Superconformal indices & instanton partition functions

(lecture 1/4)

Seok Kim

(Seoul National University)

"Partition functions and automorphic forms"

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Plan

- Lecture 1 (today): Overview & basics
- Nekrasov's instanton partition functions: motivations
- ADHM quantum mechanics
- Deriving the basic formulae for the instanton partition function

- Lecture 2: Instantons in 5d QFTs
- Instanton partition functions for various gauge theories
- 5d SCFTs: dualities and enhanced symmetries
- 5d instanton partition functions for 6d QFTs

- Lecture 3: Instantons in 6d QFTs
- 6d instanton strings and elliptic genera
- Exceptional instantons & instanton strings
- 6d (2,0) SCFT: S-duality & M5-branes

- Lecture 4: Superconformal observables
- Instantons and partition functions on $S^4, S^5, \ldots$
- 5d and 6d indices on $S^4 \times S^1$ and $S^5 \times S^1, \ldots$

N=2 supersymmetric gauge theory

- Comes with 8 supersymmetries

$$Q^A_\alpha\ ,\quad \bar{Q}^A_{\dot\alpha} \qquad A=1,2 \text{ for } SU(2)_R\ ,\quad \alpha,\dot\alpha=1,2 \text{ for } SO(3,1)\sim SU(2)_l\times SU(2)_r$$

- Vector multiplet, with given gauge group (or algebra):

$$A_\mu\ (\mu=0,1,2,3)\ ,\quad \Phi : \text{complex scalar}\ ,\quad \text{fermions}$$

in adjoint representation of gauge group G

- Hypermultiplet (matters):

$$q_A\sim(q,\tilde{q}^\dagger)\ :\quad \text{two complex scalars}\ ,\quad \text{fermions}$$

in represencation $\mathbf{R}$ of gauge group G

- N=2 SUSY Lagrangian determined once gauge group & matters are determined.

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4g_{YM}^2}\text{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2g_{YM}^2}\text{tr}(D_\mu\Phi D^\mu\Phi) + \frac{1}{4}\text{tr}[\Phi,\Phi^\dagger]^2 - |D_\mu q_A|^2 - |(\Phi)_{\mathbf{R}}q_A|^2 \\ &\quad -\frac{1}{2g_{YM}^2}\sum_{i=1}^3\left(q_A(\tau^i)^A{}_B q^{\dagger B}\right)^2 + (\text{terms involving fermions})\end{aligned}$$

- Can be extended into 5d and 6d SUSY QFTs.

$$A_\mu\ (\mu=0,\cdots,4)\ ,\quad \Phi : \text{real scalar}\ ,\quad \text{fermions}$$

$$A_\mu\ (\mu=0,\cdots,5)\ ,\quad \text{fermions}$$

in 6d, also with the so-called tensor multiplet (later)

Coulomb branch

- Vacuum with nonzero expectation value (VEV) of “vector multiplet scalar”

$$4\text{d} : v \equiv \langle 0|\Phi|0\rangle \neq 0 \qquad 5\text{d} : v = \langle \Phi + iA_4 \rangle \ \text{ on } \mathbb{R}^4 \times S^1 \qquad 6\text{d} : v = \langle A_4 + iA_5 \rangle \ \text{ on } \mathbb{R}^4 \times T^2$$

[VEV of gauge fields on circle are sometimes called “Wilson line”]

- Gauge symmetry is G is broken to $U(1)^r$, where r is the rank of G.
- Massless fields in Coulomb branch: vector multiplets for $U(1)^r$ gauge symmetry.

- Coulomb branch “effective action” of these light fields: encoded in “prepotential”

$$\mathcal{L} = \sum_{i=1}^{r} \frac{1}{4\pi} \mathrm{Im} \left[\int d^4\theta \frac{\partial \mathcal{F}(a)}{\partial a^i} \bar{a}^i + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j} W^i_\alpha W^{j\alpha} \right]$$

$$\sim \mathrm{Im} \left(\frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j} \right) \partial_\mu a^i \partial^\mu \bar{a}^j + \cdots$$

Seiberg-Witten theory

- The Coulomb branch effective theory is important in many ways.
- Solvable models in strong-coupling QFT
- It triggered developments of non-perturbative string theory
- Related to mathematical studies of 4-manifolds
- More…

- Seiberg and Witten (1994) provided natural ways of determining this low energy theory for many models, especially $F(a)$, based on,
- known behaviors of $F(a)$ near $a^i \to \infty$ (from perturbative calculations),
- suitable assumptions on singularity structures on a^i space (moduli space)

- For most theories, $F(a)$ acquires an infinite series of non-perturbative corrections

$$\mathcal{F}(a,\Lambda) = \underline{\mathcal{F}_{\text{pert}}(a, \log\Lambda)} + \sum_{k=1}^{\infty} \mathcal{F}_k(a)\Lambda^k$$

classical $\sim (\log\Lambda)^1$ or 1-loop $\sim (\log\Lambda)^0$: 1-loop exact

$$\Lambda = \# e^{-\frac{4\pi^2}{g_{YM}^2}}$$

Instantons

$$\mathcal{F}(a,\Lambda) = \mathcal{F}_{\text{pert}}(a,\log\Lambda) + \sum_{k=1}^{\infty} \mathcal{F}_k(a)\Lambda^k \qquad \Lambda = \# e^{-\frac{4\pi^2}{g_{YM}^2}}$$

- The perturbative contribution comes from standard 1-loop calculation, where the massive fields in the Coulomb branch run through the loop.
- The infinite series of non-perturbative corrections from “instantons”

- In weakly-coupled regime, $\Lambda \ll a^{\#}$, one can understand them semi-classically.

- semi-classical saddle points: $S_{\text{classical}} = 4\pi^2 k / g_{YM}^2$

$$F_{\mu\nu} = \star_4 F_{\mu\nu} \qquad k \equiv \frac{1}{8\pi^2}\int \text{tr}\,(F \wedge F) \in \mathbb{Z}$$

- Represents tunneling between various topologically distinct vacua
- $F_k(a)$ is given by quantum fluctuations of fields around k instanton background.
- In 5d or 6d, semiclassical solutions are lines wrapping S^1, or a sheet wrapping T^2
- It became an important problem to microscopically compute these coefficients.

Instanton partition function

- Nekrasov (2002) provided a first-principle method to compute these coefficients.

- Partition function of QFT on R^4 with special deformation "Omega background"

$$Z(a, \Lambda, \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} Z_k(a, \epsilon_1, \epsilon_2)\Lambda^k \quad (\text{where } Z_0 \equiv 1)$$

- Yields prepotential in a limit (can be proven in a couple of different ways)

$$Z(a, \Lambda, \epsilon_{1,2}) \xrightarrow{\epsilon_{1,2}\to 0} \exp\left[-\frac{F_{\text{inst}}(a, \Lambda)}{\epsilon_1 \epsilon_2} + (\text{less divergent terms})\right]$$

- The computational question boiled down to how to compute each coefficient Z_k
- … And whenever possible, how to deal with full series exactly.

- I'll first focus on the first question, of computing Z_k at given instanton number.

From tunneling to physical states

- It is technically helpful to consider higher dimensional setting (even to easily define & treat the Omega background). Nekrasov actually calculated it this way.

- Note that a solution to the equation below on R^4, with $F_{\mu\nu} \to 0$ fastly at infinity, is a stationary particle-like configuration on $R^{4,1}$. We call this “solitons”

$$F_{\mu\nu} = \star_4 F_{\mu\nu} \qquad k \equiv \frac{1}{8\pi^2} \int \mathrm{tr}\,(F \wedge F) \in \mathbb{Z}$$

- Euclidean QFT on $R^4 \times S^1$ has dual interpretations:

- S^1 as space: the solution is an instanton string,
- S^1 as Euclidean time, with circumference β: particle evolves through time, and the partition function computes its quantum states’ degeneracy at temperature β^{-1} (more on next slide).

- 6d: instanton strings wrapping compact spatial S^1, and runs through temporal S^1.

New interpretation: counting BPS states

- Witten index (Omega deformation & Λ as chemical potentials/fugacities)

$$Z_k = \text{Tr}_k \left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)-\epsilon_2(J_2+J_R)} e^{-a^i q_i} \cdots \right]$$

sum over Hilbert space with instanton number k

F: fermion number (defined mod 2)
$(-1)^F = 1$ for boson, -1 for fermion

$J_1, J_2 :\ U(1)^2 \subset SO(4)$ spins on $\mathbb{R}^4$

$J_R :\ U(1) \subset SU(2)_R$ 'R-charge'

$q_i :$ electric charges for $U(1)^r$

- 'Omega background' is implicitly defined this way.
- The problem boils down to properly quantizing the system and counting states.
- Needs to understand the quantum mechanics of k instanton particles.
- The 4d partition function can be computed as a small S^1 limit of 5d index.

- Elliptic genus: instanton strings wrapping spatial circle, w/ momentum along it.
- Can be computed from the 2d QFT on k instanton strings.

$$Z_k(a, q, \epsilon_{1,2}) = \text{Tr}_k \left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} q^P e^{-\epsilon_1(J_1+J_R)-\epsilon_2(J_2+J_R)} e^{-a^i q_i} \cdots \right]$$

Why is it important…?

- Of course, it can be used to derive the Seiberg-Witten solution.
- But there are surprisingly rich applications of $Z_k(a, \epsilon_{1,2})$ or $Z(q, a, \epsilon_{1,2})$…

- Higher dimensional SCFTs (5d or 6d): discovered through string theory, but microscopic formulations unknown (e.g. no known Lagrangians)
- Only limited class of observables known, including the Coulomb branch indices.

- BPS spectrum of instanton particles & strings (lectures 2 & 3)
- Can be used to explore many aspects, such as dualities, or mysterious d.o.f.

- Building blocks of curved space partition functions (w.o. a notion of Coulomb branch):

- For instance, on S^4, one finds that [Pestun] (2007)

$$Z_{S^4}(g_{YM}, \cdots) = \int [da] Z_{\mathbb{R}^4}(\Lambda = e^{2\pi i \tau}, a_i, \epsilon_{1,2}) Z_{\mathbb{R}^4}(\Lambda = e^{2\pi i \bar{\tau}}, a_i, \epsilon_{1,2}) \Big|_{\epsilon_1 = \epsilon_2 = r^{-1}}$$

- Useful observables to explore new non-perturbative physics: e.g. AGT, dualities, …

Superconformal index

- CFT can be put on $S^{d-1} \times R$, in a unique manner. For SCFTs,

- "Witten index": count BPS states preserving a pair Q, S of Poincare/conformal supercharge:

$$Z_{S^{d-1}\times\mathbb{R}} = \mathrm{Tr}\left[(-1)^F e^{-z\cdot F}\right] \quad \text{all charges } F \text{ in SCFT satisfying } [Q,F]=[S,F]=0$$

- E.g. 4d SCFT w/ N SUSY: $SU(1,1) \times \boxed{SU(1,2|\mathcal{N}) \times (\text{flavor})} \subset SU(2,2|\mathcal{N}) \times (\text{flavor})$

$$Q, S, \Delta \equiv E - \left(\frac{3}{2}R - j_1 - j_2\right) \in SU(1,1)$$

- In many field theories admitting weakly-coupled gauge theory descriptions (esp. in $d = 4$), index admits a free QFT representation, an integral over indices evaluated over free fields

$$Z_{S^{d-1}\times\mathbb{R}}(t,y,\{m\}) = \int [dU] PE\left[\sum_{i\in\text{multiplets}} f_i(t,y,\{m\})\chi_{\mathbf{R}}(U)\right]$$

$$PE[f(t,y,\{m\})\chi_{\mathbf{R}}(U)] \equiv \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(t^n,y^n,nm)\chi_{\mathbf{R}}(U^n)\right]$$

ref. talks by [Spiridonov, Gahramanov, Agarwal], for math/physical structures, or physical applications

- In 5d or 6d, no such free field representations. There are conjectured integral representations, where the integrand consists of the instanton partition functions. [H.-C.Kim, S.Kim, SK] [Lockhart, Vafa] [H.-C.Kim, S.-S.Kim, Lee] (lecture 4)

Computation from ADHM construction

- For the computation of Z_k in 5d QFT, Nekrasov does it using the ADHM construction of instantons for classical gauge groups, $G = SU(N), SO(N), Sp(N)$

- Originally [Atiyah, Drinfeld, Hitchin, Manin] [Christ, Weinberg, Stanton] developed as a method of constructing the solutions to $F_{\mu\nu} = \star_4 F_{\mu\nu}$. E.g. for SU(N),

$$A_\mu = i v^\dagger \partial v \quad (v_{(N+2k)\times N},\ v^\dagger v = \mathbf{1}_{N\times N}) \qquad U^\dagger v = 0\ ,\ \ U_{(N+2k)\times 2k} = \begin{pmatrix} \bar{q}_{N\times 2k} \\ (a_{\alpha\dot\beta})_{k\times k} - x_{\alpha\dot\beta} \otimes \mathbf{1}_{k\times k} \end{pmatrix}$$

$$D^I \equiv q_{\dot\alpha}(\tau^I)^{\dot\alpha}{}_{\dot\beta}\bar{q}^{\dot\beta} + (\tau^I)^{\dot\alpha}{}_{\dot\beta}[a_{\alpha\dot\alpha}, a^{\alpha\dot\beta}] = 0$$

- The constant matrices in U are called ADHM data. They label the instanton moduli space.

- It was soon realized that these matrices can be regarded as 1d fields, or quantum mechanical degrees of freedom on solitons, where the constant parts label the zero energy ground state solutions.

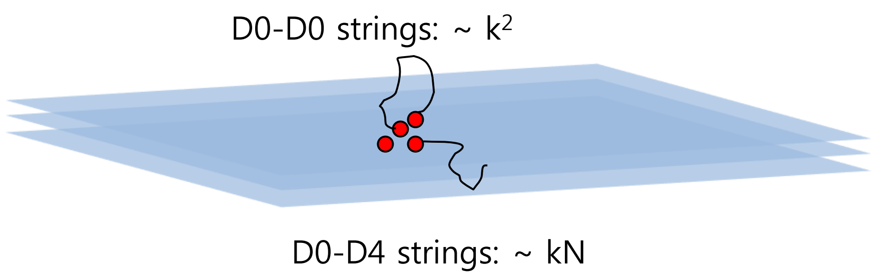
ADHM quantum mechanics

- The full ADHM quantum mechanics (GLSM):
- Promote the matrices into fields of QM, also adding more fields to make it gauge theory.

5d gauge group: $G_N = U(N), SO(N), Sp(N)$

QM gauge group: $\hat{G}_k = U(k), Sp(k), O(k)$

Intuition: k D0's & N D4's
(optionally w/ orientifold)



QM vector multiplet: $\varphi, A_t, \bar{\lambda}^A_{\dot{\alpha}}$ ($\hat{G}_k$ adjoint)

$\hat{G}_k$ adjoint/antisymmetric/symmetric hyper: $a_{\alpha\dot{\beta}}, \Psi^A_\alpha$

$G_N \times \hat{G}_k$ bi-fundamental hyper: $q_{\dot{\alpha}}, \psi^A$

$$L_{\rm QM} = \frac{1}{g_{QM}}{\rm tr}\left[\frac{1}{2}(D_t\varphi)^2 + \frac{1}{2}(D_t a_m)^2 + D_t q_{\dot{\alpha}} D_t \bar{q}^{\dot{\alpha}} + \frac{1}{2}[\varphi, a_m]^2 - (\varphi\bar{q}^{\dot{\alpha}} - \bar{q}^{\dot{\alpha}} v)(q_{\dot{\alpha}}\varphi - v q_{\dot{\alpha}}) - D^{\dot{\alpha}}{}_{\dot{\beta}} D^{\dot{\beta}}{}_{\dot{\alpha}} + \cdots\right]$$

- Z_k is the Witten index computed for these QM.
- Extra d.o.f. from 5d hypermultiplets: see, e.g. [Shadchin] 2005

[Perhaps a crucial limitation of this approach: cannot deal with exceptional gauge theories, with $G = G_2, F_4, E_6, E_7, E_8$, and other exotic hypermultiplet matters.]

ADHM details

- Instantons preserve half of SUSY: 1d reduction of 2d (0,4) SUSY
- supermultiplets: (on-shell)

$\text{supercharges}: \ Q_{A\dot{\alpha}}$

$\text{vector}: \ A_t, \ \phi, \ \lambda_{A\dot{\alpha}} \qquad \text{hyper}: \ \varphi_{\dot{\alpha}}, \ \psi_A \qquad \text{twisted hyper}: \ \varphi_A, \ \psi_{\dot{\alpha}} \qquad \text{Fermi}: \ \Psi$

- 1d gauge coupling: $[g_{QM}^2] = M^3$: Low E ~ strong coupling. High E ~ weak coupling.
- We are mostly interested in low E physics on instanton moduli space.
- Z_k, a Witten index, is insensitive to such parameters. Can easily compute in UV.

- Extra degrees of freedom in ADHM gauge theory:

- More UV degrees of freedom than those labeling moduli space: e.g. 1d vector multiplet
- May survive or be wiped out in low energy physics.

$\text{vector}: \ A_t, \ \phi, \ \lambda_{A\dot{\alpha}}$

$\text{twisted hyper}: \ \varphi_A, \ \psi_{\dot{\alpha}}$

- Understanding the subtleties of extra d.o.f. is crucial for some theories.
- After getting rid of these extra factors, we get intrinsic QFT observables.

Some calculus

- 5d $SU(N)$ super-Yang-Mills w.o. matters: 1d ADHM fields (in $(0,2)$ multiplets) are

$$\begin{aligned} \text{chiral} \quad &: \quad (q,\psi) \in (\mathbf{k},\overline{\mathbf{N}}),\ (\tilde{q},\tilde{\psi}) \in (\overline{\mathbf{k}},\mathbf{N}),\ (a,\Psi),\ (\tilde{a},\tilde{\Psi}) \in (\mathbf{adj},\mathbf{1}) \\ \text{vector} \sim \text{Fermi} \quad &: \quad (A_t,\varphi,\lambda_0),\ (\lambda) \in (\mathbf{adj},\mathbf{1}) \end{aligned}$$

representations of $U(k) \times SU(N)$ shown

- Path integral of SUSY quantum mechanics at weak-coupling

- Almost a Gaussian integral, but some zero modes should be treated exactly.
- Zero modes: constant part of scalar & S^1 Wilson line in 1d vector multiplet $\phi = \varphi + iA_\tau$
- Holding zero modes fixed, integrate over other fields. Each multiplet contributes as

$$\text{chiral}: \ \left[2\sinh\left(\frac{\rho\cdot\phi + J\epsilon_+ + Fz}{2}\right)\right]^{-1} \qquad \text{vector}: \ 2\sinh\left(\frac{\alpha\cdot\phi}{2}\right) \qquad \text{Fermi}: \ 2\sinh\left(\frac{\rho\cdot\phi + J\epsilon_+ + Fz}{2}\right)$$

- Reasoning: $(2\sinh x/2)^{-1}$ from infinite tower of modes of a scalar on S^1, whose chemical potential factor is x. (and an inverse factor from fermion fields)

$$\left[2\sinh\frac{x}{2}\right]^{-1} = x^{-1}\prod_{n\neq 0}\left(1+\frac{x}{2\pi n i}\right)^{-1} \sim x^{-1}\prod_{n\neq 0}\left(\frac{x}{\beta}+\frac{2\pi n i}{\beta}\right)^{-1}$$

- Final question: integral over zero modes. Turns out to be a contour integral.

Results

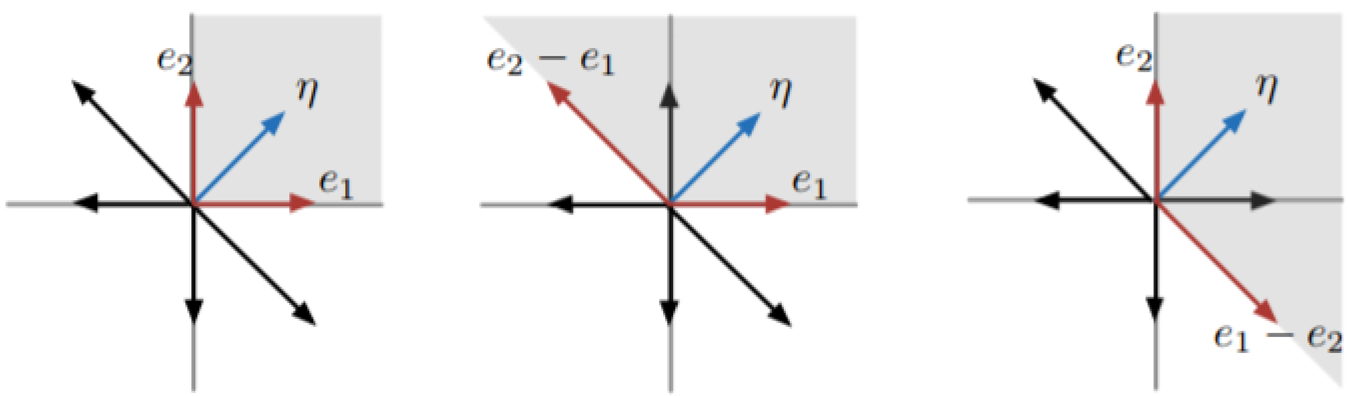
- Residue sum rules: keep only a subset of them. "Jeffrey-Kirwan residue" (JK-Res)

- For rank r group $\hat{G}$, pick an r-dimensional auxiliary vector η.
- In all examples we shall meet in instanton counting, we can rearrange the integrand into sum of various terms, where each term only has "simple pole"

$$\text{JK-Res}(\mathbf{Q}_*, \eta)\frac{d\phi_1 \wedge \cdots \wedge d\phi_r}{Q_{j_1}(\phi)\cdots Q_{j_r}(\phi)} = \begin{cases} |\det(Q_{j_1}, \cdots, Q_{j_r})|^{-1} & \text{if } \eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r}) \\ 0 & \text{otherwise} \end{cases}$$

$$Z = \frac{1}{|W|}\sum_{\phi_*} \text{JK-Res}(\mathbf{Q}(\phi_*), \eta)\ Z_{\text{1-loop}}(\phi, \epsilon_+, z)$$

- E.g. for $\hat{G} = U(2)$ (i.e. at $k = 2$), let us pick $\eta = (1,1)$.
- Possible charge pairs with nonzero JK-Res are



- With a "rare" exceptional cases, choice of η will not matter for most models.

SU(N): labeling the poles

- For SU(N) k instantons, the integral is given by $Z_k = \frac{1}{k!}\oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} Z_{\text{1-loop}}$

$$Z_{\text{1-loop}} = \frac{\prod_{I\neq J} 2\sinh\frac{\phi_{IJ}}{2} \cdot \prod_{I,J=1}^{k} 2\sinh\frac{\phi_{IJ}+2\epsilon_+}{2}}{\prod_{I=1}^{k}\prod_{i=1}^{N} 2\sinh\frac{\epsilon_+\pm(\phi_I-a_i)}{2}\prod_{I,J=1}^{k} 2\sinh\frac{\phi_{IJ}+\epsilon_1}{2}\cdot 2\sinh\frac{\phi_{IJ}+\epsilon_2}{2}}$$

- Again convenient to take $\eta = (1,\cdots,1)$
- All charges responsible for poles are of the form e_i, $e_i - e_j$ $(i,j = 1,\cdots,k)$.
- k-some of charge vectors spanning η can be collected in to N or less groups:

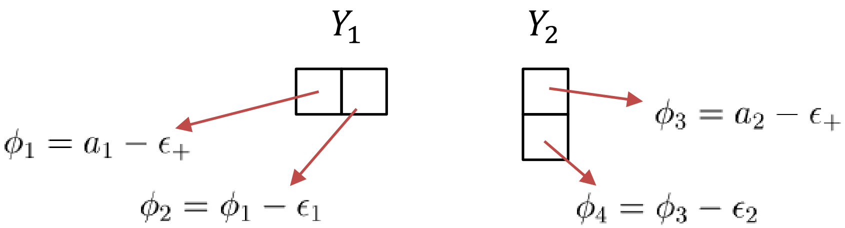
$$e_1,\ e_2 - e_1, \cdots,\ e_{k_1} - e_{k_1-1};$$
$$e_{k_1+1},\ e_{k_1+2} - e_{k_1+1}, \cdots,\ e_{k_1+k_2} - e_{k_1+k_2-1};$$
$$\cdots$$
$$e_{k_1+\cdots+k_{N-1}+1},\ e_{k_1+\cdots+k_{N-1}+2} - e_{k_1+\cdots+k_{N-1}+1}, \cdots,\ e_{k_1+\cdots+k_N} - e_{k_1+\cdots k_N-1}$$

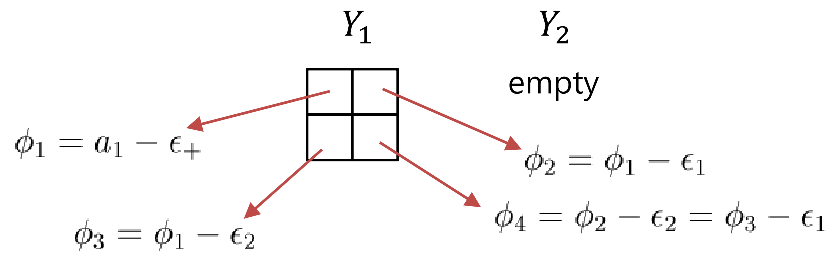
up to the Weyl symmetry of U(k),which permutes various e_i's , and $k_i \geq 0$, $k_1 + \cdots + k_N = k$.

- When e_I is picked, comes from one of $\epsilon_+ + \phi_I - a_i = 0$: Two different ϕ_I, ϕ_J's cannot come with same a_i since $\sinh(\phi_{IJ}/2) = 0$ on numerator (not a pole). So no more than N groups.
- Each chosen $e_{I+1} - e_I$ comes from either of $\phi_{I+1,I} + \epsilon_{1,2} = 0$.
- These leads the to so-called "colored Young diagram classification" of poles for SU(N).

Young diagrams

- Prepare N Young diagrams, with total box number is k. E.g. at N=2, k=4





Start from the upper/left corner.
Move left by ϵ_1, and down by ϵ_2.

Paths ambiguous between two boxes. It means more than $r(=4)$ factors vanish on denominator. Young diagrams label poles without redundancy.

- For a given Young diagram, pole location:

$$\phi(s) = a_i - \epsilon_+ - (n-1)\epsilon_1 - (m-1)\epsilon_2 \;, \quad s = (m,n) = (\text{row}, \text{column}) \in Y_i$$

- Residue (after some calculus & numerator/denom. cancelations):

$$Z_k = (-1)^{kN} \sum_{\sum_i |Y_i| = k} \prod_{i=1}^{N} \prod_{s \in Y_i} \frac{1}{\prod_{j=1}^{N} 2\sinh\frac{E_{ij}(s)}{2} \cdot 2\sinh\frac{E_{ij}(s) - 2\epsilon_+}{2}}$$

$$E_{ij}(s) = a_i - a_j - \epsilon_1 h_i(s) + \epsilon_2 (v_j(s) + 1)$$

[Same formula can be derived from other methods: Higgs branch localization, or closely related fixed point formulae, or topological string/vertex calculus. Different methods have different virtues and generalizations. E.g. our gauge theory & contour integral method can be easily extended to other gauge groups.]

Comments on 4d/6d results

- 4d: 0d matrix integral: matrices without modes on S^1
- Replace all $2\sinh x/2$ factors by x, both in contour integral and residue sum
- Yields the 4d instanton partition function for its original use (SW theory)

- 6d gauge theory: 2d QFT. Matrices' modes on T^2 labeled by 2 towers of integers
- Replace all

$$2\sinh\frac{x}{2} \to \frac{i\theta_1(\tau|\frac{x}{2\pi i})}{\eta(\tau)} \equiv q^{\frac{1}{12}} \cdot 2\sinh\frac{x}{2}\prod_{n=1}^{\infty}(1-q^n y)(1-q^n y^{-1})$$

$$q \equiv e^{2\pi i\tau} \ , \quad y \equiv e^{-x}$$

- Note: There are many new 2d subtleties. For instance, 2d uplifts for many models won't make sense, due to gauge anomalies, etc. So before uplifting the 1d formulae, should first check if 2d ADHM makes sense as a QFT. (3rd lecture)