Calabi-Yau manifolds and sporadic groups

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Based on:

Calabi-Yau manifolds and sporadic groups [arXiv:1711.09698]

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INSTITUTE FOR THEORETICAL PHYSICS

FWF Der Wissenschaftsfonds.

Outline

Overview of the talk

Where does the moon shine?

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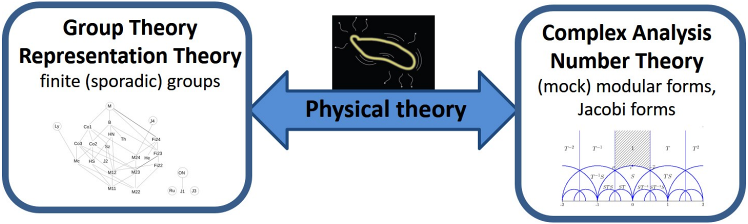
- A better answer would be John MaKay's observation in 70's

$$196884 = 1 + 196883$$

- Broadly, "Moonshine" refers to some connection between two apparently different mathematical objects which a priori has nothing to do with each other.



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  - Though the interplay between String theory and maths have been very fruitful to geometry, not many results are known on the number theory side. Moonshine seems to be a great opportunity to develop the number theoretic aspects of string theory.
  - We get to construct systems with large sym. groups.
  - Structure of BPS spectra etc.

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  - Structure of BPS spectra etc.
- Lot is going on now...
  - There are more & more examples. The meaning of "Moonshine" is everchanging.
  - The "Origin" problem is becoming clearer.
  - Of particular interest is symmetries of "$K3$–ish" string compactifications.

Overview of the talk

Finite Simple groups

- Just like prime factorization of natural numbers we can think about finite groups in terms of "Building blocks".
- Notion of "Composite series" of normal subgroups builds finite groups out of a set of "primes"–finite simple groups.
- The full classification is probably one of the greatest work of mathematics in 20^{th} century. [Atlas, *Robert Wilson*.]
- There are 18 Infinite families, e.g.
  - Alternating group of n elements A_n

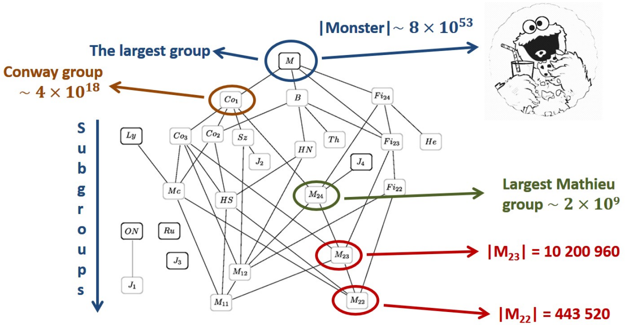
$$A_3 \ : \ (123) \longleftrightarrow (231) \longleftrightarrow (312)$$

  - Cyclic group of prime order C_p

$$C_p = \mathbb{Z}_p = \left\langle e^{\frac{2\pi i}{p}} \right\rangle$$

Sporadic groups

There are 26 so called sporadic groups which don't fall into the infinite families.



Modular forms

For τ in the UHP $\supset$ (Fnd. domain) and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$:

- Modular form of weight k:

$$\phi_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \phi_k(\tau)$$

- Jacobi form of weight k and index m with $\lambda, \mu \in \mathbb{Z}$:

$$\begin{aligned} \phi_{k,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi_{k,m}(\tau,z) \\ \phi_{k,m}(\tau, z+\lambda\tau+\mu) &= (-1)^{2m(\lambda+\mu)} e^{-2\pi i m(\lambda^2\tau+2\lambda z)} \phi_{k,m}(\tau,z) \end{aligned}$$

- $\tau \to \tau+1$ and $z \to z+\mu$ allows for a Fourier expansion:

$$\phi_{k,m}(q=e^{2\pi i\tau}, y=e^{2\pi i z}) = \sum_{n,r} c(n,r) q^n y^r \quad r^2 > 4nm$$

Eisenstein series

The Eisenstein series have the following Fourier decomposition

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240q + 2160q^2 + \dots\,,$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} = 1 - 504q - 16632q^2 + \dots\,.$$

Standard examples of holomorphic modular forms. Unfortunately, the space of holomorphic modular forms is too restrictive, it is just the ring of monomials $E_4^{\alpha} E_6^{\beta}$

Generalizations

- Multipliers: Allow for a phase $\psi : SL_2(\mathbb{Z}) \rightarrow C^*$ in transformation, e.g., Dedekind eta fn. $\eta(\tau) = e^{\frac{2\pi i}{24}}\eta(\tau+1)$

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- Poles: Allow the function to have exponetial growth near the cusps. (Weakly holomorphic). e.g. the $J(\tau)$ function (Hauptmodul), For a gives pole structure at cusp $i\infty$ and up to a constant it is a unique function which maps the fundamental domain (an S^2) to compactified $\mathbb{C}$ (an S^2).

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- Subgroups: Consider a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of the modular group for which we impose the transformation property. e.g. Twined partition function $Tr < g >$ for Monster.

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- Now, we have a zoo of interesting species.

Jacobi Theta functions

The Jacobi theta functions $\theta_i(\tau, z)$, $i = 1, \cdots, 4$ are defined as

$$\theta_1(\tau, z) = -\mathrm{i} \sum_{n+\frac{1}{2}\in\mathbb{Z}} (-1)^{n-\frac{1}{2}} y^n q^{\frac{n^2}{2}}$$

$$= -\mathrm{i} q^{\frac{1}{8}} \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)\left(1 - y^{-1}q^n\right) ,$$

$$\theta_2(\tau, z) = \sum_{n+\frac{1}{2}\in\mathbb{Z}} y^n q^{\frac{n^2}{2}}$$

$$= q^{\frac{1}{8}} \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)\left(1 + y^{-1}q^n\right) ,$$

Jacobi Theta functions

$$\theta_3(\tau, z) = \sum_{n \in \mathbb{Z}} y^n q^{\frac{n^2}{2}}$$
$$= \prod_{n=1}^{\infty} (1 - q^n) \left(1 + y q^{n-\frac{1}{2}}\right) \left(1 + y^{-1} q^{n-\frac{1}{2}}\right) ,$$
$$\theta_4(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n y^n q^{\frac{n^2}{2}}$$
$$= \prod_{n=1}^{\infty} (1 - q^n) \left(1 - y q^{n-\frac{1}{2}}\right) \left(1 - y^{-1} q^{n-\frac{1}{2}}\right) ,$$

Jacobi forms

$$\begin{aligned}
\phi_{0,1}(\tau,z) &= 4\left(\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2+\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\right)\\
&= \frac{1}{y}+10+y+\mathcal{O}(q)\,,\\
\phi_{-2,1}(\tau,z) &= \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\\
&= -\frac{1}{y}+2-y+\mathcal{O}(q)\,,\\
\phi_{0,\frac{3}{2}}(\tau,z) &= 2\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\\
&= \frac{1}{\sqrt{y}}+\sqrt{y}+\mathcal{O}(q)\,.
\end{aligned}$$

Monster moonshine

- The irreducible representations of the Monster group have dimensions 1, 196 883, 21 296 876, $\cdots$
- The J–function, that appears in many places in string theory, enjoys the expansion

$$J(q) = \frac{1}{q} + \underbrace{196884}_{1 + 196883}\, q + \underbrace{21493760}_{1 + 196883 + 21296876}\, q^2 + \ldots$$

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- Concrete realization : The (left-moving) bosonic string compactified on a $\mathbb{Z}_2$ orbifold of R^{24}/Λ with Λ the Leech lattice (even, self–dual) has as its 1–loop partition function the $J(q)$–function. [*Frenkel, Lepowsky, Meurman '88*]

Monster moonshine

no massless q^0 states

$$Z(q) = \mathrm{Tr}_{\mathrm{H}}\, q^{L_0 - \frac{c}{24}} = J(q) = \frac{1}{q} + 196884\, q + 21493760\, q^2 + \ldots$$

tachyon of the bosonic string

supermassive string states

- The symmetry group of the compactification space $R^{24}/\Lambda/\mathbb{Z}_2$ is the Monster group.
- Virasoro algebra : Expand the $J(q)$–function in terms of Virasoro characters (traces of Verma modules)

$$ch_{h=0}(q) = \frac{q^{-c/24}}{\prod_{n=2}^{\infty}(1-q^n)}\ ; \quad ch_h(q) = \frac{q^{h-c/24}}{\prod_{n=1}^{\infty}(1-q^n)}$$

Monster moonshine

$$J(q) = \frac{1}{q} + 196884\,q + 21493760\,q^2 + \ldots$$
$$= \boxed{1}\,\mathrm{ch}_0(q) + \boxed{196883}\,\mathrm{ch}_2(q) + \boxed{21296876}\,\mathrm{ch}_3(q) + \ldots$$

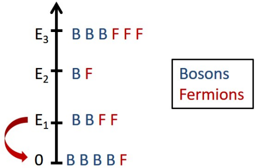
- Other realizations in terms of 23 Niemeier lattices. Construct from ADE root systems with glueing vectors. They are related to Umbral moonshine. Adds const. to $J(q)$.
- Interesting for mathematicians not so interesting for physicists
  - Compactification of the bosonic string:
    - We have a tachyon (instability).
    - Spacetime theory has no fermions.
  - Additionally, only two spacetime dimensions are non–compact.
- There exists various supersymmetric generalizations of mainly Extremal CFT constructions with different "Moonshine".

Overview of the talk

Elliptic genus

Elliptic genus : Defined for a SCFT with $N = (2, 2)$ or more SUSY. An index is invariant under deformations of the theory, e.g. masses going to zero in Witten Index.

$$Z_{\text{Witten}} = \mathrm{Tr}(-1)^F = n_B - n_F = \mathrm{Tr}\left((-1)^F q^H\right)$$



$$Z_{\text{elliptic}}(q, y) = \mathrm{Tr}_{\text{RR}}\left((-1)^{F_L} q^{L_0 - \frac{c}{24}} y^{J_0} (-1)^{F_R} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right)$$

Chemical potential for U(1) in left-moving N=2 theory

No dependence on $\bar{q}$!

$N = 2$ Characters

- For central charge $c = 3d$ and heighest weight state $|\Omega\rangle$ with eigenvalues h, l w.r.t. L_0 and J_0.

$$\mathrm{ch}^{\mathcal{N}=2}_{d,h-\frac{c}{24},\ell}(\tau,z) = \mathrm{tr}_{\mathcal{H}_{h,\ell}}((-1)^F q^{L_0-\frac{c}{24}} e^{2\pi i z J_0})$$

- In the Ramond sector unitarity requires $h \geq \frac{c}{24} = \frac{d}{8}$.
- Massless (BPS) representations exist for $h = \frac{d}{8}$; $\ell = \frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-2, \ldots, -(\frac{d}{2}-1), -\frac{d}{2}$. For $\frac{d}{2} > \ell \geq 0$

$$\mathrm{ch}^{\mathcal{N}=2}_{d,0,\ell\geq 0}(\tau,z) = (-1)^{\ell+\frac{d}{2}} \frac{(-i)\theta_1(\tau,z)}{\eta(\tau)^3} y^{\ell+\frac{1}{2}} \sum_{n\in\mathbb{Z}} q^{\frac{d-1}{2}n^2+(\ell+\frac{1}{2})n} \frac{(-y)^{(d-1)n}}{1-yq^n}$$

- Massive (non-BPS) representations exist for $h > \frac{d}{8}$; $\ell = \frac{d}{2}, \frac{d}{2}-1, \ldots, -(\frac{d}{2}-1), -\frac{d}{2}$ and $\ell \neq 0$ for $d =$ even.

$$\mathrm{ch}^{\mathcal{N}=2}_{d,h-\frac{c}{24},\ell>0}(\tau,z) = (-1)^{\ell+\frac{d}{2}} q^{h-\frac{d}{8}} \frac{i\theta_1(\tau,z)}{\eta(\tau)^3} y^{\ell-\frac{1}{2}} \sum_{n\in\mathbb{Z}} q^{\frac{d-1}{2}n^2+(\ell-\frac{1}{2})n} (-y)^{(d-1)n}$$

Weak Jacobi forms

- The space of weak Jacobi forms of even weight k and integer index m is generated by [*Zagier et. al. '85; Gritsenko '99*]

$$E_4(\tau), E_6(\tau), \phi_{-2,1}(\tau, z), \phi_{0,1}(\tau, z)$$

- Simple combinatorics gives the space $J_{0,m}$ of Jacobi forms of weight 0 and index m, is generated by m functions for $m = 1, 2, 3, 4, 5$. In particular, we have

$$\begin{aligned}
J_{0,1} &= \langle \phi_{0,1} \rangle\,, \\
J_{0,2} &= \langle \phi_{0,1}^2, E_4\, \phi_{-2,1}^2 \rangle\,, \\
J_{0,3} &= \langle \phi_{0,1}^3, E_4\, \phi_{-2,1}^2\, \phi_{0,1}, E_6\, \phi_{-2,1}^3 \rangle\,, \\
J_{0,4} &= \langle \phi_{0,1}^4, E_4\, \phi_{-2,1}^2\, \phi_{0,1}^2, E_6\, \phi_{-2,1}^3\, \phi_{0,1}, E_4^2\, \phi_{-2,1}^4 \rangle\,, \\
J_{0,5} &= \langle \phi_{0,1}^5, E_4\, \phi_{-2,1}^2\, \phi_{0,1}^3, E_6\, \phi_{-2,1}^3\, \phi_{0,1}^2, E_4^2\, \phi_{-2,1}^4\, \phi_{0,1}, E_4\, E_6\, \phi_{-2,1}^5 \rangle
\end{aligned}$$

Weak Jacobi forms

- The functions $J_{0,\frac{d}{2}}$ above appear in the elliptic genus of Calabi-Yau $d = 2, 4, 6, 8, 10$ target manifolds.
- Coefficients can be fixed in terms of a few topological numbers of the CY d–fold.

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- Coefficients can be fixed in terms of a few topological numbers of the CY d–fold.
- Weight zero, half integer index Jacobi forms, follows from

$$J_{2k,m+\frac{1}{2}} = \phi_{0,\frac{3}{2}} J_{2k,m-1}\,, \qquad m \in \mathbb{Z}$$

- In particular, $\phi_{0,\frac{3}{2}}$ and $\phi_{0,\frac{3}{2}}\phi_{0,1}$ are, up to rescaling, the unique Jacobi forms of weight 0 and index $\frac{3}{2}$ and $\frac{5}{2}$, respectively.
- Generally, the space $J_{0,m+\frac{3}{2}}$ is spanned by m functions for $m = 1, 2, 3, 4, 5$ and these functions are the ones given in previous slide multiplied by $\phi_{0,\frac{3}{2}}$.
- Summary : Space of Jacobi forms $J_{0,\frac{d}{2}}$ is generated by very few functions for small d. Carries little info. about the CY.

Calabi–Yaus

- $\mathcal{Z}_{CY_d}(\tau, z) = \sum_{p=0}^{d} (-1)^p \chi_p(CY_d) y^{\frac{d}{2}-p} + \mathcal{O}(q)$
- Various signed indices $\chi_p(CY_d) = \sum_{r=0}^{d} (-1)^r h^{p,r}$.
- Euler no. : $\mathcal{Z}_{CY_d}(\tau, 0) = \chi_{CY_d} = \sum_{p=0}^{d} (-1)^p \chi_p(CY_d)$.

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- For larger d many different Calabi-Yau d-folds will give rise to the same elliptic genus since the number of Calabi-Yau manifolds grows much faster with d than the number of basis elements of $J_{0,\frac{d}{2}}$.
- Question : If one finds interesting expansion coefficients in higher dimensional manifolds, i.e the expansion coefficients are given in terms of irreducible representations of a particular sporadic group, does this imply that all manifolds with such elliptic genus are connected to the particular sporadic group, or only a few or none?

Calabi-Yaus

- Calabi-Yau 1-fold : For the standard torus T^2 the elliptic genus vanishes, $\mathcal{Z}_{T^2}(\tau, z) = 0$. The same holds true for any even dimensional torus $\mathcal{Z}_{T^{2n}}(\tau, z) = 0$, $\forall n \in \mathbb{N}$. This is due to the fermionic zero modes in the right moving Ramond sector.

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- Calabi-Yau 2-fold : Non-trivial example is $K3$ surface. Elliptic genus is a Jacobi form that appears in Mathieu Moonshine.

  [*Tachikawa et.al. '10 ; Gaberdiel et.al. '10, Mukai '88*]

  $\mathcal{Z}_{K3}(\tau, z) = 2\phi_{0,1}(\tau, z) = -20\,\mathrm{ch}^{\mathcal{N}=2}_{2,0,0}(\tau, z) + 2\,\mathrm{ch}^{\mathcal{N}=2}_{2,0,1}(\tau, z) - \sum_{n=1}^{\infty} A_n \mathrm{ch}^{\mathcal{N}=2}_{2,n,1}(\tau, z)$

  The coefficients are irreps of M_{24}.

$$\begin{aligned} 20 &= 23 - 3\cdot 1\,, \\ -2 &= -2\cdot 1\,, \\ A_1 &= 45 + \underline{45}\,, \\ A_2 &= 231 + \underline{231}\,, \\ A_3 &= 770 + \underline{770} \end{aligned}$$

Calabi-Yaus

- Calabi-Yau 3-fold : Unfortunately, rather uninteresting expansion in $N = 2$ characters
  $\mathcal{Z}_{CY_3}(\tau, z) = \frac{\chi_{CY_3}}{2}\ \phi_{0,\frac{3}{2}} = \frac{\chi_{CY_3}}{2}\left(\mathrm{ch}^{\mathcal{N}=2}_{3,0,\frac{1}{2}}(\tau, z) + \mathrm{ch}^{\mathcal{N}=2}_{3,0,-\frac{1}{2}}(\tau, z)\right)$
  Doesn't mean no connection to moonshine, e.g. By Heterotic Type II duality connection between Vafa's New SUSY index (One loop correction to prepotential in Het. side) connects to Gromov-Witten inv. on Type II side. [*Wrase '14 ; Aradhita's talk*]

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- Calabi-Yau 4-folds : Coefficients of $J_{0,2}$ is fixed by Euler no. χ_{CY_4} and $\chi_0 = \sum_r (-1)^r h^{0,r} = h^{0,0} + h^{0,4} = 2$ (for gen. CY_4)
$$\mathcal{Z}_{CY_4}(\tau, z) = \frac{\chi_{CY_4}}{144}\left(\phi^2_{0,1} - E_4\, \phi^2_{-2,1}\right) + \chi_0 E_4\, \phi^2_{-2,1}$$
Obvious e.g. is $\mathcal{Z}_{K3\times K3}(\tau, z) = 4\phi^2_{0,1}$ (not a gen. CY_4), exhibits an $\mathrm{M}_{24}\times\mathrm{M}_{24}$ symmetry. Many, other connections.
[work in progress, Cheng et.al.]

Calabi-Yau 5-folds

- Elliptic genus is proportional to $\phi_{0,\frac{3}{2}}\phi_{0,1}$ and we can fix the prefactor in terms of the Euler number χ_{CY_5}.

$$\begin{aligned}\mathcal{Z}_{CY_5}(\tau,z) &= \frac{\chi_{CY_5}}{24}\phi_{0,\frac{3}{2}}\phi_{0,1}\\ &= -\frac{\chi_{CY_5}}{48}\Bigg[22\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)\\ &\qquad -2\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &\qquad +\sum_{n=1}^{\infty}A_n\left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right)\Bigg]\end{aligned}$$

- In particular, for CY 5-folds with $\chi_{CY_5}=-48$ we find essentially the same expansion coefficients as in Mathieu moonshine, while for $\chi_{CY_5}=-24$ we find essentially the same coefficients as for Enriques moonshine.

Twined Elliptic Genus

- Since, the elliptic genus is effectively the same for a huge class of 5-folds, it stands to reason that we should check the Twinings by elements of M_{24}.
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- We didn't expect to weed out the huge class of potential CY's we scanned.
- Toric code : The Calabi-Yau manifolds we are interested in are hypersurfaces in weighted projective ambient spaces. A particular Calabi-Yau d-fold that is a hypersurface in the weighted projective space $\mathbb{CP}^{d+1}_{w_1 \ldots w_{d+2}}$ is determined by a solution of $p(\Phi_1, \ldots, \Phi_{d+2}) = 0$, where the Φ_i denote the homogeneous coordinates of the weighted projective space and p is a transverse polynomial of degree $m = \sum_i w_i$.

Twined Elliptic Genus

- Mapping [*Benini et.al.*] : Two-dim. gauged linear sigma model with $N = (2,2)$ SUSY.
  - $U(1)$ gauge field under which the chiral multiplets Φ_i have charge w_i. Additionally, one extra chiral multiplet X with $U(1)$ charge $-m$.
  - Invariant superpotential $W = Xp(\Phi_1, \ldots, \Phi_{d+2})$
  - The F-term equation $\partial W/\partial X = p = 0$ restricts us to the Calabi-Yau hypersurface above.
  - R-charge : Zero for Φ_i and 2 for X.

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  - R-charge : Zero for Φ_i and 2 for X.
- Refined Elliptic genus : Extra chemical potential $x = e^{2\pi i u}$

$$\mathcal{Z}_{\mathrm{ref}}(\tau, z, u) = \mathrm{Tr}_{RR}\left((-1)^{F_L} y^{J_0} q^{L_0-\frac{d}{8}} x^Q (-1)^{F_R} \bar{q}^{\bar{L}_0-\frac{d}{8}}\right)$$

Twined Elliptic Genus

- Contribution to Refined elliptic genus :
  - Each chiral multiplet of $U(1)$ charge Q and $\mathcal{R}$-charge R

$$\mathcal{Z}^{\Phi}_{\rm ref}(\tau, z, u) = \frac{\theta_1\left(\tau, \left(\frac{R}{2} - 1\right) z + Qu\right)}{\theta_1\left(\tau, \frac{R}{2} z + Qu\right)}$$

  - Abelian vector field

$$\mathcal{Z}^{vec}_{\rm ref}(\tau, z) = \frac{\mathrm{i}\eta(\tau)^3}{\theta_1(\tau, -z)}$$

  - Combined :

$$\mathcal{Z}_{\rm ref}(\tau, z, u) = \frac{\mathrm{i}\eta(\tau)^3}{\theta_1(\tau, -z)} \frac{\theta_1\left(\tau, -mu\right)}{\theta_1\left(\tau, z - mu\right)} \prod_{i=1}^{d+2} \frac{\theta_1\left(\tau, -z + w_i u\right)}{\theta_1\left(\tau, w_i u\right)}$$

- Standard elliptic genus is obtained by integrating over u. The integral localizes to sum over contour integrals around poles of u in the integrand.

Twined Elliptic Genus

$$\begin{aligned}\mathcal{Z}_{CY_d}(\tau,z) &= \sum_{k,\ell=0}^{m-1} \frac{e^{-2\pi i \ell z}}{m} \prod_{i=1}^{d+2} \frac{\theta_1\left(\tau, \frac{w_i}{m}(k+\ell\tau+z)-z\right)}{\theta_1\left(\tau, \frac{w_i}{m}(k+\ell\tau+z)\right)} \\ &= \sum_{k,\ell=0}^{m-1} \frac{y^{-\ell}}{m} \prod_{i=1}^{d+2} \frac{\theta_1\left(q, e^{\frac{2\pi i w_i k}{m}} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m}-1}\right)}{\theta_1\left(q, e^{\frac{2\pi i w_i k}{m}} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m}}\right)}\end{aligned}$$

Twined Elliptic Genus

$$\mathcal{Z}_{CY_d}(\tau, z) = \sum_{k,\ell=0}^{m-1} \frac{e^{-2\pi i \ell z}}{m} \prod_{i=1}^{d+2} \frac{\theta_1\left(\tau, \frac{w_i}{m}(k+\ell\tau+z)-z\right)}{\theta_1\left(\tau, \frac{w_i}{m}(k+\ell\tau+z)\right)}$$

$$= \sum_{k,\ell=0}^{m-1} \frac{y^{-\ell}}{m} \prod_{i=1}^{d+2} \frac{\theta_1\left(q, e^{\frac{2\pi i w_i k}{m}} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m}-1}\right)}{\theta_1\left(q, e^{\frac{2\pi i w_i k}{m}} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m}}\right)}$$

Twine the elliptic genus by an Abelian symmetry g :

$$g : \Phi_i \to e^{2\pi i \alpha_i} \Phi_i \,, \qquad i = 1, 2, \ldots, d+2$$

It effectively, leads to a shift of the original z coordinate (i.e. the second argument) of the θ_1-functions for each Φ_i by α_i.

Twining for CY_5

- A list of 5 757 727 CY 5-folds that can be described by reflexive polytopes is given on the TU website.
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Twining for CY_5

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- Out of these 5 757 727 CY 5-folds only 19 353 are described by transverse polynomials in weighted projective spaces.
- For generic χ_{CY_5} (the constant sitting in front of $\mathcal{Z}_{CY_5}(\tau, z)$) we can perform the twining. e.g. For the hypersurface in the weighted projective space $\mathbb{CP}^6_{1,1,1,3,5,9,10}$ with $\chi = -170\,688$ and

$$\mathcal{Z}_{CY_5}(\tau, z) = 3556 \cdot \left[22 \left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau, z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau, z) \right) \cdots \right]$$

- For the $\mathbb{Z}_2$ symmetry

$$\mathbb{Z}_2 : \begin{cases} \Phi_1 \to -\Phi_1 \,, \\ \Phi_2 \to -\Phi_2 \,, \end{cases}$$

Twining for CY_5

- Corresponding twined elliptic genus

$$\begin{aligned}\mathcal{Z}^{tw,2\mathrm{A}}_{CY_5}(\tau,z) \;=\; 14\cdot\Big[&2\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)\\ &-2\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &+6\left(\mathrm{ch}^{\mathcal{N}=2}_{5,1,\frac{3}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,1,-\frac{3}{2}}(\tau,z)\right)+\ldots\Big]\end{aligned}$$

  which is a twined constant, 14 instead of 3556, multiplied by the 2A series of M_{24}.
- Generically, in most cases the $\mathbb{Z}_2$ twining produced a linear combination of $1A$ and $2A$ conjugacy classes of M_{24} hence killing the scope of M_{24} symmetry. Cases, which reproduced say $2A$ were lifted by higher order twinings.

Overview of the talk

Final comments

- It is not absolutely settled as to whether the Mathieu moonshine in $K3$ is a property of the manifold or not? Same questions can be asked for higher dim Calabi-Yau and it seems to be property of the Jacobi form rather than the manifold.

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- We also study CY 5-folds with Euler number different from ± 48, whose elliptic genus expansion agrees with the K3 elliptic genus expansion only up to a prefactor, is that the product spaces $\mathrm{K3}\times\mathrm{CY}_3$ have an elliptic genus that likewise agrees with the K3 elliptic genus expansion only up to a prefactor.

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- The conclusions we drew rules out single copy of Mathieu moonshine but in some cases the reasoning allows for multiple copies but I agree it is preposterous.
- If the goal is to connect the weight zero Jacobi forms to interesting jacobi forms coming from Umbral moonshine, it seems product of CYs doesn't work.

THANK YOU