BLACK HOLE DEGENERACIES FROM MATHIEU MOONSHINE

Based on *Dyon degeneracies from Mathieu moonshine symmetry*
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with Justin R. David

Introduction and Results

- We construct the Siegel modular forms associated with the theta lift of twisted elliptic genera of $K3$ orbifolded with g' corresponding to the conjugacy classes of the Mathieu group M_{24}
- These forms satisfy the required properties for them to be generating functions of $1/4$ BPS dyons of type II string theories
- Inverse of these Siegel modular forms admit a Fourier expansion with integer coefficients and the correct sign as predicted from black hole physics (as conjectured by Sen)
- The correct sign is observed for dyons for all the 7 CHL compactifications and also some non-geometric orbifolds of $K3$

Elliptic genus

- Consider the Elliptic genus of $K3$.

$$F(K3;\tau,z) = \mathrm{Tr}_{RR}\left((-1)^{F^{K3}+\bar{F}^{K3}} e^{2\pi i z F^{K3}} e^{2\pi i\tau(L_0-c/24)} \bar{e}^{-2\pi i\bar{\tau}(\bar{L}_0-\hat{c}/24)}\right)$$

$$= \sum_{m\geq 0,l} c(4m-l^2) e^{2\pi i m\tau} e^{2\pi i l z}$$

The trace is taken over the Ramond sector.

The elliptic genus is holomorphic in τ, z.

Only the ground states of the right movers are counted.

Evaluating the index we obtain

$$F(K3;\tau,z) = 8\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right]$$

Few simple CFT models of $K3$ comes as an orbifold on T^4:

$$T^4/\mathbb{Z}_N \text{ for } N = 2,3,4,6$$

$$\mathbb{Z}_2 : (y_1, y_2, y_3, y_4) \to -(y_1, y_2, y_3, y_4)$$

The Hodge diamond of $K3$ is given by

$$h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} = 1,$$
$$h_{(1,1)} = 20$$

Twisted Elliptic genus

There exists $\mathbb{Z}_N$ quotients of $K3$ for which the Hodge diamond of $K3/\mathbb{Z}_N$ becomes

$$h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} = 1,$$
$$h_{(1,1)} = 2\left(\frac{24}{N+1} - 2\right) = 2k$$

N	$h_{(1,1)}$	k
1	20	10
2	12	6
3	8	4
5	4	2
7	2	1

Let us refer to these $\mathbb{Z}_N$ action by g'.

• Let g' be action of this quotient,
the twisted elliptic genus of $K3$ is defined as

$$
\begin{aligned}
& F^{(r,s)}(\tau, z) \\
= \; & \frac{1}{N} \mathrm{Tr}^{K3}_{RR;g'^r} \left((-1)^{F^{K3}+\bar{F}^{K3}} g'^s e^{2\pi i z F^{K3}} e^{2\pi i \tau (L_0 - c/24)} \bar{q}^{-2\pi i \bar{\tau}(\bar{L}_0 - c/24)} \right) \\
= \; & \sum_{b=0}^{1} \sum_{m \geq 0 \in \mathbb{Z}/N, l \in 2\mathbb{Z}+b} c_b^{(r,s)}(4m - l^2) e^{2\pi i m \tau} e^{2\pi i l z} \\
& 0 \leq r, s, \leq (N-1).
\end{aligned}
$$

These twisted elliptic genera for the $\mathbb{Z}_N$ quotients of $K3$ by g' with $N = 2, 3, 5, 7$ have been written down in
David, Jatkar, Sen (2006)

For the $N = 2$ orbifold the twisted indices are

$$F^{(0,0)}(\tau, z) = 4\left[\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}\right],$$

$$F^{(0,1)}(\tau, z) = 4\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2}, \quad F^{(1,0)}(\tau, z) = 4\frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2},$$

$$F^{(1,1)}(\tau, z) = 4\frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2}.$$

Relation with Mathieu Moonshine

One can write the elliptic genus in terms of the characters of the short and the long representations of the $\mathcal{N}=4$ super conformal algebra

$$Z_{K3}(\tau, z) = 24\mathrm{ch}_{h=\frac{1}{4}, l=0}(\tau, z) + \sum_{n=0}^{\infty} A_n^{(1A)} \mathrm{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(\tau, z).$$

$$\begin{aligned} \mathrm{ch}_{h=\frac{1}{4}, l=0}(\tau, z) &= -i\frac{e^{\pi i z}\theta_1(\tau, z)}{\eta(\tau)^3} \sum_{n=-\infty}^{\infty} \frac{e^{\pi i \tau n(n+1)} e^{2\pi i(n+\frac{1}{2})}}{1 - e^{2\pi i(n\tau + z)}}, \\ \mathrm{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(\tau, z) &= e^{2\pi i \tau(n-\frac{1}{8})} \frac{\theta_1(\tau, z)^2}{\eta(\tau)^2}. \end{aligned}$$

The first few values of $A_n^{(1A)}$ are given by

$$A_n^{(1A)} = -2,\ 90,\ 462,\ 1540,\ 4554,\ 11592, \ldots$$

These coefficients are sums of dimensions of the irreps of the group M_{24}.
Eguchi, Ooguri, Tachikawa (2010)

$$A_n^{(1A)} = \mathrm{Tr}(\mathbf{1})_n, \qquad n > 0$$

Similarly the twining character $F^{(0,1)}$ for $2A$ orbifold admits the decomposition

$$2F^{(0,1)}(\tau,z) = 8\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,z) + \sum_{n=0}^{\infty} A_n^{(2A)} \mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau,z).$$

The first few values of $A_n^{(2A)}$ are given by

$$A_n^{(2A)} = -2,\ -6,\ 14,\ -28,\ 42,\ -56,\ 86,\ -138,\ \ldots$$

These coefficients can be read off from McKay-Thompson series constructed out of trace of the elements g'_{2A} in the $2A$ conjugacy class of the Mathieu group M_{24}.

$$A_n^{(2A)} = \mathrm{Tr}(g'_{2A})_n$$

Cheng (2010), Gaberdiel, Hohenegger, Volpato (2010)

A few facts

- The group $M_{24} \subset S_{24}$ is of order 244823040 $\sim 2 \times 10^8$
- If one of the elements in M_{24} remain fixed one gets the subgroup M_{23}
- M_{24} admits 26 conjugacy classes of which 16 belong to M_{23}

Conjucay Class	Order	Cycle shape	Cycle
1A	1	1^{24}	()
2A	2	$1^8 \cdot 2^8$	(1, 8)(2, 12)(4, 15)(5, 7)(9, 22)(11, 18)(14, 19)(23, 24)
3A	3	$1^6 \cdot 3^6$	(3, 18, 20)(4, 22, 24)(5, 19, 17)(6, 11, 8)(7, 15, 10)(9, 12, 14)
5A	4	$1^4 \cdot 5^4$	(2, 21, 13, 16, 23)(3, 5, 15, 22, 14)(4, 12, 20, 17, 7)(9, 18, 19, 10, 24)
7A	7	$1^3 \cdot 7^3$	(1, 17, 5, 21, 24, 10, 6)(2, 12, 13, 9, 4, 23, 20)(3, 8, 22, 7, 18, 14, 19)
7A	7	$1^3 \cdot 7^3$	(1, 21, 6, 5, 10, 17, 24)(2, 9, 20, 13, 23, 12, 4)(3, 7, 19, 22, 14, 8, 18)
11A	11	$1^2 \cdot 11^2$	(1, 3, 10, 4, 14, 15, 5, 24, 13, 17, 18)(2, 21, 23, 9, 20, 19, 6, 12, 16, 11, 22)
23A	23	$1^1 \cdot 23^1$	(1, 7, 6, 24, 14, 4, 16, 12, 20, 9, 11, 5, 15, 10, 19, 18, 23, 17, 3, 2, 8, 22, 21)
23B	23	$1^1 \cdot 23^1$	(1, 4, 11, 18, 8, 6, 12, 15, 17, 21, 14, 9, 19, 2, 7, 16, 5, 23, 22, 24, 20, 10, 3)
4B	4	$1^4 \cdot 2^2 \cdot 4^4$	(1, 17, 21, 9)(2, 13, 24, 15)(3, 23)(4, 14, 5, 8)(6, 16)(12, 18, 20, 22)
6A	6	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$	(1, 8)(2, 24, 11, 12, 23, 18)(3, 20, 10)(4, 15)(5, 19, 9, 7, 14, 22)(6, 16, 13)
8A	8	$1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$	(1, 13, 17, 24, 21, 15, 9, 2)(3, 16, 23, 6)(4, 22, 14, 12, 5, 18, 8, 20)(7, 11)
14A	14	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1, 12, 17, 13, 5, 9, 21, 4, 24, 23, 10, 20, 6, 2)(3, 18, 8, 14, 22, 19, 7)(11, 15)
14B	14	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1, 13, 21, 23, 6, 12, 5, 4, 10, 2, 17, 9, 24, 20)(3, 14, 7, 8, 19, 18, 22)(11, 15)
15A	15	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2, 13, 23, 21, 16)(3, 7, 9, 5, 4, 18, 15, 12, 19, 22, 20, 10, 14, 17, 24)(6, 8, 11)
15B	15	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2, 23, 16, 13, 21)(3, 12, 24, 15, 17, 18, 14, 4, 10, 5, 20, 9, 22, 7, 19)(6, 8, 11)

Table: Conjugacy classes of $M_{23} \subset M_{24}$ (Type 1)

Conjucay Class	Order	Cycle shape	Cycle
2B	4	2^{12}	(1, 8)(2, 10)(3, 20)(4, 22)(5, 17)(6, 11)(7, 15)(9, 13)(12, 14)(16, 18)(19, 23)(21, 24)
3B	9	3^{8}	(1, 10, 3)(2, 24, 18)(4, 13, 22)(5, 19, 15)(6, 7, 23)(8, 21, 12)(9, 16, 17)(11, 20, 14)
12B	144	12^{2}	(1, 12, 24, 23, 10, 8, 18, 6, 3, 21, 2, 7)(4, 9, 11, 15, 13, 16, 20, 5, 22, 17, 14, 19)
6B	36	6^{4}	(1, 24, 10, 18, 3, 2)(4, 11, 13, 20, 22, 14)(5, 17, 19, 9, 15, 16)(6, 21, 7, 12, 23, 8)
4C	16	4^{6}	(1, 23, 18, 21)(2, 12, 10, 6)(3, 7, 24, 8)(4, 15, 20, 17)(5, 14, 9, 13)(11, 16, 22, 19)
10A	20	$2^{2} \cdot 10^{2}$	(1, 8)(2, 18, 21, 19, 13, 10, 16, 24, 23, 9)(3, 4, 5, 12, 15, 20, 22, 17, 14, 7)(6, 11)
21A	63	$3^{1} \cdot 21^{1}$	(1, 3, 9, 15, 5, 12, 2, 13, 20, 23, 17, 4, 14, 10, 21, 22, 19, 6, 7, 11, 16) (8, 18, 24)
21B	63	$3^{1} \cdot 21^{1}$	(1, 12, 17, 22, 16, 5, 23, 21, 11, 15, 20, 10, 7, 9, 13, 14, 6, 3, 2, 4, 19)(8, 24, 18)
4A	8	$2^{4} \cdot 4^{4}$	(1, 4, 8, 15)(2, 9, 12, 22)(3, 6)(5, 24, 7, 23)(10, 13)(11, 14, 18, 19)(16, 20)(17, 21)
12A	24	$2^{1} \cdot 4^{1} \cdot 6^{1} \cdot 12^{1}$	(1, 15, 8, 4)(2, 19, 24, 9, 11, 7, 12, 14, 23, 22, 18, 5)(3, 13, 20, 6, 10, 16)(17, 21)

Table: Conjugacy classes of $M_{24} \notin M_{23}$ (Type 2)

Using the McKay thompson series associated with each of these 26 conjugacy classes one can write down the twining character $F^{(0,1)}$ for each of the 26 classes.

Closed form expressions for these were given by Cheng (2010), Eguchi (2010), Gaberdiel, Hohenegger, Volpato (2010)

Therefore M_{24} symmetry of the elliptic genus, points to the existence of 26 quotients of $K3$.

Transformation property of twisted elliptic genus

Modular transformations relate these elements by

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=\exp\left(2\pi i\frac{cz^2}{c\tau+d}\right)F^{(cs+ar,ds+br)}(\tau,z)$$

with

$$a,b,c,d\in\mathbb{Z},\qquad ad-bc=1.$$

The indices $cs+ar$ and $ds+br$ belong to $\mathbb{Z}$ mod N.

Explicit closed form expressions for the all the components of the twisted elliptic genus was obtained for [all the classes in type 1 and the first two classes in type 2].
Chattopadhyaya, David (2017)

To determine the twisted elliptic genus in the sectors unrelated to the $F^{(0,1)}$ we use the cycle shape of the conjugacy class in M_{23} and for 2B and 3B we used a torus model ($\sum_{r,s} F^{(r,s)} = 0$)

Using the twisted elliptic genus we can compute the Hodge numbers and identify the classes $4B, 6A, 8A$ together with $2A, 3A, 5A, 7A$ to that of CHL orbifolds.

Many of these are not geometric quotients.

For eg. the $11A$ conjugacy class, using the twisted elliptic genus we can obtain what would have correspond to the hodge number $h^{(1,1)}$.
It turns out this vanishes. Thus even the Kähler form of $K3$ is projected out.

The non-geometric ones are
$11A, 14A/B, 15A/B, 23A/B, 2B, 3B$.

There is an explicit CFT construction in terms of 6 $SU(2)$ WZW theories at level 1 whose twisted elliptic genera is given by the $F^{(0,1)}$ of the 2B orbifold.

Twisted elliptic genera and counting Black Hole degeneracy

Consider type II B/A theory on $K3 \times T^2/\mathbb{Z}_N$ where the $\mathbb{Z}_N$ action is g' on $K3$ and a shift of $1/N$ on one of the circles of T^2.

These compactifications preserve $\mathcal{N} = 4$ supersymmetry in $d = 4$.

This gives a class of new $\mathcal{N} = 4$ string vacua.

Each of these vacua admit $1/4$ BPS states.
These are dyons with both electric and magnetic charges.

For large charges they can be identified with supersymmetric black hole solutions.

The generating function for the degeneracy (index) of dyons in these $\mathcal{N} = 4$ theories is given by

$$-B_6 = -(-1)^{Q\cdot P}\int_{\mathcal{C}} \mathrm{d}\rho \mathrm{d}\sigma \mathrm{d}v\ e^{-\pi i(N\rho Q^2+\sigma/NP^2+2vQ\cdot P)}\frac{1}{\tilde{\Phi}(\rho,\sigma,v)},$$

where $\mathcal{C}$ is a contour in the complex 3-plane. Q, P refer to the electric and magnetic charge of the dyons.

Dijkgraaf, Verlinde, Verlinde (1996), Jatkar Sen (2005), David, Jatkar, Sen (2006), David, Sen (2006), Dabholkar Nampuri (2006)

The contour $\mathcal{C}$ is defined over a 3 dimensional subspace of the 3 complex dimensional space $(\rho = \rho_1 + i\rho_2, \sigma = \sigma_1 + i\sigma_2, v = v_1 + iv_2)$.

$$\rho_2 = M_1, \qquad \sigma_2 = M_2, \qquad v_2 = -M_3,$$
$$0 \leq \rho_1 \leq 1, \qquad 0 \leq \sigma_1 \leq N, \qquad 0 \leq v_1 \leq 1.$$
$$M_1, M_2 >> 0, \quad M_3 << 0, \qquad |M_3| << M_1, M_2$$

$\tilde{\Phi}(\rho, \sigma, v)$ is the Siegel modular form associated with the twisted elliptic genus is given by

$$\tilde{\Phi}(\rho, \sigma, v) = e^{2\pi i(\tilde{\alpha}\rho + \tilde{\beta}\sigma + v)}$$

$$\prod_{b=0,1} \prod_{r=0}^{N-1} \prod_{\substack{k' \in \mathcal{Z} + \frac{r}{N}, l \in \mathcal{Z}, \\ j \in 2\mathcal{Z} + b \\ k', l \geq 0, j < 0 k' = l = 0}} (1 - e^{2\pi i(k'\sigma + l\rho + jv)})^{\sum_{s=0}^{N-1} e^{2\pi i s l/N} c_b^{r,s}(4k'l - j^2)}.$$

where

$$\tilde{\beta} = \frac{1}{N}, \qquad \tilde{\alpha} = 1$$

Here N is the order of the orbifold action.
This Siegel modular form transforms as a weight k form under appropriate sub-groups of $Sp(2, \mathbb{Z})$.

The modular property is defined as follows. Let

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$

Then

$$\tilde{\Phi}_k((C\Omega + D)^{-1}(A\Omega + B)) = [\det(C\Omega + D)]^k \tilde{\Phi}_k(\Omega)$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{4\times 4} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A, B, C, D are 2×2 matrices with integer elements.

The weight k is related to the low lying coefficients of the twisted elliptic genus and is given by

$$k = \frac{1}{2} \sum_{0}^{N-1} c^{(0,s)}(0).$$

$-B_6$

In the context of N= 4 supersymmetric string theories in four dimensions the 6th helicity trace index B_6 which corresponds to 12 broken supersymmetries (1/4 BPS dyons) can be given by

$$B_6 = \frac{1}{6!}\mathrm{Tr}((-1)^{2h}(2h)^6)$$

where h is the third component of the angular momentum of a state in the rest frame, and the trace is taken over all states carrying a given set of charges.

From the above definition we require $-B_6$ to be positive for single centered black holes.

Also from the analysis of Kiritsis 97 we have $-B_6 \sim e^{S_{BH}}$, where S_{BH} is the extremal black hole entropy.

Two tests for this degeneracy formula

Test 1

Comparison of the statistical entropy with the Wald entropy.

Using a saddle point analysis of the contour determining the degeneracy we can find the degeneracy and entropy for large charges

$$S(Q,P) = \pi\sqrt{Q^2P^2-(Q\cdot P)^2} + \ln(h^{(k+2)}(\tau)) + \ln(h^{(k+2)}(-\bar{\tau})) - (k+2)\ln(2\tau_2)$$

with

$$\tau_1 = \frac{Q\cdot P}{P^2}, \qquad \tau_2 = \frac{1}{P^2}\sqrt{Q^2P^2-(Q\cdot P)^2}$$

The leading term in the asymptotic formula for the entropy is the Hawking Bekenstein entropy of the corresponding black hole.

The subleading term gives the contribution of entropy from the (Gauss Bonnet term) in the effective action using the Wald formula.

We checked the degeneracies given by the Siegel modular form constructed from the twisted elliptic genus matches the entropy given by the Wald formula

The weights of the Siegel modular forms are given by

Type 1	pA	4B	6A	8A	14A	15A
Weight	$\frac{24}{p+1} - 2$	3	2	1	0	0

Table: Weight of Siegel modular forms corresponding to classes in M_{23}

Type 2	2B	3B
Weight	0	-1

Table: Weight of Siegel modular forms corresponding to the classes $\notin M_{23}$

The modular functions which determine the sub-leading corrections are given by

Conjugacy Class	$h^{(k+2)}(\rho)$
pA	$\eta^{k+2}(\rho)\eta^{k+2}(p\rho)$
4B	$\eta^4(4\rho)\eta^2(2\rho)\eta^4(\rho)$
6A	$\eta^2(\rho)\eta^2(2\rho)\eta^2(3\rho)\eta^2(6\rho)$
8A	$\eta^2(\rho)\eta(2\rho)\eta(4\rho)\eta^2(8\rho)$
14A	$\eta(\rho)\eta(2\rho)\eta(7\rho)\eta(14\rho)$
15A	$\eta(\rho)\eta(3\rho)\eta(5\rho)\eta(15\rho)$
2B	$\frac{\eta^8(4\rho)}{\eta^4(2\rho)}$
3B	$\frac{\eta^3(9\rho)}{\eta(3\rho)}$

Table: $p \in \{1, 2, 3, 5, 7, 11, 23\}$

Test 2

Secondly and perhaps a more stringent test:

The coefficients $-B_6(Q, P)$ certainly must be integers.
It was conjectured that:
from the fact that for single centered black holes,
due to spherical symmetry and the regularity of the horizon,
the only angular momentum it caries is from the fermionic zero modes.

$-B_6(Q, P)$ for single centered black holes must be positive.
Sen (2010)

The sufficient condition which ensures this property is that for charges which satisfy

$$Q \cdot P \geq 0, \qquad (Q \cdot P)^2 < Q^2 P^2, \qquad Q^2, P^2 > 0.$$

the coefficient $-B_6(Q, P)$ evaluated from the Fourier expansion of the Siegel modular form should be positive.

This gives a non-trivial condition on the Fourier expansion of the inverse of Siegel modular forms which are generating functions for the index $-B_6(Q, P)$

For the case of $1A$, (compactification of type II on $K3 \times T^2$) for a specific class of charges, this conjecture has been proved by Bringmann, Murthy (2013)

For the orbifolds corresponding to classes pA, $p = 2, 3, 5, 7,$ it has been verified by explicit computation of the Fourier coefficients of $-B_6(Q, P)$ for low lying charges.
Sen (2010)

We constructed the twisted elliptic genus for orbifolds corresponding to all the conjugacy classes in type 1 and the first two classes in type 2

Using this we can explicitly evaluate the Fourier coefficients which evaluate $-B_6$ of the dyons for low lying charges.

Results for 11A, 4B, 2B are listed.

$(Q^2,\ P^2) \quad \backslash Q \cdot P$	-2	0	1	2	3
(1/2, 2)	-512	176	8	0	0
(1/2, 4)	-1536	896	80	0	0
(1/2, 6)	-4544	3616	480	0	0
(1/2, 8)	11752	12848	2176	24	0
(1,4)	-4592	5024	832	16	0
(1,6)	-13408	22464	36786	224	0
(1,8)	-33568	88320	26176	1760	0
(3/2, 6)	-37330	112316	36786	2998	38
(3/2, 8)	-80896	491920	196960	23616	592

Table: Some results for the index $-B_6$ for the $4B$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q \cdot P$	-2	0	1	2	3
(2/11, 2)	-50	10	0	0	0
(2/11, 4)	-100	30	0	0	0
(2/11, 6)	-200	82	1	0	0
(4/11, 6)	-400	276	18	0	0
(6/11, 6)	-800	806	83	0	0
(6/11, 8)	-1438	2064	314	2	0
(6/11, 10)	-2584	4962	937	16	0
(6/11, 12)	-4328	11132	2558	72	0
(6/11, 22)	-34000	366378	139955	12760	114

Table: Some results for the index $-B_6$ for the 11A orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q\cdot P$	-2	0	1	2	3
(1/2, 2)	320	288	24	0	0
(1/2, 4)	0	512	256	0	0
(1/2, 6)	-752	1120	888	48	0
(1/2, 8)	384	3328	2048	384	0
(1,4)	32	4416	2240	32	0
(1,6)	-2304	22464	13248	224	0
(1,8)	5920	42944	27328	5920	64
(3/2, 6)	-2008	102380	66172	9032	28
(3/2, 8)	59392	372736	243712	59392	2048

Table: Some results for the index B_6 for the $2B$ orbifold of $K3$ for different values of Q^2, P^2 and $Q\cdot P$

As a check of the mathematica program used to evaluate these Fourier coefficients, we verified the coefficients listed by Sen(2010) for the orbifolds $pA, p = 2, 3, 5, 7$.

It is interesting to note that the non-geometric orbifolds $11A, 23A, 23B, 2B, 3B$ also satisfy the positivity constraints.

Remarks

The test for positivity of $-B_6$ was also carried out for two orbifolds of $K3$ proposed by (Paquette, Volpato, Zimet 2017) and some of the values turned out to be negative.

Other applications.

Compactifications of heterotic string on $K3 \times T^2/\mathbb{Z}_N$ $([g'] \in M_{24},\ O(g') = N)$ generalize the compactification of heterotic on $K3 \times T^2$.

These examples provide a class of $\mathcal{N} = 2$ string vaccua dual to type II compactification on Calabi-Yau manifolds.

The spectrum and the one loop effective action in the gauge sector of these compactifications were explored in
Datta, David, Lust (2015), Chattopadhyaya, David (2016)

Recently we have also evaluated the one loop effective action in the gravitational sector.
Chattopadhyaya, David 2017 (1712.08791)

These couplings F_g appear as the following terms in the effective action

$$S = \int F_g(y, \bar{y}) F^{2g-2} R^2$$

These contain information of Gopakumar-Vafa invariants which capture important toplogical data of the Calabi-Yau manifolds.

The gravitational amplitudes in these orbifold models leads to a generalization (twisted versions) of these invariants.

The coupling F_g can be schematically given by,

$$F_g \sim \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} [\Theta_{2k}^{(r,s)}(\tau, \bar{\tau}, y, \bar{y}) f^{(r,s)}(\tau) \mathcal{P}_{2k+2}(\tau)]$$

where, $f^{(r,s)}(\tau)$ can be evaluated from the twisted elliptic genus of $K3$ and involves $\Gamma_0(N)$ modular functions.
$\mathcal{P}_{2k+2}$ is a weakly holomorphic modular form,
$\Theta_{2k}^{(r,s)}(\tau, \bar{\tau}, y, \bar{y}) \sim 2k$ derivatives acting on Siegel Narain theta function.

One can get the GV invariants and the Euler characters of these Calabi Yau manifolds from F_g, by extracting its holomorphic piece.

The integral can be done by "unfolding-technique" or Borcherds-Harvey-Moore reduction.

Results χ

Orbifold	$N_h - N_v$	χ
1A	-240	-480
2A	16	32
3A	138	276
4B	200	400
5A	260	520
6A	262	524
7A	321	642
8A	322	644

Table: List of Euler character for the dual Calabi-Yau manifolds for the CHL cases

Observations

Gopakumar Vafa invariants for all twisted and untwisted sectors for the 16 models are integers as expected.

At special points in the moduli space there exists singularities (poles) called conifold singularities, and these special points are present only at the twisted sectors of the g' orbifolds.

Strength of these are determined by the genus zero Gopakumar Vafa invariants at the corresponding points in the moduli space.

THANK YOU