A.V.Kotikov and V.I. Shilin (JINR, Dubna), S. Teber, (UPMC, LPTHE, Paris).

Helmholtz International Summer School
"Quantum Field Theory at the Limits: from Strong Fields to Heavy Quarks",
July 18 – 30, Dubna

Critical behaviour of (2+1)-dimensional QED: $1/N_f$-corrections in the Landau gauge

OUTLINE

1. Introduction

2. Results:

3. Conclusions and Prospects

1. Introduction t

Quantum Electrodynamics in $2+1$ dimensions (QED_3) has been extensively studied during more than three decades now.

Originally, the interest in QED_3 came from its similarities to $(3+1)$-dimensional QCD and the fact that phenomena such as dynamical chiral symmetry breaking (DχSB) and mass generation may be studied systematically in such a toy model (Pisarski, 1984, 1991), (Appelquist et al., 1984, 1988, 1999), (Nash, 1989), (Atkinson et al., 1998), (Dagotto et al., 1989, 1998), (Bashir et al., 2007, 2009), (Giombi, 2016), (Di Pietro, 2016)

Later, a strong interest in QED_3 arose in connection with planar condensed matter physics systems having relativistic-like low-energy excitations such as some two-dimensional antiferromagnets
(A. Marston and L. Ioffe, 1989)
and graphene
(F.F. Semenoff and Wallace, 1984)
[see reviews (Kotov, 2012) (S. Miransky, 2015)]

In all cases, the understanding of the phase structure of QED_3 is a crucial pre-requisite to understand non-perturbative dynamic phenomena in more realistic particle and condensed matter physics models.

Despite the fact that a large number of investigations have been carried out to study DχSB in QED$_3$, very different results have been obtained.

Pisarski solved the Schwinger-Dyson (SD) gap equation
(Pisarski, 1984)
using a leading order (LO) $1/N$-expansion and found that a fermion mass is generated for all values of N, decreasing exponentially with N and vanishing only in the limit $N \to \infty$.
Later, he confirmed his finding by a renormalization group analysis
(Pisarski, 1991)
Support of Pisarski's result was given by
(Pennington, 1991, 1992)
who adopted a more general non-perturbative approach to solving the SD equations.

On the other hand, in a more refined analysis of the gap equation at LO of the $1/N$-expansion, (Appelquist et al., 1988) have shown that the theory exhibits a critical behaviour as the number N of fermion flavours approaches $N_c = 32/\pi^2$; that is, a fermion mass is dynamically generated only for $N < N_c$.

Contrary to all previous results, an alternative non-perturbative study by (Atkinson et al., 1998) suggested that chiral symmetry is unbroken at sufficiently large N.

The theory has also been simulated on the lattice (K.K.Dagotto, 1989, 1990), (Azcoiti, 1993, 1996), (K.N.Karthik. 2016)

The conclusions of (K.K.Dagotto, 1989, 1990) are in the agreement with the existence of a critical N as predicted in the analysis of (Appelquist et al., 1988).

The second paper (Azcoiti, 1993, 1996) finds DχSB for all N.

The recent third one (K.N.Karthik. 2016): no sign of DχSB at all.

Even in the case where a finite N_c is found, its value is subject to uncertainty with estimates ranging from $N_c = 1$ to $N_c = 4$ (see (Appelquist et al., 1984) for a review).

Moreover, (Appelquist et al., 1999) found an upper bound, $N_c < 3/2$.

More recently, (K.T. Giombi et al., 2016) found that $N_c < 4.4$ and (K.S. DiPietro et al., 2016) that $N_c < 9/4$.

Clearly, all these disagreements reflect our poor understanding of this problem.

The purpose of the present work is to include $1/N$ corrections to the LO result of (Appelquist et al., 1988)

Because the critical value N_c is not large, the contribution of such higher orders in the $1/N$ expansion can be essential and their proper study may lead to a better understanding of the problem.

The well-known results of (D. Nash, 1989) demonstrated a quite strong stability of the $1/N$ expansion. The results was obtained using a different gauge parameters for various part of calculations.

The last years witnessed a strong progress in the study of the gauge dependence of DχSB in various models. (A. Ahmad, 2016).

The progress is related to the use of the Landau-Khalatnikov-Fradkin transformation.

(L.D. Landau and I.M. Khalatnikov, 1956), (E.S. Fradkin, , 1956)

On the School there was the talk of Muhammad Jamil Aslam about continuation of LKF-transformations to QCD.

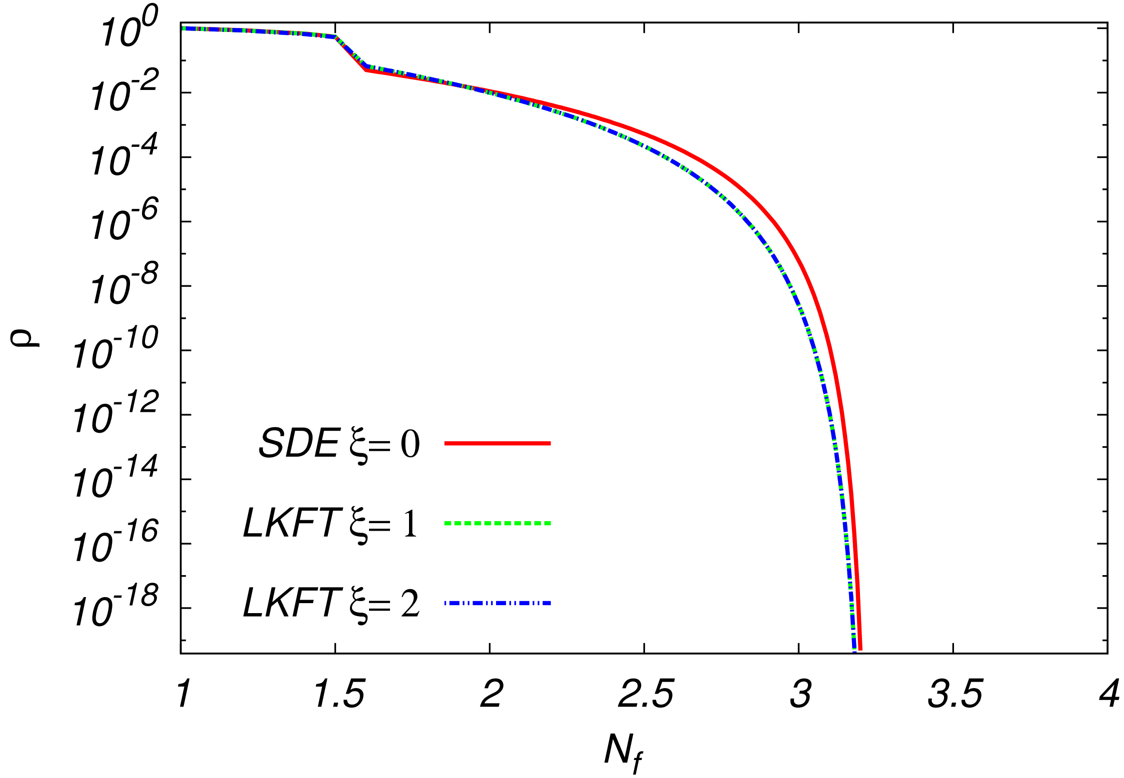

Figure 1: A dependence of the (normalized) dynamical mass on the gauge parameter.

In the case of QED_3 in the $1/N$-expansion, the application of this transformation (R. Bashir et al., 2009) has revealed the almost complete lack of gauge dependence for N_c. This confirms that we can limit our analysis to the case of the Landau gauge.

2. Model and Schwinger-Dyson equations

The Lagrangian of massless QED_3 with N flavours of fermions reads

$$L = \overline{\Psi}(i\hat{\partial} - e\hat{A})\Psi - \frac{1}{4}F_{\mu\nu}^2 \,, \tag{1}$$

where Ψ is taken to be a four component complex spinor. In the massless case the model contains infrared divergences.

The latter soften when the model is analysed in a $1/N$ expansion.
(P. Appelquist, 1981), (T.Jackiw and P. Appelquist, 1981)
Since the theory is super-renormalizable, the mass scale is given by the dimensionful coupling constant: $a = Ne^2/8$, which is kept fixed as $N \to \infty$.

In the four component case, we can introduce the matrices γ_3 and γ_5 which anticommute with γ_0, γ_1 and γ_2. Then, the massless case is invariant under the transformations: $\Psi \to \exp(i\alpha_1\gamma_3)\Psi$ and $\Psi \to \exp(i\alpha_2\gamma_5)\Psi$. Together with the identity matrix and $[\gamma_3, \gamma_5]$, we have a $U(2)$ symmetry for each spinor and the full global "chiral" (or rather flavour) symmetry is $U(2N)$. A mass term will break this symmetry to $U(N) \times U(N)$.

Following (P. Appelquist, 1988), we now study the solution of the SD equation. The inverse fermion propagator has the form

$$S^{-1}(p) = [1 + A(p)] \; (i\hat{p} + \Sigma(p)) \; , \tag{2}$$

where $A(p)$ is the wave-function renormalization and $\Sigma(p)$ is the dynamically generated parity-conserving mass which is taken to be the same for all the fermions.

Notice that in our definition of $\Sigma(p)$, the choice of the free vertex corresponds to the so-called central Ball-Chiu vertex
(C. Ball, 1989)
for the "more standard" definition $\tilde{\Sigma}(p) = \Sigma(p)[1 + A(p)]$.

With these conventions, the SD equation for the fermion propagator may be decomposed into scalar and vector components as follows:

$$\tilde{\Sigma}(p) = \frac{2a}{N}\,\mathsf{Tr}\int \frac{d^3k}{(2\pi)^3}\frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k)\Gamma^\nu(p,k)}{[1+A(k)]\left(k^2+\Sigma^2(k)\right)}\,, \tag{3}$$

$$A(p)p^2 = -\frac{2a}{N}\,\mathsf{Tr}\int \frac{d^3k}{(2\pi)^3}\frac{D_{\mu\nu}(p-k)\hat{p}\gamma^\mu\hat{k}\Gamma^\nu(p,k)}{[1+A(k)]\left(k^2+\Sigma^2(k)\right)}\,, \tag{4}$$

where $D_{\mu\nu}(p)$ is the photon propagator in the Landau gauge:

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2\,[1+\Pi(p)]}\,, \tag{5}$$

$\Pi(p)$ is the polarization operator and $\Gamma^\nu(p,k)$ is the vertex function.

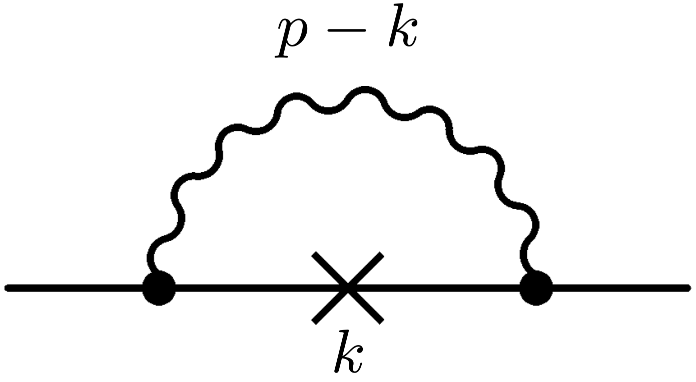

Figure 2: LO diagram to the dynamically generated mass $\Sigma(p)$. The crossed line denotes mass insertion.

3. Leading order

The LO approximations in the $1/N$ expansion are given by:

$$A(p) = 0, \quad \Pi(p) = a/|p|, \quad \Gamma^{\nu}(p,k) = \gamma^{\nu}\,, \tag{6}$$

where the fermion mass has been neglected in the calculation of $\Pi(p)$.

A single diagram contributes to the mass gap equation at LO, see Fig. 2, and the latter reads:

$$\Sigma(p) = \frac{16a}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{\big(k^2 + \Sigma^2(k)\big)\,[(p-k)^2 + a\,|p-k|]}\,. \tag{7}$$

Performing the angular integration in Eq. (7) yields:

$$\Sigma(p) = \frac{4a}{\pi^2 N|p|} \int_0^\infty d|k| \, \frac{|k|\Sigma(|k|)}{k^2 + \Sigma^2(|k|)} \ln\left(\frac{|k| + |p| + a}{|k - p| + a}\right) . \qquad (8)$$

The study of the last equation (P. Appelquist, 1988) has revealed the existence of a critical number of fermion flavours N_c, such that for $N > N_c$, $\Sigma(p) = 0$.

As it was argued in (P. Appelquist, 1988), QED$_3$ is strongly damped for $|p| > a$, *i.e.*, all relevant physics occur at $|p|/a < 1$. Hence, only the lowest order terms in $|p|/a$ have to be kept on the r.h.s. with a hard cut-off at $|p| = a$. Moreover, considering N close to N_c, the value of $\Sigma(|k|)$ can be made arbitrarily small. Thus, $k^2 + \Sigma^2(|k|)$ can be replaced by k^2 on the r.h.s. which then further simplifies as:

$$\Sigma(p) = \frac{8}{\pi^2 N} \int_0^a d|k| \, \frac{\Sigma(|k|)}{\text{Max}(|k|, |p|)} . \qquad (9)$$

Following (P. Appelquist, 1988), the mass function may then be parametrized as:

$$\Sigma(k) = B\,(k^2)^{-\alpha}\,, \tag{10}$$

(with an arbitrary B value) where the index α has to be self-consistently determined. Substituting (10) in Eq. (9), the gap equation reads:

$$1 = \frac{2\beta}{L} \quad \text{where} \quad \beta = \frac{1}{\alpha\,(1/2 - \alpha)} \quad \text{and} \quad L \equiv \pi^2 N\,. \tag{11}$$

Solving the gap equation, the following values of α are obtained:

$$\alpha_{\pm} = \frac{1}{4}\left(1 \pm \sqrt{1 - \frac{32}{L}}\right), \tag{12}$$

which reproduces the solution given by (P. Appelquist, 1988):

$N_c = 32/\pi^2 \approx 3.24$ ($i.e.$, $L_c = 32$),

such that $\Sigma(p) = 0$ for $N > N_c$ and

$$\Sigma(0) \simeq \exp[-2\pi/(N_c/N - 1)^{1/2}], \tag{13}$$

for $N < N_c$.

Thus, DχSB occurs when α becomes complex, that is for $N < N_c$.

As it was shown in (A.V.K. 1993, 2012), the same result for $\Sigma(p)$ can be obtained in another way. Taking the limit of large a, the linearized version of Eq. (7) has the following form:

$$\Sigma(p) = \frac{16}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{k^2\,|p-k|}\,. \tag{14}$$

With the help of the ansatz (10), one can then see that the r.h.s. of Eq. (14) may be calculated with the help of the standard rules of perturbation theory for massless Feynman diagrams as in (D.Kazakov, 1983)

Indeed, the computation of Eq. (14) is straightforward and reads:

$$\Sigma^{(\mathrm{LO})}(p) = \frac{8B}{N} \frac{(p^2)^{-\alpha}}{(4\pi)^{3/2}} \frac{2\beta}{\pi^{1/2}} \,. \tag{15}$$

This immediately yields the gap equation (11) and, hence, the results of Eq. (12) together with the critical value $N_c = 32/\pi^2$ at which the index α becomes complex.

Similarly, such rules allow for a straightforward evaluation of the wave function renormalization. At LO,

$$A(p)p^2 = -\frac{2}{N}\mathsf{Tr}\int\frac{d^Dk}{(2\pi)^D}\frac{(g_{\mu\nu}-\frac{(p-k)_\mu(p-k)_\nu}{(p-k)^2})\hat{p}\gamma^\mu\hat{k}\gamma^\nu}{k^2|p-k|}\,, \quad (16)$$

where the integral has been dimensionally regularized with $D = 3-2\varepsilon$.

Taking the trace and computing the integral on the r.h.s. yields:

$$A(p) = \frac{\Gamma(1+\varepsilon)(4\pi)^\varepsilon\mu^{2\varepsilon}}{p^{2\varepsilon}}C_1 = \frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}C_1 + \mathrm{O}(\varepsilon)\,, \quad (17)$$

where the $\overline{MS}$ parameter $\overline{\mu}$ has the standard form $\overline{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$ with the Euler constant γ_E and

$$C_1 = +\frac{4}{3\pi^2 N}\left(\frac{1}{\varepsilon}+\frac{7}{3}-2\ln 2\right)\,. \quad (18)$$

The corresponding anomalous scaling dimension of the fermion field then reads:

$$\eta = \mu^2 \frac{d}{d\mu^2} A(p) = \frac{4}{3\pi^2 N},$$

and coincides with the one in (D. Gracey, 1993)

Some rules for calculations $(\lambda = D/2 - 1)$

On the School there were some discussions about some rules for calculations in the talks of Andrey Grozin and Michal Czakon

Loop

$$\int d^D x \frac{1}{x^{2\alpha}(x-y)^{2\beta}} = \frac{1}{(4\pi)^{D/2}} \frac{1}{y^{2(\alpha+\beta-\lambda-1)}} A(\alpha,\beta),$$

where

$$A(\alpha,\beta) = \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-\lambda-1)}, \quad a(\alpha) = \frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}, \quad \tilde{\alpha} = D/2 - \alpha$$

Unical triangle $(\Sigma_1^3 \alpha_i = D)$

$$T(\alpha_1, \alpha_2, \alpha_3) \equiv \int d^D x \frac{1}{(x-y_1)^{2\alpha_1}(x-y_2)^{2\alpha_2}(x-y_3)^{2\alpha_2}}$$

$$= \frac{1}{(4\pi)^{D/2}} A(\alpha_1, \alpha_2) \frac{1}{y_{12}^{2\tilde{\alpha}_3} y_{13}^{2\tilde{\alpha}_2} y_{23}^{2\tilde{\alpha}_1}}, \quad y_{13}^{2\alpha} \equiv (y_1 - y_3)^{2\alpha}$$

IBP procedure for triagle $T(\alpha_1, \alpha_2, \alpha_3)$:

$$\int d^D x \frac{d}{dx^\mu}\left[\frac{(x-y_1)^\mu}{(x-y_1)^{2\alpha_1}(x-y_2)^{2\alpha_2}(x-y_3)^{2\alpha_2}}\right]$$
$$= \int d^D x [\frac{D}{(x-y_1)^{2\alpha_1}(x-y_2)^{2\alpha_2}(x-y_3)^{2\alpha_2}}$$
$$+(x-y_1)^\mu \frac{d}{dx^\mu}\left[\frac{1}{(x-y_1)^{2\alpha_1}(x-y_2)^{2\alpha_2}(x-y_3)^{2\alpha_2}}\right]] = 0$$

produces the relation

$$(D - 2\alpha_1 - \alpha_2 - \alpha_3) T(\alpha_1, \alpha_2, \alpha_3)$$
$$= \alpha_2 [T(\alpha_1 - 1, \alpha_2 + 1, \alpha_3) - y_{12}^2 T(\alpha_1, \alpha_2 + 1, \alpha_3)]$$
$$+ \alpha_3 [T(\alpha_1 - 1, \alpha_2, \alpha_3 + 3) - y_{13}^2 T(\alpha_1, \alpha_2, \alpha_3 + 1)]$$

Gegenbauer Polynomials

Following (Chetyrkin, Kataev,Tkachev, 1980) D-space integration can be represented in the form

$$d^D x = \frac{1}{2} S_{D-1} (x^2)^{\lambda} dx^2 d\hat{x},$$

where $\hat{x} = \vec{x}/\sqrt{x^2}$, and $S_{D-1} = 2\pi^{\lambda+1}/\Gamma(\lambda+1)$ is the surface of the unit hypersphere in R_D. The Gegenbauer polynomials $C_n^{\delta}(t)$ are defined as

$$(1 - 2rt + r^2)^{-\delta} = \sum_{n=0}^{\infty} C_n^{\delta}(t) r^n \quad (r \leq 1), \quad C_n^{\delta}(1) = \frac{\Gamma(n+2\delta)}{n!\Gamma(2\delta)}$$

whence the expansion for the propagator is:

$$\frac{1}{(x_1 - x_2)^{2\delta}} = \sum_{n=0}^{\infty} C_n^{\delta}(\hat{x_1}\hat{x_2}) \, [\frac{(x_1^2)^{n/2}}{(x_2^2)^{n/2+\delta}} \Theta(x_2^2 - x_1^2) \; + \; (x_1^2 \longleftrightarrow x_2^2)],$$

where

$$\Theta(y) = \begin{cases} 1, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases}$$

Orthogonality of the Gegenbauer polynomials is expressed by the equation

$$\int C_n^\lambda(\hat{x}_1\hat{x}_2)\ \hat{x}_2\ C_m^\lambda(\hat{x}_2\hat{x}_3) \ = \ \frac{\lambda}{n+\lambda}\delta_n^m\ C_n^\lambda(\hat{x}_1\hat{x}_3),$$

where δ_n^m is the Kroneker symbol.

The following formulae are useful:

$$C_n^\delta(x) \ = \ \sum_{p\geq 0} \frac{(2x)^{n-2p}(-1)^p\Gamma(n-p+\delta)}{(n-2p)!p!\Gamma(\delta)} \qquad \text{and}$$

$$\frac{(2x)^n}{n!} \ = \ \sum_{p\geq 0} C_{n-2p}^\delta(x)\frac{(n-2p+\delta)\Gamma(\delta)}{p!\Gamma(n-p+\delta+1)}$$

Substituding the latter equation for $\delta = \lambda$ to the first one, we have the following equation after the separate analysis at odd and even n:

$$C_n^\delta(x) \ = \ \sum_{k=0}^{[n/2]} C_{n-2p}^\lambda(x)\frac{(n-2k+\lambda)\Gamma(\lambda)}{k!\Gamma(\delta)}\frac{\Gamma(n+\delta-k)\Gamma(k+\delta-\lambda)}{\Gamma(n+\lambda+1-k)\Gamma(\delta-\lambda)}$$

The rules of calculations have the following form (A.V.K, 1995)

:

$$\int d^D x \frac{1}{x^{2\alpha}(x-y)^{2\beta}} \Theta(x^2 - y^2)$$
$$= \frac{1}{(4\pi)^{D/2}} \frac{1}{y^{2(\alpha+\beta-\lambda-1)}} \sum_{m=0}^{\infty} \frac{B(m,0|\beta,\lambda)}{m+\alpha+\beta-\lambda-1}$$
$$\stackrel{(\beta=\lambda)}{=} \frac{1}{(4\pi)^{D/2}} \frac{1}{y^{2(\alpha-1)}} \frac{1}{\Gamma(\lambda+1)} \frac{1}{(\alpha-1)}$$

$$\int d^D x \frac{1}{x^{2\alpha}(x-y)^{2\beta}} \Theta(y^2 - x^2)$$
$$= \frac{1}{(4\pi)^{D/2}} \frac{1}{y^{2(\alpha+\beta-\lambda-1)}} \sum_{m=0}^{\infty} \frac{B(m,0|\beta,\lambda)}{m-\alpha+\lambda+1}$$
$$\stackrel{(\beta=\lambda)}{=} \frac{1}{(4\pi)^{D/2}} \frac{1}{y^{2(\alpha-1)}} \frac{1}{\Gamma(\lambda+1)} \frac{1}{(\lambda+1-\alpha)}$$

where

$$B(m,n|\beta,\lambda) \;=\; \frac{\Gamma(m+\beta+n)}{m!\Gamma(m+n+1+\lambda)\Gamma(\beta)}\,\frac{\Gamma(m+\beta-\lambda)}{\Gamma(\beta-\lambda)}$$

Hereafter we add this specific case $\beta=\lambda$ to our rules.

The sum of above diagrams does not contain Θ-terms and should reproduce the above result for Loop in then form of Γ-functions..

It can be obtained by usage of the transformation of ${}_3F_2$-hypergeometric function with unit argument:

$$\begin{aligned} {}_3F_2(a,b,c;e,f;1) \;&=\; \frac{\Gamma(1-a)\Gamma(e)\Gamma(f)\Gamma(c-b)}{\Gamma(e-b)\Gamma(f-b)\Gamma(1+b-a)\Gamma(c)}\,\cdot \\ {}_3F_2(b,b-e+1,b-f+1;&1+b-c,1+b-a;1) \;+\; (b \longleftrightarrow c) \end{aligned}$$

When $e = b + 1$ (it is our case), the ${}_3F_2$-function can be represented as the sum of another ${}_3F_2$-function and a term containing only Γ-function products:

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+a)\Gamma(k+c)}{k!\Gamma(k+f)} \frac{1}{k+b} = \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(c-b)}{\Gamma(f-b)\Gamma(1+b-a)}$$

$$- \frac{\Gamma(1-a)\Gamma(a)}{\Gamma(f-c)\Gamma(1+c-f)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(k+c-f+1)\Gamma(k+c)}{k!\Gamma(k+1+c-a)} \frac{1}{k+c-b}$$

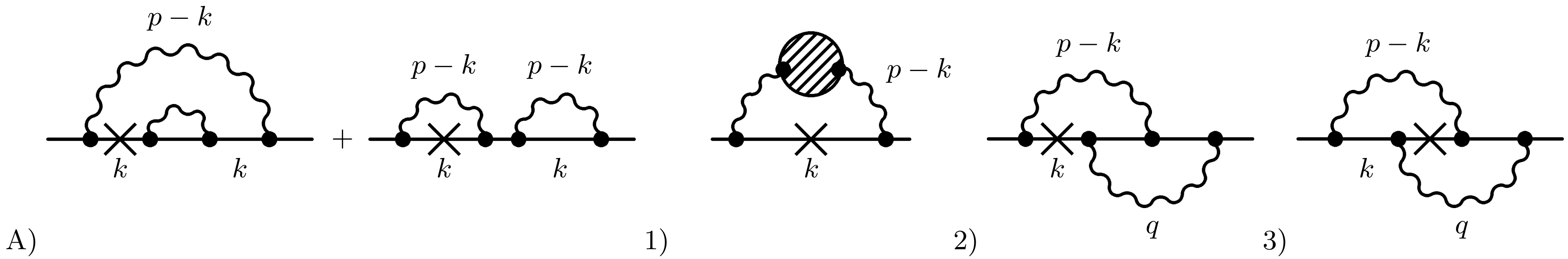

Figure 3: NLO diagrams to the dynamically generated mass $\Sigma(p)$. .

3. Next-to-leading order

The use at which the standard rules for computing massless Feynman diagrams allowed us to derive LO results suggests the possibility to extend these computations beyond LO. We therefore consider the NLO contributions to the dynamically generated mass and parametrize them as:

$$\Sigma^{(\mathrm{NLO})}(p) = \left(\frac{8}{N}\right)^2 B\,\frac{(p^2)^{-\alpha}}{(4\pi)^3}\,\left(\Sigma_A + \Sigma_1 + 2\,\Sigma_2 + \Sigma_3\right)\,, \qquad (19)$$

where each NLO contribution is represented graphically in Fig. 3.

Because we are dealing with the linearized gap equation, each contribution contains a single mass insertion. Adding these contributions to the LO result, Eq. (15), the gap equation has the following general form:

$$1 = \frac{2\beta}{L} + \frac{\pi}{L^2}\left[\Sigma_A + \Sigma_1 + 2\,\Sigma_2 + \Sigma_3\right]. \tag{20}$$

After very tedious and lengthy calculations, all NLO contributions could be evaluated exactly using the rules for computing massless Feynman diagrams. For the most complicated scalar diagrams, see $I_1(\alpha)$ and $I_2(\alpha)$ below, the Gegenbauer-polynomial technique has been used following (A.V.K., 1995) We now summarize our results.

The contribution Σ_A, see Fig. 3 A), originates from the LO value of $A(p)$ and is singular. Using dimensional regularization, it reads:

$$\overline{\Sigma}_A = +\frac{16}{3}\frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\,\beta\left(\frac{1}{\varepsilon} + \Psi_1 + \frac{4}{3} - \frac{\beta}{4}\right) + \mathrm{O}(\varepsilon)\,, \qquad (21)$$

where $\overline{\Sigma}_i = \pi\Sigma_i$, $(i = 1, 2, 3.A)$ and

$$\Psi_1 = \Psi(\alpha) + \Psi(1/2 - \alpha) - 2\Psi(1) + \frac{3}{1/2 - \alpha} - 2\ln 2\,, \qquad (22)$$

and Ψ is the digamma function.

The contribution of diagram 1) in Fig. 3 is finite and reads:

$$\overline{\Sigma}_1 = -4\hat{\Pi}\beta, \quad \hat{\Pi} = \frac{92}{9} - \pi^2\,, \qquad (23)$$

The contribution of diagram 2) in Fig. 3 is again singular. Dimensionally regularizing it yields:

$$2\,\overline{\Sigma}_2 = -\frac{16}{3}\frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\,\beta\left(\frac{1}{\varepsilon}+\Psi_1+\frac{7}{3}+\frac{5\beta}{8}\right)-2\hat{\Sigma}_2+O(\varepsilon)\,, \qquad (24)$$

where

$$\begin{aligned}\hat{\Sigma}_2 &= (1-4\alpha)\beta[\Psi'(\alpha)-\Psi'(1/2-\alpha)] \\ &-\frac{\pi}{2\alpha}\,\tilde{I}_1(\alpha)-\frac{\pi}{2(1/2-\alpha)}\tilde{I}_1(\alpha+1)\,,\end{aligned} \qquad (25)$$

and Ψ' is the trigamma function. Notice that the singularities in $\overline{\Sigma}_A$ and $\overline{\Sigma}_2$ cancel each other and their sum is therefore finite:

$$\overline{\Sigma}_A + 2\,\overline{\Sigma}_2 = -\frac{2}{3}\beta(7\beta+8)-2\hat{\Sigma}_2\,. \qquad (26)$$

This cancellation corresponds to the one of the logarithms, $\ln(p/\alpha)$, in (D. Nash, 1989)

The dimensionless integral $\tilde{I}_1(\alpha)$ appearing in Eq. (25) is defined as:

$$I_1(\alpha) \equiv \frac{(p^2)^{-\alpha}}{(4\pi)^3}\,\tilde{I}_1(\alpha) \tag{27}$$
$$= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{|p-k_1|k_1^{2\alpha}(k_1-k_2)^2(p-k_2)^2|k_2|}\,,$$

and obeys the following relation (it can be obtained by analogy with the ones in (D. Kazakov, 1983)):

$$\tilde{I}_1(\alpha+1) = \frac{(\alpha-1/2)^2}{\alpha^2}\tilde{I}_1(\alpha) - \frac{1}{\pi\alpha^2}[\Psi'(\alpha) - \Psi'(1/2-\alpha)] \tag{28}$$

Using (A.V.K., 1995) the integral $\tilde{I}_1(\alpha)$ can be represented in the form of a two-fold series

$$\tilde{I}_1(\alpha) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{B(l,n,1,1/2)}{(n+1/2)\,\Gamma(1/2)} \times [\frac{2}{n+1/2}\left(\frac{1}{l+n+\alpha} + \frac{1}{l+n+3/2-\alpha}\right) + \frac{1}{(l+n+\alpha)^2} + \frac{1}{(l+n+3/2-\alpha)^2}], \quad (29)$$

where

$$B(m,n,\alpha,1/2) = \frac{\Gamma(m+n+\alpha)\Gamma(m+\alpha-1/2)}{m!\Gamma(m+n+3/2)\Gamma(\alpha)\Gamma(\alpha-1/2)}. \quad (30)$$

Finally, the contribution of diagram 3) in Fig. 3 is finite and reads:

$$\overline{\Sigma}_3 = \hat{\Sigma}_3 + 3\beta^2,$$
$$\hat{\Sigma}_3 = (1/2-\alpha)\pi\tilde{I}_2(1+\alpha) + \frac{\pi}{2}\tilde{I}_2(\alpha) + (\alpha-2)\pi\tilde{I}_3(\alpha)\,. \quad (31)$$

The dimensionless integrals are defined as:

$\tilde{I}_2(\alpha) = \tilde{I}(\gamma = 1/2, \alpha)$ and $\tilde{I}_3(\alpha) = \tilde{I}(\gamma = -1/2, 1+\alpha)$, where:

$$I(\gamma,\alpha) \equiv \frac{(p^2)^{-\alpha-\gamma+1/2}}{(4\pi)^3}\,\tilde{I}(\gamma,\alpha)$$
$$= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{(p-k_1)^{2\gamma}k_1^2(k_1-k_2)^{2\alpha}(p-k_2)^2|k_2|}\,. \quad (32)$$

They satisfy the following relations:

$$\tilde{I}_2(\alpha) = \tilde{I}_2(3/2-\alpha),$$
$$\tilde{I}_3(\alpha) = \frac{2}{4\alpha-1}(\alpha\tilde{I}_2(1+\alpha) - (1/2-\alpha)\tilde{I}_2(\alpha)) - \frac{\beta^2}{\pi}\,, \quad (33)$$

and, thus, only one of them is independent.

Using (A.V.K., 1985), the integral $\tilde{I}_2(\alpha)$ can be represented in the form of a three-fold series:

$$\tilde{I}_2(\alpha) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B(m,n,\beta,1/2) \sum_{l=0}^{\infty} B(l,n,1,1/2) \times C(n,m,l,\alpha), \tag{34}$$

$$C(n,m,l,\alpha) = \frac{1}{(m+n+\alpha)(l+n+\alpha)} + \frac{1}{(m+n+\alpha)(l+m+n+1)} + \frac{1}{(m+n+1/2)(l+m+n+\alpha)} + \frac{1}{(m+n+1/2)(l+n+3/2-\alpha)} + \frac{1}{(n+l+\alpha)(l+m+n+\alpha)} + \frac{1}{(l+n+3/2-\alpha)(l+n+m+\alpha)}\,. \tag{35}$$

Combining all of the above results, the gap equation (20) may be written in an explicit form as:

$$1 = \frac{2\beta}{L} + \frac{1}{L^2}\left[8S(\alpha) - \frac{5}{3}\beta^2 - \frac{16}{3}\beta - 4\hat{\Pi}\beta\right], \tag{36}$$

where

$$S(\alpha) = (\hat{\Sigma}_3(\alpha) - 2\hat{\Sigma}_2(\alpha))/8\,. \tag{37}$$

At this point, we consider Eq. (36) directly at the critical point $\alpha = 1/4$, *i.e.*, at $\beta = 16$. This yields:

$$L_c^2 - 32L_c - 8(S - 64 - 8\hat{\Pi}) = 0\,, \tag{38}$$

where $S = S(\alpha = 1/4)$. Solving Eq. (38), we have two standard solutions:

$$L_{c,\pm} = 16 \pm \sqrt{D}, \quad D = 8(S - 32 - 8\hat{\Pi})\,. \tag{39}$$

It turns out that the "$-$" solution is unphysical and has to be rejected because $L_{c,-} < 0$. So, the physical solution is $L_c = L_{c,+}$.

In order to provide a numerical estimate for N_c, we have used the series representations in order to evaluate the integrals:
$\pi \tilde{I}_1(\alpha = 1/4) \equiv R_1$ and
$\pi \tilde{I}_2(\alpha = 1/4 + i\delta) \equiv R_2 - iP_2\delta + O(\delta^2)$
where $\delta \to 0$ regulates an artificial singularity in
$\pi \tilde{I}_3(\alpha = 1/4) = R_2 + P_2/4$.

With 10000 iterations for each series, we obtain the following numerical estimates:

$$R_1 = 163.7428, \quad R_2 = 209.175, \quad P_2 = 1260.720\,. \tag{40}$$

From these results, we may then obtain the numerical value of

$$S = R_1 - R_2/8 - 7P_2/128$$

which, combined with the one of $\hat{\Pi}$, yields $L_c = 32.45$ and therefore $N_c = 3.29$. This result shows that the inclusion of the $1/N$ corrections increases the critical value of N_c by only 1.5% with respect to its LO value.

Conclusion

We have included $O(1/N^2)$ contributions to the SD equation exactly and found that the critical value N_c increased by 1.5% with respect to the LO result.

- Our analysis is in nice agreement with (D. Nash, 1989) and therefore gives further evidence in favour of the solution found by (P. Appelquist, 1988).
- Our results are in support of the fact that the $1/N$ expansion of the kernel of the SD equation describes reliably the critical behaviour of the theory.
- Our good agreement with (D. Nash, 1989) is nice but rather strange because the two analyses are done in quite different ways.

- While we have used the Landau gauge (in accordance with recent results (R.S. Bashir, 2009) showing the gauge invariance of N_c), Nash worked with an arbitrary gauge fixing parameter, ξ.
- He has resummed the most important NLO terms ($\propto \beta^2$ in our definition) which, together with the LO ones, lead to a gauge invariant result for N_c. This result is larger by a factor $4/3$ than the pure LO one (P. Appelquist, 1988).
- The rest of the NLO terms ($\propto \beta$) were evaluated (mostly numerically) in the Feynman gauge, which modifies N_c another time and gives the final result of Nash: $N_c = 3.28$, which in-turn is very close to LO one.

- For these reasons, and despite the surprising closeness of the final results, our analysis substantially differs from that of Nash and intermediate expressions are difficult to compare.
- In order to clear up the beautiful agreement we have with Nash's results (D. Nash, 1989), we plan to take into account of all ξ-dependent terms in our forthcoming publication.
- Really we have calculated already the ξ-dependent terms (through srong discussions with Valery Gusynin from Kiev). We have been waiting for publishing the him similar paper. It came today (e-Print: arXiv:1607.08582 [hep-ph]). So, I think we will prepare our paper just after our vacations.