Lecture 3: Instantons in 6d QFTs

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“Partition functions and automorphic forms”

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Plan

- Lecture 3:

- 6d instanton strings and elliptic genera

[Haghighat, Klemm, Lockhart, Vafa] [H.-C.Kim, SK, Park] [Del Zotto, Lockart] [Haghighat, Kozcaz, Iqbal, Lockhart, Vafa] [J. Kim, SK, K. Lee, Park, Vafa] ……

- Exceptional instantons and instanton strings

[H.-C.Kim, J.Kim, SK, K.-H.Lee, Park], [Del Zotto, Lockhart] [J. Kim, K. Lee, Park] …

- Strong coupling studies of 6d SCFT: example of 6d (2,0) theories

[Haghighat, Kozcaz, Iqbal, Lockhart, Vafa] [SK, Nahmgoong] …

6d instanton strings

- 6d Yang-Mills theories describe 6d SCFTs as follows.
- Has $N = (1{,}0)$ SUSY (8 Hermitian), which is chiral.
- 6d SCFTs may have (Abelian) tensor, vector, hypermultiplets. Tensor multiplet consists of

$$B_{\mu\nu} \text{ with } H = dB = \star dB \ , \quad \Psi^A \ , \ \boxed{\Phi} \longrightarrow \text{VEV}$$

- Tensor branch: VEV to scalar. Tensor branch effective action (e.g. w/ one tensor multiplet)

$$S^{\rm bos}_{\rm v+t} = \int \left[\frac{1}{2} d\Phi \wedge \star d\Phi + \frac{1}{2} H \wedge \star H\right] + \sqrt{c} \int \left[-\Phi \mathrm{tr}(F \wedge \star F) + B \wedge \mathrm{tr}(F \wedge F)\right]$$

$$H \equiv dB + \sqrt{c}\ \mathrm{tr}\left(AdA - \frac{2i}{3} A^3\right)$$

- Scalar VEV ~ $1/g_{YM}^2$ of 6d super-Yang-Mills. (Breaks scale invariance spontaneously)

- Instanton string = self-dual string: (like M-strings or E-strings of lecture 3)
- the term $\sim B \wedge \mathrm{tr}(F \wedge F)$: instanton number is electric source to a string.
- Bianchi identify: $dH \sim \mathrm{tr}(F \wedge F)$: magnetic source as well.
- They are same types of strings that arose in the last lecture (M-strings, E-strings)

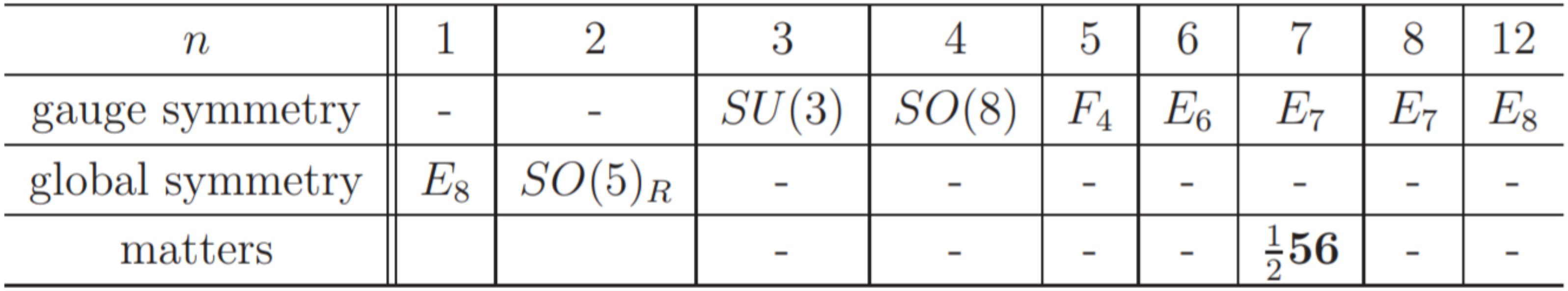
Consistent 6d gauge theories

- For instance, consider a simple gauge group (~ 1 tensor multiplet), and without any hypermultiplet matters (or non-Higgsable matters).

- All allowed theories are listed below:

- Gauge anomaly cancelation condition: Allowed gauge groups (without matters) are $G = SU(2), SU(3), G_2, SO(8), F_4, E_6, E_7, E_8$

- Free of “global anomalies”: $SU(2)$ and G_2 are inconsistent [Witten] [Bershadsky, Vafa]



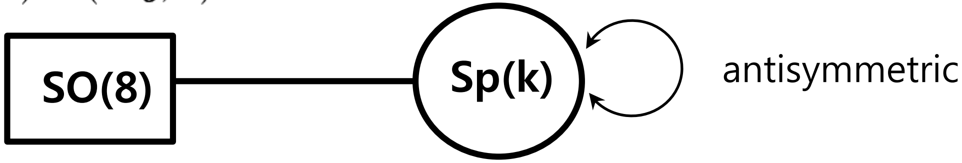
n	1	2	3	4	5	6	7	8	12
gauge symmetry	-	-	$SU(3)$	$SO(8)$	F_4	E_6	E_7	E_7	E_8
global symmetry	E_8	$SO(5)_R$	-	-	-	-	-	-	-
matters			-	-	-	-	$\frac{1}{2}\mathbf{56}$	-	-

- Many interesting 6d Yang-Mills and 6d SCFTs can be formed by making “quivers” of these theories. [Morrison, Taylor] [Heckman, Morrison, Vafa], [Heckman, Morrison, Rudelius, Vafa]

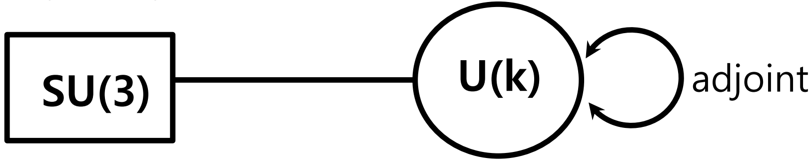
Classical vs. exceptional instantons

- Many gauge theories come w/ exceptional gauge groups.
- So Nekrasov’s ADHM approach seems unavailable here.
- Without matters, only SU(3) & SO(8) are classical: can we use 2d ADHMs?

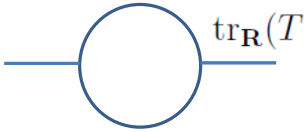
$$\begin{aligned} \text{chiral} &: (q,\psi)\in(\mathbf{2k},\mathbf{8}), (a,\Psi),\ (\tilde{a},\tilde{\Psi})\in(\mathbf{anti}_2,\mathbf{1}) \\ \text{vector}\sim\text{Fermi} &: (A_t,\varphi,\lambda_0),\ (\lambda)\in(\mathbf{adj},\mathbf{1}) \end{aligned}$$



$$\begin{aligned} \text{chiral} &: (q,\psi)\in(\mathbf{k},\overline{\mathbf{3}}),\ (\tilde{q},\tilde{\psi})\in(\overline{\mathbf{k}},\mathbf{3}),\ (a,\Psi),\ (\tilde{a},\tilde{\Psi})\in(\mathbf{adj},\mathbf{1}) \\ \text{vector}\sim\text{Fermi} &: (A_t,\varphi,\lambda_0),\ (\lambda)\in(\mathbf{adj},\mathbf{1}) \end{aligned}$$



- SO(8) is OK, but surprisingly, SU(3) bad: suffers from 2d U(k) gauge anomaly.



$$\mathrm{tr}_{\mathbf{R}}(T^aT^b)=D_{\mathbf{R}}\delta^{ab}$$

$$D_{\mathbf{k}}=1$$

$$D_{\mathbf{adj}}=2k,$$

$$\sim\ 2\cdot3\cdot1+2\cdot2k-2\cdot2k=6\neq0$$

- Distinction between classical/exceptional becomes vaguer here.

SU(3) instanton strings

- "Experimental data" from topological strings [Haghighat, Klemm, Lockhart, Vafa] (2014)
- F-theory (~type IIB strings) on $R^{5,1} \times CY_3$, where $T^2 \to CY_3 \to B_4$
- F-theory on $R^{4,1} \times S^1 \times CY_3$: dual to to M-theory on $R^{4,1} \times CY_3$
- Topological strings compute BPS degeneracies of M2-branes wrapping 2-cycles on CY_3, dual to self-dual strings of the 6d SCFTs in F-theory setting.

- From ADHM gauge theory side, can we improve & cure the bad SU(3) ADHM?
- Yes ! [H.-C. Kim, SK, Park] (2016): add fields to cancel anomlies & to fit experimental data

$$\begin{array}{rcl}
(\phi, \chi) & : & \text{chiral multiplet in } (\bar{\mathbf{k}}, \bar{\mathbf{3}}) \\
(b, \xi) + (\tilde{b}, \tilde{\xi}) & : & \text{two chiral multiplet in } (\overline{\mathbf{anti}}, \mathbf{1}) \\
(\hat{\lambda}, \hat{G}) & : & \text{complex Fermi multiplet in } (\mathbf{sym}, \mathbf{1}) \\
(\check{\lambda}, \check{G}) & : & \text{complex Fermi multiplet in } (\mathbf{sym}, \mathbf{1}) \\
(\zeta, G_\zeta) & : & \text{complex Fermi multiplet in } (\bar{\mathbf{k}}, \mathbf{1}) \ .
\end{array}$$

- Anomaly: SU(k) from ADHM ~ $2 \cdot 3 \cdot 1 + 2 \cdot 2k - 2 \cdot 2k = 6 \neq 0$

  from others ~ $+3 \cdot 1 + 2(k-2) - (k+2) - (k+2) - 1 = -6$

$$D_{\text{sym}} = k + 2$$
$$D_{\text{anti}} = k - 2$$

U(1) $+3 \cdot 2 \cdot 1^2 \cdot k + 3 \cdot 1^2 \cdot k + 2 \cdot 2^2 \cdot \dfrac{k^2 - k}{2} - 2^2 \cdot \dfrac{k^2 + k}{2} - 2^2 \cdot \dfrac{k^2 + k}{2} - 1^2 \cdot k = 0$

SU(3) partition functions

- k SU(3) strings' partition function (elliptic genus):

$$Z_k^{SU(3)} = (-1)^{\frac{k^2-k}{2}}\eta^{6k}\sum_{\vec{Y};|\vec{Y}|=k}\prod_{i=1}^{3}\prod_{s\in Y_i}\frac{\theta_1(2u(s))\theta_1(2\epsilon_+-2u(s))\theta_1(\epsilon_++u(s))}{\prod_{j=1}^{3}\theta_1(E_{ij})\theta_1(E_{ij}-2\epsilon_+)\theta_1(\epsilon_+-u(s)-v_j)}$$

$$\times\prod_{i\leq j}^{3}\prod_{s_{i,j}\in Y_{i,j};s_i<s_j}\frac{\theta_1(u(s_i)+u(s_j))\theta_1(2\epsilon_+-u(s_i)-u(s_j))}{\theta_1(\epsilon_{1,2}-u(s_i)-u(s_j))}$$

$$E_{ij}=v_i-v_j-\epsilon_1h_i(s)+\epsilon_2(v_j(s)+1)$$

$$u(s)=v_i-\epsilon_+-(m-1)\epsilon_1-(n-1)\epsilon_2$$

- E.g. at $k=1$,

$$Z_1=\frac{1}{\theta(\epsilon_{1,2})}\sum_{i=1}^{3}\frac{\theta(a_i)\theta(2a_i-4\epsilon_+)}{\prod_{j(\neq i)}\theta(a_{ij})\theta(a_{ij}-2\epsilon_+)\theta(a_j+2\epsilon_+)}$$

- 1d limit, $\theta(z)=i\theta_1(\tau|z/2\pi i)/\eta(\tau)\rightarrow 2\sinh z/2$: e.g. at $k=1$

$$q^{1/2}Z_1\rightarrow\frac{1}{2\sinh\frac{\epsilon_{1,2}}{2}}\sum_{i=1}^{3}\frac{2\sinh\frac{a_i}{2}\cdot 2\sinh(a_i-2\epsilon_+)}{\prod_{j(\neq i)}2\sinh\frac{a_{ij}}{2}\cdot 2\sinh\frac{a_{ij}-2\epsilon_+}{2}\cdot 2\sinh\frac{a_j+2\epsilon_+}{2}}$$

$$\overset{?}{=}\frac{1}{2\sinh\frac{\epsilon_{1,2}}{2}}\sum_{i=1}^{3}\frac{1}{\prod_{j(\neq i)}2\sinh\frac{a_{ij}}{2}\cdot 2\sinh\frac{a_{ij}-2\epsilon_+}{2}}$$

Yes!

- We honestly checked the identity at k=1,2,3 (using computers). We also have a "physics" argument (based on RG flow and IR decoupling) that it should hold generally.

Full elliptic genus

- k=1: data from topological strings [Haghighat, Klemm, Lockart, Vafa]

$$\log Z_1(\tau,\epsilon_{1,2},v_i)=\sum_{g\geq 0,n\geq 0}(\epsilon_1\epsilon_2)^{g-1}(\epsilon_1+\epsilon_2)^n F_{g,n}(\tau,v_i)$$

$$\begin{aligned}F_{0,0} &= -\left[\frac{\theta_1(2v_1)\theta_1(v_1)}{\theta_1(v_{12})^2\theta_1(v_{13})^2\theta_1(v_2)\theta_1(v_3)}+(1,2,3\to 2,3,1)+(1,2,3\to 3,1,2)\right]\\ &= e^{-\pi i\tau+2\pi i v_{12}+2\pi i v_{23}}\sum_{d_0,d_1,d_2=0}^{\infty}N_{d_0,d_1,d_2}\left(\frac{e^{2\pi i\tau}}{e^{2\pi i v_{12}}e^{2\pi i v_{23}}}\right)^{d_0}e^{2\pi d_1 v_{12}}e^{2\pi d_2 v_{23}}\end{aligned}$$

$d_1 \setminus d_2$	0	1	2	3	4	5
0	1	3	5	7	9	11
1	3	4	8	12	16	20
2	5	8	9	15	21	27
3	7	12	15	16	24	32
4	9	16	21	24	25	35
5	11	20	27	32	35	36

Table 1: q^0

$d_1 \setminus d_2$	0	1	2	3	4	5
0	3	4	8	12	16	20
1	4	16	36	60	84	108
2	8	36	56	96	144	192
3	12	60	96	120	180	252
4	16	84	144	180	208	288
5	20	108	192	252	288	320

Table 2: q^1

$d_1 \setminus d_2$	0	1	2	3	4	5
0	5	8	9	15	21	27
1	8	36	56	96	144	192
2	9	56	149	288	465	651
3	15	96	288	456	735	1080
4	21	144	465	735	954	1371
5	27	192	651	1080	1371	**1632**

Table 3: q^2

$d_1 \setminus d_2$	0	1	2	3	4	5
0	7	12	15	16	24	32
1	12	60	96	120	180	252
2	15	96	288	456	735	1080
3	16	120	456	1012	1788	**2796**
4	24	180	735	1788	**2823**	**4356**
5	32	252	1080	**2796**	**4356**	**5760**

Table 4: q^3

black numbers: from top. strings, reproduced by our "theory"

red: "theory" predictions

Exceptional instantons

- In a sense, SU(3) instanton strings is at the border of classical/exceptional
- \- No D-brane model.
- \- We “cooked up” ADHM-like descriptions, w.o. any D-brane guidance.
- \- At least for Coulomb branch partition functions, equivalent to OR cures standard ADHM
- \- ADHM-like description for other exceptional instantons or instanton strings?

- The 6d “Higgsing” sequence:

$$
\begin{aligned}
(SU(3)) \quad &\leftarrow \quad (G_2, n_{\mathbf{7}}=1) \leftarrow (SO(7), n_{\mathbf{7}}=0, n_{\mathbf{8}}=2) \\
&\leftarrow (SO(8), n_{\mathbf{8}_v}=n_{\mathbf{8}_s}=n_{\mathbf{8}_c}=1) \\
&\leftarrow \begin{cases} (SO(N), n_{\mathbf{N}}=N-7, n_{\mathbf{S}}=\frac{16}{d_{\mathbf{S}}})_{N=9,\cdots,12} \\ (F_4, n_{\mathbf{26}}=2) \leftarrow (E_6, n_{\mathbf{27}}=3) \leftarrow (E_7, n_{\frac{1}{2}\mathbf{56}}=5) \leftarrow (E_8, n_{\text{inst}}=9) \end{cases}
\end{aligned}
$$

- Indeed “exceptional”, in that the unHiggsing sequence terminates.
- Can we build-up ADHM-like models for some of these exceptional instantons…?
- Yes, at least (so far) for some low-rank cases [H.-C.Kim, J.Kim, SK, K.-H. Lee, Park] (2018)

G_2 with n_7 matters

- Start from SU(3) ADHM, and add more 1d/2d fields (reduced UV symmetry)

$$\begin{aligned} A_\mu, \lambda_0, \lambda \ &: \ \mathcal{N}=(0,4)\ U(k) \text{ vector multiplet} \\ q_i, \tilde{q}^i \ &: \ (\mathbf{k}, \bar{\mathbf{3}}) + (\bar{\mathbf{k}}, \mathbf{3}) \quad (i=1,2,3) \\ a, \tilde{a} \ &: \ (\mathbf{adj}, \mathbf{1})\ . \end{aligned}$$

$$\begin{aligned} \phi_i, \phi_4 \ &: \ \text{chiral in } (\bar{\mathbf{k}}, \bar{\mathbf{3}})_{J=\frac{1}{2}} + (\bar{\mathbf{k}}, \mathbf{1})_{J=\frac{1}{2}} \\ b, \tilde{b} \ &: \ \text{chiral in } (\overline{\mathbf{anti}}_2, \mathbf{1})_{J=\frac{1}{2}} \\ \hat{\lambda} \ &: \ \text{Fermi in } (\mathbf{sym}_2, \mathbf{1})_{J=0} \\ \check{\lambda} \ &: \ \text{Fermi in } (\mathbf{sym}_2, \mathbf{1})_{J=-1}\ . \end{aligned}$$

$$\Psi_a\ ,\ \tilde{\Psi}_a\ :\ (\mathbf{k}, \mathbf{1}) + (\bar{\mathbf{k}}, \mathbf{1})\ \ , \qquad a = 1, \cdots, n_7$$

- The partition function (1d, for instance)

$$Z_k = (-1)^k \sum_{\vec{Y};|\vec{Y}|=k} \prod_{i=1}^{3} \prod_{s\in Y_i} \frac{2\sinh(\phi(s)) \cdot 2\sinh(\epsilon_+ - \phi(s))}{\prod_{j=1}^{3}\left(2\sinh\frac{E_{ij}(s)}{2} \cdot 2\sinh\frac{E_{ij}(s)-2\epsilon_+}{2} \cdot 2\sinh\frac{\epsilon_+-\phi(s)-v_j}{2}\right) \cdot 2\sinh\frac{\epsilon_+-\phi(s)}{2}}$$
$$\cdot \prod_{i\leq j}^{3} \prod_{s_{i,j}\in Y_{i,j};s_i<s_j} \frac{2\sinh\frac{\phi(s_i)+\phi(s_j)}{2} \cdot 2\sinh\frac{\phi(s_i)+\phi(s_j)-2\epsilon_+}{2}}{2\sinh\frac{\epsilon_{1,2}-\phi(s_i)-\phi(s_j)}{2}} \cdot \prod_{i=1}^{3} \prod_{s\in Y_i} \prod_{a=1}^{n_7} 2\sinh\frac{m_a \pm \phi(s)}{2}$$

- Uplifts to elliptic genus, by replacing $2\sinh\frac{z}{2} \to \theta(z) = \frac{i\,\theta_1(z/2\pi i)}{\eta}$, at $n_7 = 1$
- Again, tested against many “data” known from different calculations

$SO(7)$ with n_8 matters

- Start from SU(4) ADHM, and add more matters:

$$\begin{aligned} \text{chiral } \phi_i &: (\bar{\mathbf{k}}, \bar{\mathbf{4}})_{J=\frac{1}{2}} \\ \text{chiral } b, \tilde{b} &: (\overline{\mathbf{anti}}_2, \mathbf{1})_{J=\frac{1}{2}} \\ \text{Fermi } \hat{\lambda} &: (\mathbf{sym}_2, \mathbf{1})_{J=0} \\ \text{Fermi } \check{\lambda} &: (\mathbf{sym}_2, \mathbf{1})_{J=-1} \end{aligned}$$

$$\Psi_a, \tilde{\Psi}_a \; : \; (\mathbf{k}, \mathbf{1}) + (\bar{\mathbf{k}}, \mathbf{1}) \qquad (a = 1, \cdots, n_8)$$

- Regarded as “exceptional” instantons: matters in 8 cannot be engineered by D-branes./open strings
- Instanton partition functions:

$$Z_k = \sum_{\vec{Y};|\vec{Y}|=k} \prod_{i=1}^{4} \prod_{s\in Y_i} \frac{2\sinh(\phi(s)) \cdot 2\sinh(\phi(s)-\epsilon_+)}{\prod_{j=1}^{4} 2\sinh\frac{E_{ij}(s)}{2} \cdot 2\sinh\frac{E_{ij}(s)-2\epsilon_+}{2} \cdot 2\sinh\frac{\epsilon_+-\phi(s)-v_j}{2}}$$

$$\times \prod_{i\leq j}^{4} \prod_{s_{i,j}\in Y_{i,j};s_i<s_j} \frac{2\sinh\frac{\phi(s_i)+\phi(s_j)}{2} \cdot 2\sinh\frac{2\epsilon_+-\phi(s_i)-\phi(s_j)}{2}}{2\sinh\frac{\epsilon_{1,2}-\phi(s_i)-\phi(s_j)}{2}} \cdot \prod_{i=1}^{4} \prod_{s\in Y_i} \prod_{a=1}^{n_8} 2\sinh\frac{m_a \pm \phi(s)}{2}$$

- Uplifts to 2d elliptic genus at $n_8 = 2$.
- Both G_2 and $SO(7)$ results are tested against “data” known from string theory.

Open problems

- Build ADHM-like QFTs (in 1d/2d) for all other $G = F_4, E_6, E_7, E_8$ …!
- Still in progress, in the spirit of “model building”
- There are many results known about such instanton strings, including the elliptic genera (e.g. at low instanton numbers)

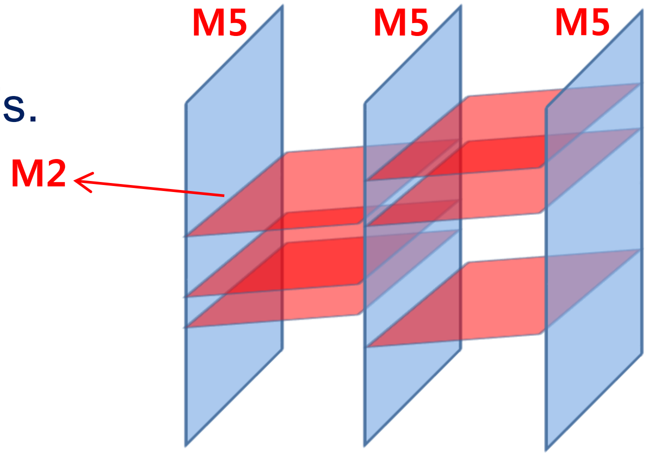
[Del Zotto, Lockhart] [Haghight, Klemm, Lockhart, Vafa] [Hayashi, Ohmori] [K. Lee, J. Kim, Park]
[Del Zotto, Gu, Huang, Kashani-Poor, Klemm, Lockhart] [H.-C. Kim, SK, Park] [Shimizu, Tachikawa] …..

- Though the question of instanton strings is quite natural and looks classic, it has been seriously considered only recently.

- Part of a bigger question on the descriptions of 6d self-dual strings, irrespective of “Yang-Mills instanton” picture. E.g. M-strings, E-strings of lecture 2.
- Their elliptic genera are coefficients of $Z_{R^4 \times T^2}$.
- In the remaining time, I’ll explain what nice physics one can do by having such descriptions.

M-strings & M5-brane SCFT

- The 6d (2,0) theory is one of the most enigmatic QFTs we are aware of.
- Instanton partition function computes bound state degeneracies of D0 (instanton) with charged particles (F1)
- Open M2's w/ momentum suspended between M5's.



- 2d QFT on M-strings known

[Haghighat, Kozcaz, Iqbal, Lockhart, Vafa]

$$Z(q,v_i,\epsilon_{1,2},m) = Z_{\rm pert}\sum_{k=0}^{\infty} q^k Z_k(v_i,\epsilon_{1,2},m) = e^{-\varepsilon_0} Z_{U(1)}^N \sum_{n_1,\cdots,n_{N-1}=0}^{\infty} e^{-n_i(v_i-v_{i+1})} Z_{(n_i)}(\tau,m,\epsilon_{1,2})$$

- Coefficients are elliptic genera of M-strings at given string numbers n_i

$$Z_{(n_i)} = \sum_{Y_1,\cdots,Y_{N-1};|Y_i|=n_i} \prod_{i=1}^{N}\prod_{s\in Y_i} \frac{\theta_1(\tau|\frac{E_{i,i+1}(s)-m+\epsilon_-}{2\pi i})\theta_1(\tau|\frac{E_{i,i-1}(s)+m+\epsilon_-}{2\pi i})}{\theta_1(\tau|\frac{E_{i,i}(s)+\epsilon_1}{2\pi i})\theta_1(\tau|\frac{E_{i,i}(s)-\epsilon_2}{2\pi i})}$$

$$E_{i,j}(s=(a,b)) = (Y_{i,a}-b)\epsilon_1 - (Y_{j,b}^T - a)\epsilon_2$$

Elliptic genera & M5 physics

- The really interesting regime of 6d (2,0) theory is different from our setting.
- Tensor/Coulomb branch vs. symmetric phase: expect N^3 d.o.f. in the latter phase
- 6d QFT on $R^{4,1} \times S^1$: “instanton soliton” picture for small S^1 (radius $\ll 1/T$). But more interesting is the “decompactification” limit, radius $\gg 1/T$.
- At least, we try to asymptotically approach the interesting point.

- Euclidean QFT on $R^4 \times T^2$: complex structure $\tau = \frac{i}{2\pi RT}$, where $q = e^{2\pi i\tau}$.
- $q \to 0$ is easy, dominated by 5d perturbative part. $q \to 1$ challenging.
- 4d N=4 SYM for small torus: duality in S-modular transformation
- Finite torus? Is there S-duality? Can we use it to explore decompactifying region?
- Yes, to certain extent, with M-string expansions.

$$Z_{(n_i)}\left(-\frac{1}{\tau},\frac{m}{\tau},\frac{\epsilon_{1,2}}{\tau}\right) = \exp\left[\frac{1}{4\pi i\tau}\left(\epsilon_1\epsilon_2\sum_{i,j=1}^{N-1}\Omega^{ij}n_in_j + 2(m^2-\epsilon_+^2)\sum_{i=1}^{N-1}n_i\right)\right]Z_{(n_i)}(\tau,m,\epsilon_{1,2})$$

$$\Omega^{ii} = 2\ , \quad \Omega^{i,i+1} = \Omega^{i,i-1} = -1$$

- For technical reasons, study $Z(\tau,a,m,\epsilon_{1,2}) \sim \exp[-f(\tau,a,m)/\epsilon_1\epsilon_2]$ at $\epsilon_{1,2} \to 0$.

Modular anomaly & S-duality anomaly

- τ dependence via quasi-modular forms: $\theta_1(\tau|z) = 2\pi i z\ \eta(\tau)^3 \exp\left[\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau)(2\pi i z)^{2k}\right]$

- 3 generators: $E_2(-1/\tau) = \tau^2\left(E_2 + \boxed{\frac{6}{\pi i \tau}}\right)\ ,\quad E_4(-1/\tau) = \tau^4 E_4(\tau)\ ,\quad E_6(-1/\tau) = \tau^6 E_6(\tau)$

  → causes modular anomaly of $Z_{(n_i)}(\tau, m, \epsilon_{1,2})$

- modular anomaly equation:

$$\frac{\partial}{\partial E_2} Z_{(n_i)}(\tau, m, \epsilon_{1,2} : E_2) = \frac{1}{24}\left[\epsilon_1\epsilon_2\Omega^{ij} n_i n_j - 2\Omega^{ij}(m^2 - \epsilon_+^2)\rho_i n_j\right] Z_{(n_i)}$$

$$\hat{Z}(\tau, v, m, \epsilon_{1,2}) = \sum_{n_1,\cdots,n_r=0}^{\infty} e^{-\sum_{i=1}^{r} n_i\alpha_i(v)} Z_{(n_i)}$$

$$\frac{\partial \hat{Z}}{\partial E_2} = \frac{1}{24}\left[\epsilon_1\epsilon_2\Omega^{ij}\partial_i\partial_j + 2(m^2 - \epsilon_+^2)\Omega^{ij}\rho_i\partial_j\right]\hat{Z} \equiv \frac{1}{24}\left[\epsilon_1\epsilon_2\Omega^{ij}\partial_i\partial_j + 2\Omega^{ij} I_i(m, \epsilon_+)\partial_j\right]\hat{Z}$$

- some manipulations: $Z_{S-dual} \equiv \hat{Z}\,\boxed{\exp\left[\#\frac{\Omega^{ij} I_i v_j}{\epsilon_1\epsilon_2} + \#\frac{\Omega^{ij} I_i I_j}{\epsilon_1\epsilon_2} E_2(\tau)\right]}$

$$\frac{\partial Z_{\text{S-dual}}}{\partial E_2} = \frac{\epsilon_1\epsilon_2}{24}\Omega^{ij}\partial_i\partial_j Z_{\text{S-dual}}$$: heat equation

- dividing modular anomaly: "standard" anomaly + anomaly of "standard" anomaly

Prepotential

- S-duality of Z_{S-dual} : $(\delta = \frac{6}{\pi i\, \tau})$ $Z_{\text{S-dual}}\left(-\frac{1}{\tau}, v, \frac{m}{\tau}, \frac{\epsilon_{1,2}}{\tau}; E_2(-\frac{1}{\tau})\right) = Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}, E_2(\tau) + \delta)$

$$Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}; E_2(\tau) + \delta) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dv_i'\ K(v, v') Z_{\text{S-dual}}(\tau, v', m, \epsilon_{1,2}; E_2(\tau))$$

$$K(v, v') = \left(\frac{i\tau}{\epsilon_1\epsilon_2}\right)^{\frac{N}{2}} \exp\left[-\frac{\pi i\tau}{\epsilon_1\epsilon_2}(v - v')^2\right]$$

- Expand the Gaussian, rescale v, & absorb into classical prepotential: Fourier transform

- Small $\epsilon_{1,2}$: RHS computed by saddle point approx. $Z \sim \exp\left[-\frac{f(\tau, v, m)}{\epsilon_1\epsilon_2}\right]$, $Z_{\text{S-dual}} \sim \exp\left[-\frac{f_{\text{S-dual}}(\tau, v, m)}{\epsilon_1\epsilon_2}\right]$
- 4d limit: S-dual of $\mathrm{F}_{\mathrm{S-dual}} = \pi i\tau\ \mathrm{tr}(v^2) + f_{S-dual}$ is its Legendre transformation (EM duality)
- Natural 6d extension $\tau^2 F_{\text{S-dual}}\left(\tau_D = -\frac{1}{\tau}, v_D = v + \frac{1}{2\pi i\tau}\frac{\partial f}{\partial v}, \frac{m}{\tau}\right) = F_{\text{S-dual}}(\tau, v, m) - v\frac{\partial F_{\text{S-dual}}}{\partial v}(\tau, v, m)$

- "S-duality anomaly": extra anomaly. E.g. for ADE (2,0),

$$f(\tau, v, m) = f_{\text{S-dual}}(\tau, v, m) + r\, f_{U(1)}(\tau, m) + \frac{c_2|G|}{288} m^4 E_2(\tau)$$

$$f_{U(1)} = m^2\left(\frac{1}{2}\log m - \frac{3}{4} + \frac{\pi i}{2} + \log\phi(\tau)\right) + \sum_{n=1}^{\infty} \frac{m^{2n+2} B_{2n}}{2n \cdot (2n+2)!} E_{2n}(\tau)$$

- note: The last S-duality anomaly vanishes in the 4d limit (after restoring S^1 radius)

Asymptotic free energy of 6d (2,0) theories

- Use “dual weak-coupling setting”: anomalous part + 5d perturbative part
- Results:

\- N M5’s:

$$-\log Z \sim \frac{f(\tau \to 0, v, m)}{\epsilon_1\epsilon_2} \sim \frac{i}{\epsilon_1\epsilon_2\tau}\left[\boxed{\frac{N^3m^4}{48\pi} - \frac{\pi N m^2}{12}} \boxed{\mp i\frac{N^2m^3}{12}}\right] \qquad \text{for } \mathrm{Im}(m) \gtrless 0$$

from S-duality anomaly

from “low T” dual perturbative part

\- Indeed, large N free energy is proportional to N^3. Effects of light instantons (~D0-branes)

\- ADE (2,0)’s:

$$-\log Z \sim \frac{i}{\epsilon_1\epsilon_2\tau}\left[\frac{(c_2|G|+r)m^4}{48\pi} - \frac{\pi r m^2}{12} \mp i\frac{|G|m^3}{12}\right]$$

- A check: imaginary part of O(m^4) computed from 6d chiral anomaly [Di Pietro, Komargodski]

\- 6d background gauge fields in $U(1) \subset SU(2)_L \subset SO(5)_R$: $(2\pi)^4 I_8 \to \frac{N^3}{24}F^4$

\- Small temporal S^1: 5d CS terms determined by this anomaly are

$$S \leftarrow -\frac{iN^3\beta}{192\pi^3}\int (A_6^4 a\wedge da \wedge da + 4A_6^3\mathcal{A}\wedge da\wedge da + 6A_6^2\mathcal{A}\wedge d\mathcal{A}\wedge da + 4A_6\mathcal{A}\wedge d\mathcal{A}\wedge d\mathcal{A})$$

\- background:

$$ds^2(\mathbb{R}^4 \times T^2) = \sum_{a=1,2}\left|dz_a - \frac{2i\epsilon_a}{\beta}z_a dy\right|^2 + (dx-\mu dy)^2 + dy^2 \;= e^{2\phi}(dy+a)^2 + h_{ij}dx^i dx^j$$

$$a = \frac{1}{1+\mu^2+\frac{4\epsilon_a^2|z_a|^2}{\beta^2}}\left(-\mu dx - \frac{2\epsilon_a|z_a|^2}{\beta}d\phi_a\right) \qquad A_6 = \frac{2m}{\beta} \qquad \mathcal{A} = -A_6 a \qquad \tau = \frac{\beta}{4\pi}(\mu+i)$$

Remarks

- I tried to give you some flavors on the recent studies in 6d instanton strings, and related developments in 6d SCFTs

- instanton strings & broader notion of 6d self-dual strings: key object in 6d SCFT
- 2d constraints (e.g. anomaly) lead to new ADHM-like descriptions, even for some exceptional instantons, which in turn become useful to study exceptional instanton particles.

- Partition function of self-dual strings is for S^1-compactified 6d theory in the tensor branch, but it still gives interesting information on 6d SCFT itself (e.g. N^3 scaling of free energy)
- Collective degrees of freedom of 5d Yang-Mills instantons at strong coupling

- More roles of instanton partition functions: curved space observables of SCFTs (4th lecture)