor

$$F(-q^2, -p^2, -p'^2) = C_V(-q^2, \mu) J(-p^2, \mu) J(-p'^2, \mu) S(\Lambda_s^2, \mu)$$

$$\Lambda_s^2 = \frac{(-p^2)(-p'^2)}{-q^2}$$

$$\frac{d \log F(-q^2, -p^2, -p'^2)}{d \log \mu} = 0 = \frac{d \log C_V(-q^2, \mu)}{d \log \mu}$$

$$+ \frac{d \log J(-p^2, \mu)}{d \log \mu} + \frac{d \log J(-p'^2, \mu)}{d \log \mu} + \frac{d \log S(\Lambda_s^2, \mu)}{d \log \mu}$$

Resummed form factor

$$F(-q^2,-p^2,-p'^2) = C_V(-q^2,\mu)J(-p^2,\mu)J(-p'^2,\mu)S(\Lambda_s^2,\mu)$$

$$\Lambda_s^2 = \frac{(-p^2)(-p'^2)}{-q^2}$$

$$\frac{d\log F(-q^2,-p^2,-p'^2)}{d\log\mu} = 0 = \frac{d\log C_V(-q^2,\mu)}{d\log\mu}$$

$$+\frac{d\log J(-p^2,\mu)}{d\log\mu} + \frac{d\log J(-p'^2,\mu)}{d\log\mu} + \frac{d\log S(\Lambda_s^2,\mu)}{d\log\mu}$$

$$\frac{d\log C_V(-q^2,\mu)}{d\log\mu} = \Gamma(\alpha_s(\mu))\log\frac{-q^2}{\mu^2} + \gamma_V(\alpha_s(\mu))$$

$$\frac{d\log J(-p^2,\mu)}{d\log\mu} = -\Gamma(\alpha_s(\mu))\log\frac{-p^2}{\mu^2} - \gamma_J(\alpha_s(\mu))$$

$$\frac{d\log S(\Lambda_s^2,\mu)}{d\log\mu} = \Gamma(\alpha_s(\mu))\log\frac{\Lambda_s^2}{\mu^2} + \gamma_S(\alpha_s(\mu))$$

Resummed form factor

$$\Gamma(\alpha_s(\mu))\left[\log\frac{-q^2}{\mu^2} - \log\frac{-p^2}{\mu^2} - \log\frac{-p'^2}{\mu^2} + \log\frac{(-p^2)(-p'^2)}{(-q^2)\mu^2}\right]$$
$$+\gamma_V(\alpha_s(\mu)) - 2\gamma_J(\alpha_s(\mu)) + \gamma_S(\alpha_s(\mu)) = 0$$

Resummed form factor

$$\Gamma(\alpha_s(\mu))\left[\log\frac{-q^2}{\mu^2} - \log\frac{-p^2}{\mu^2} - \log\frac{-p'^2}{\mu^2} + \log\frac{(-p^2)(-p'^2)}{(-q^2)\mu^2}\right] + \gamma_V(\alpha_s(\mu)) - 2\gamma_J(\alpha_s(\mu)) + \gamma_S(\alpha_s(\mu)) = 0$$