Blowup Equations for Refined Topological String

Kaiwen Sun (SISSA)

JINR, Dubna

February 1, 2018

Blowup Equations for Refined Topological String

Min-xin Huang,[§] **Kaiwen Sun,**[†] **Xin Wang**[§]

[§] *Interdisciplinary Center for Theoretical Study, Department of Modern Physics, University of Science and Technology of China, 96 Jinzhai Road, Hefei, Anhui 230026, China*

[†] *Department of Mathematics, University of Science and Technology of China, 96 Jinzhai Road, Hefei, Anhui 230026, China*

E-mail: minxin@ustc.edu.cn, skw@mail.ustc.edu.cn, wxin@mail.ustc.edu.cn

ABSTRACT: Göttsche-Nakajima-Yoshioka K-theoretic blowup equations characterize the Nekrasov partition function of five dimensional $\mathcal{N}=1$ supersymmetric gauge theories compactified on a circle, which via geometric engineering correspond to the refined topological string theory on $SU(N)$ geometries. In this paper, we study the K-theoretic blowup equations for general local Calabi-Yau threefolds. We find that both vanishing and unity blowup equations exist for the partition function of refined topological string, and the crucial ingredients are the $\mathbf{r}$ fields introduced in our previous paper. These blowup equations are in fact the functional equations for the partition function and each of them results in infinite identities among the refined free energies. Evidences show that they can be used to determine the full refined BPS invariants of local Calabi-Yau threefolds. This serves an independent and sometimes more powerful way to compute the partition function other than the refined topological vertex in the A-model and the refined holomorphic anomaly equations in the B-model. We provide a procedure to determine all the vanishing and unity $\mathbf{r}$ fields from the polynomial part of refined topological string at large radius point. We also find that certain form of blowup equations exist at generic loci of the moduli space.

To Sheldon Katz on his 60th anniversary

Outline

- Background
  - Local Calabi-Yau and local mirror symmetry
  - Refined topological string
- Two motivations
  - Quantization of mirror curve
    - Exact Nekrasov-Shatashivili (NS) quantization
    - Grassi-Hatsuda-Mariño (GHM) conjecture
    - Compatibility formulae
  - Göttsche-Nakajima-Yoshioka (GNY) K-theoretic blowup equations
- Blowup equations for refined topological string
  - Vanishing blowup equations
  - Unity blowup equations
- Examples
  - Resolved conifold
  - Local $\mathbb{P}^2$
  - Local $\frac{1}{2}K3$/E-strings/6d minimal (1,0) SCFT
- Outlook

Calabi-Yau

- Compact CY
  - Quintic $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = \lambda z_1 z_2 z_3 z_4 z_5$
  - Related to SUGRA, black holes
  - Gromov-Witten (GW) invariants, Gopakumar-Vafa (GV) invariants, Donaldson-Thomas (DT) invariants, Pandharipande-Thomas (PT) invariants
  - Partition function of topological string
- Local CY
  - Non-compact
  - Anti-canonical line bundle on a surface
  - Resolved conifold, local $\mathbb{P}^2$, local $\mathbb{P}^1 \times \mathbb{P}^1 \dots$
  - Related to supersymmetric gauge theories
  - Refined BPS invariants
  - Partition function of refined topological string

Local toric CY (A model)

- A toric Calabi-Yau threefold is a toric variety given by the quotient,

$$M = (\mathbb{C}^{k+3} \backslash \mathcal{SR})/G, \qquad (0.1)$$

where $G = (\mathbb{C}^*)^k$ and $\mathcal{SR}$ is the Stanley-Reisner ideal of G. The quotient is specified by a matrix of charges Q_i^α, $i = 0, \cdots, k+2$, $\alpha = 1, \cdots, k$. The group G acts on the homogeneous coordinates x_i as

$$x_i \to \lambda_\alpha^{Q_i^\alpha} x_i, \qquad i = 0, \cdots, k+2, \qquad (0.2)$$

where $\alpha = 1, \ldots, k$, $\lambda_\alpha \in \mathbb{C}^*$ and $Q_i^\alpha \in \mathbb{Z}$.

- To avoid R symmetry anomaly, one requires

$$\sum_{i=1}^{k+3} Q_i^\alpha = 0, \quad \alpha = 1, \ldots, k. \qquad (0.3)$$

Given the matrix of charges Q_i^α, one can introduce the vectors,

$$\nu^{(i)} = \left(1, \nu_1^{(i)}, \nu_2^{(i)}\right), \qquad i = 0, \cdots, k+2, \tag{0.4}$$

satisfying the relations

$$\sum_{i=0}^{k+2} Q_i^\alpha \nu^{(i)} = 0. \tag{0.5}$$

Then mirror curve can be written as

$$H(\mathrm{e}^x, \mathrm{e}^p) = \sum_{i=0}^{k+2} x_i \exp\left(\nu_1^{(i)} x + \nu_2^{(i)} p\right). \tag{0.6}$$

The mirror Calabi-Yau itself is

$$H(\mathrm{e}^x, \mathrm{e}^p) = uv, \tag{0.7}$$

with $(u, v, x, p) \in \mathbb{C}^4$.

Refined topological string

Topological string has a refinement on local Calabi-Yau, corresponding to the $N = 2$ gauge theory on Omega background

$$\epsilon_1 = -\epsilon_2 = g_s \;\to\; \text{general } \epsilon_1, \epsilon_2.$$

The refined free energy $F_{\mathrm{ref}}(\mathbf{t}; \epsilon_1, \epsilon_2)$ contains two parts:

$$F_{\mathrm{ref}}^{\mathrm{pert}}(\mathbf{t}; \epsilon_1, \epsilon_2) = \frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{6}\sum_{i,j,k=1}^{n_X} a_{ijk} t_i t_j t_k + 4\pi^2 \sum_{i=1}^{n_X} b_i^{\mathrm{NS}} t_i\right) + \sum_{i=1}^{n_X} b_i t_i - \frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}\sum_{i=1}^{n_X} b_i^{\mathrm{NS}} t_i \;,$$

and

$$F_{\mathrm{ref}}^{\mathrm{inst}}(\mathbf{t}, \epsilon_1, \epsilon_2) = \sum_{j_L, j_R \geq 0} \sum_{\mathbf{d}} \sum_{w=1}^{\infty} (-1)^{2j_L+2j_R} N_{j_L, j_R}^{\mathbf{d}} \frac{\chi_{j_L}(q_L^w)\chi_{j_R}(q_R^w)}{w(q_1^{w/2} - q_1^{-w/2})(q_2^{w/2} - q_2^{-w/2})} e^{-w\mathbf{d}\cdot\mathbf{t}} \;,$$

where

$$q_{1,2} = e^{\epsilon_{1,2}} \;, \quad q_{L,R} = e^{(\epsilon_1 \mp \epsilon_2)/2} \;,$$

and

$$\chi_j(q) = \frac{q^{2j+1} - q^{-2j-1}}{q - q^{-1}} \;.$$

The refined topological string free energy can be expand as

$$F(\mathbf{t},\epsilon_1,\epsilon_2)=\sum_{n,g=0}^{\infty}(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}\mathcal{F}^{(n,g)}(\mathbf{t}) \tag{0.8}$$

The traditional topological string free energy can be obtained by taking the unrefined limit,

$$\epsilon_1=-\epsilon_2=g_s. \tag{0.9}$$

$$F_{\mathrm{GV}}(\mathbf{t},g_s)=F(\mathbf{t},g_s,-g_s). \tag{0.10}$$

The NS free energy can be obtained by taking the NS limit,

$$F^{\mathrm{NS}}(\mathbf{t},\hbar)=\lim_{\epsilon_1\to 0}\epsilon_1 F(\mathbf{t},\epsilon_1,\hbar). \tag{0.11}$$

$F_0(\mathbf{t})$ are determined by making an appropriate choice of cycles on the curve, α_i, β_i, $i=1,\cdots,s$, then we have

$$t_i=\oint_{\alpha_i}pdx,\qquad \frac{\partial F_0}{\partial t_i}=\oint_{\beta_i}pdx,\qquad i=1,\cdots,s. \tag{0.12}$$

Refined BPS invariants

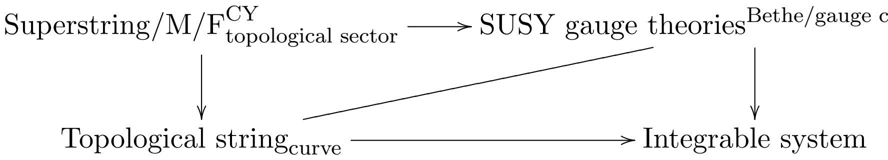
- M-theory compactified on local Calabi-Yau threefold X
- the BPS particles in the 5D susy gauge theory arising from M2-branes wrapping the holomorphic curves within X.
- The homology class $\beta \in H_2(X, \mathbb{Z})$ which can be represented by a degree vector $\mathbf{d}$
- The BPS particles are also classified by their spins (j_L, j_R) under the 5D little group $SU(2)_L \times SU(2)_R$.
- The multiplicities $N^{\mathbf{d}}_{j_L,j_R}$ of the BPS particles are called the refined BPS invariants.
- There exists a integral vector $\mathbf{B}$ such that non-vanishing BPS invariants $N^{\mathbf{d}}_{j_L,j_R}$ occur only at

$$2j_L + 2j_R + 1 \equiv \mathbf{B} \cdot \mathbf{d} \qquad \mod 2\,.$$

How to compute?

- Refined topological vertex
  - toric CY
  - complicated when non-toric
- Refined holomorphic anomaly equations
  - holomorphic ambiguity
- Blowup Equations
  - not necessarily toric
  - correponding to the target physics
  - directly related to GV/BPS formalism
  - easy to determine the refined BPS invariants

Relation Web



Examples:

- 4d $N = 2/5$d $N = 1$ pure $SU(2)$ gauge theory $\leftrightarrow$ topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ $\leftrightarrow$ sine-Gordon model
- 4d $N = 2/5$d $N = 1$ pure $SU(N)$ gauge theory $\leftrightarrow$ topological string on $SU(N)$ geometries $\leftrightarrow$ N periodic Toda chain

Relation Web

- Classical
  - Coulomb parameter $\mathbf{a}$ $\sim$ Kähler parameter $\mathbf{t}$ $\sim$ classical periods of integrable system
  - Seiberg-Witten prepotential $\mathcal{F}_{SW}(\mathbf{a})$ $\sim$ genus zero free energy $F_0(\mathbf{t})$ $\sim$ classical action
- Quantum
  - quantum Coulomb parameter $\mathbf{a}(\hbar)$ $\sim$ quantum A-periods $\mathbf{t}(\hbar)$ $\sim$ quantum periods of integrable system
  - Nekrasov-Shatashivili free energy $\mathcal{F}_{NS}(\mathbf{a},\hbar)$ $\sim$ NS free energy $F_{NS}(\mathbf{t},\hbar)$ $\sim$ Yang-Yang function
- Refined
  - Nekrasov partition function $\mathcal{Z}_{Nek}(\mathbf{a},\epsilon_1,\epsilon_2)$ $\sim$ refined free energy $F_{ref}(\mathbf{t},\epsilon_1,\epsilon_2)$

Non-perturbative Top String & Quantizing mirror curve

Non-perturbative description from various correspondences

- SCFT (Lockhart, Vafa)
- Matrix models (Mariño, . . .)
- ABJM theories (Kapustin, Mariño, Putrov, Hatsuda, . . .)
- Integrable systems (Aganagic, Dijkgraaf, Klemm, Mariño, Vafa, Cheng, Krefl, . . .)
- Chern-Simons theories, SUSY gauge theories, string/M, conformal blocks, OSV formulas, resurgence, . . .

Quantization of the mirror curve of local Calabi-Yau,

$$H(\mathrm{e}^x, \mathrm{e}^p) = \sum x_i \exp\left(\nu_1^{(i)} x + \nu_2^{(i)} p\right) = 0$$

with Heisenberg relation

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar.$$

e.g: for local $\mathbb{P}^2$

$$\exp(\hat{H}) = \exp(\hat{x}) + \exp(\hat{p}) + \exp(-\hat{x} - \hat{p})$$

Exact Nekrasov-Shatashvili Quantization

Bethe/Gauge correspondence: SUSY vacua equation/Bethe ansatz

$$\exp\left(\partial_{a_I}\mathcal{W}(\vec{a};\hbar)\right)=1.$$

Exact NS Quantization Conditions for mirror curve Σ_g,

$$\mathrm{Vol}_i(\mathbf{t},\hbar)=\hbar C_{ij}\frac{(F^{\mathrm{NS}}_{poly}+F^{\mathrm{NS}}_{p}+F^{\mathrm{NS}}_{np})(\mathbf{t},\hbar)}{\partial t_j}=2\pi\hbar\left(n_i+\frac{1}{2}\right),\ i=1,\cdots,g,$$

where

$$F^{\mathrm{NS}}_{p}(\mathbf{t},\hbar)=F^{\mathrm{NS}}_{\mathrm{inst}}(\mathbf{t}+\mathrm{i}\pi\mathbf{B},\hbar),\qquad F^{\mathrm{NS}}_{np}(\mathbf{t},\hbar)=\frac{\hbar}{2\pi}F^{\mathrm{inst}}_{\mathrm{NS}}\left(\frac{2\pi\mathbf{t}}{\hbar}+\mathrm{i}\pi\mathbf{B},\frac{4\pi^2}{\hbar}\right),$$

$$F^{\mathrm{NS}}_{\mathrm{inst}}(\mathbf{t},\hbar)=\sum_{j_L,j_R}\sum_{w,\mathbf{d}}N^{\mathbf{d}}_{j_L,j_R}\frac{\sin\frac{\hbar w}{2}(2j_L+1)\sin\frac{\hbar w}{2}(2j_R+1)}{2w^2\sin^3\frac{\hbar w}{2}}\mathrm{e}^{-w\mathbf{d}\cdot\mathbf{t}}.$$

- pole cancellation
- self S-duality!
- consistent with Lockhart-Vafa partition function of non-perturbative topological string!

(II) Grassi-Hatsuda-Mariño conjecture

Generalized grand potential (inspired from ABJM theories)

$$\mathsf{J}(\boldsymbol{\mu},\boldsymbol{\xi},\hbar)=\mathsf{J}^{\mathrm{WKB}}(\boldsymbol{\mu},\boldsymbol{\xi},\hbar)+\mathsf{J}^{\mathrm{WS}}(\boldsymbol{\mu},\boldsymbol{\xi},\hbar),$$

with

$$\mathsf{J}^{\mathrm{WKB}}=\frac{t_i(\hbar)}{2\pi}\frac{\partial F^{\mathrm{NS}}(\mathbf{t}(\hbar),\hbar)}{\partial t_i}+\frac{\hbar^2}{2\pi}\frac{\partial}{\partial\hbar}\left(\frac{F^{\mathrm{NS}}(\mathbf{t}(\hbar),\hbar)}{\hbar}\right)+\frac{2\pi}{\hbar}b_i t_i(\hbar)+A(\boldsymbol{\xi},\hbar),$$

$$\mathsf{J}^{\mathrm{WS}}=F^{\mathrm{GV}}\left(\frac{2\pi}{\hbar}\mathbf{t}(\hbar)+\pi\mathrm{i}\mathbf{B},\frac{4\pi^2}{\hbar}\right).$$

GHM conjecture: the generalized spectral determinant of the inverse operator of quantum mirror curve is given by

$$\Xi(\mathbf{t};\hbar)=\sum_{\mathbf{n}\in\mathbb{Z}^g}\exp\left(\mathsf{J}(\boldsymbol{\mu}+2\pi\mathrm{i}\mathbf{n},\boldsymbol{\xi},\hbar)\right).$$

Quantum Riemann theta function defined from

$$\Xi(\mathbf{t};\hbar)=\exp\left(\mathsf{J}(\boldsymbol{\mu},\boldsymbol{\xi},\hbar)\right)\Theta(\mathbf{t};\hbar).$$

Then GHM quantization condition for the mirror curve written as

$$\Theta(\mathbf{t};\hbar)=0.$$

The Equivalence

There exist a set of constant integral vectors $\mathbf{r}^a$, $a = 1, \cdots, w_\Sigma$, where $w_\Sigma \geq g_\Sigma$, such that the intersections of the theta divisors of all w_Σ quantum Riemann theta functions $\Theta(\mathbf{t} + \mathrm{i}\pi\mathbf{r}^a, \hbar)$ coincide with the spectra solved by the exact NS quantization conditions.

$$\left\{\Theta(\mathbf{t}+\mathrm{i}\pi\mathbf{r}^a, \hbar) = 0,\ a = 1, ., w_\Sigma\right\} \Leftrightarrow \left\{\mathrm{Vol}_i(\mathbf{t}, \hbar) = 2\pi\hbar\left(n_i+\frac{1}{2}\right),\ i = 1, ., g_\Sigma\right\}$$

All the vector $\mathbf{r}^a$ are the representatives of the $\mathbf{B}$ field, which means for all triples of degree $\mathbf{d}$, spin j_L and j_R such that the refined BPS invariants $N^{\mathbf{d}}_{j_L,j_R}$ is non-vanishing, they must satisfy

$$(-1)^{2j_L+2j_R-1} = (-1)^{\mathbf{r}^a\cdot\mathbf{d}}, \quad a = 1, \cdots, w_\Sigma.$$

Compatibility formulae

The above equivalence is guaranteed by some novel identities:

Identities (Huang, KS, Wang)

For an arbitrary toric Calabi-Yau threefold with Kähler moduli $\mathbf{t}$ *and charge matrix* C_{ij}*, the following identities hold:*

$$\sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\left(\sum_{i=1}^{g} n_i \pi \mathrm{i} + F_{unref}\left(\mathbf{t}+\mathrm{i}\hbar\mathbf{n}\cdot C+\frac{1}{2}\mathrm{i}\hbar\mathbf{r}^a, \hbar\right) - \mathrm{i}n_i C_{ij}\frac{\partial}{\partial t_j}F_{NS}(\mathbf{t},\hbar)\right) \equiv 0,$$

- highly nontrivial!
- impose infinite constraints among the refined BPS invariants!
- verified to high orders for local del Pezzo surfaces, resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold and $SU(N)$ geometries,
- proved for some geometries at $\hbar = 2\pi/k$.

Blowup equations

Relations between the Nekrasov partition functions $Z^{\mathrm{Nek}}(\mathbf{a}, \epsilon, \epsilon_2)$ on $\mathbb{C}^2$ and $Z^{\mathrm{Nek}}_{k,d}(\mathbf{a}, \epsilon, \epsilon_2)$ on $\widehat{\mathbb{C}^2}$

- Nakajima-Yoshioka blowup equations
  - 4D $\mathcal{N} = 2$ $SU(N)$ pure gauge theories
- Nakajima-Yoshioka K-theoretic blowup equations
- used to prove Nekrasov's conjecture
  - 5D $\mathcal{N} = 1$ $SU(N)$ pure gauge theories
- Göttsche-Nakajima-Yoshioka K-theoretic blowup equations
  - 5D $\mathcal{N} = 1$ $SU(N)$ gauge theories with 5D Chern-Simons term
  - Chern-Simons level $m = 0, 1, \ldots, N$.
  - Corresponding to refined topological strings of $X_{N,m}$ geometries
  - $X_{2,0}$ is just local $\mathbb{P}_1 \times \mathbb{P}_1$, $X_{2,1}$ is local $\mathbb{F}_1$

To connect the Nekrasov partition function of gauge theory and the refined topological string partition funtion, we need to define the twisted partition function of refined topological string

$$\widehat{Z}_{\mathrm{ref}}(\mathbf{t};\epsilon_1,\epsilon_2)=\exp\left(F_{\mathrm{ref}}^{\mathrm{pert}}(\mathbf{t};\epsilon_1,\epsilon_2)+F_{\mathrm{ref}}^{\mathrm{inst}}(\mathbf{t}+\pi\mathrm{i}\mathbf{B};\epsilon_1,\epsilon_2)\right).$$

It turns out this is also the most natural object to write down the functional equations.

Generalized K-theoreitc blowup equations

Conjecture (Huang, KS, Wang)

For an arbitrary local Calabi-Yau threefold X with mirror curve of genus g, suppose there are $b=\dim H_2(X,\mathbb{Z})$ irreducible curve classes corresponding to Kähler moduli **t** *in which $b-g$ classes correspond to mass parameters* **m**, *and denote* **C** *as the intersection matrix between the b curve classes and the g irreducible compact divisor classes, then there exist infinite constant integral vectors $\mathbf{r}\in\mathbb{Z}^b$ such that the following functional equations for the twisted partition function of refined topological string on X hold:*

$$\sum_{\mathbf{n}\in\mathbb{Z}^g}(-)^{|\mathbf{n}|}\ \widehat{Z}_{\mathrm{ref}}\left(\epsilon_1,\epsilon_2-\epsilon_1;\mathbf{t}+\epsilon_1\mathbf{R}\right)\cdot\widehat{Z}_{\mathrm{ref}}\left(\epsilon_1-\epsilon_2,\epsilon_2;\mathbf{t}+\epsilon_2\mathbf{R}\right)$$

$$=\begin{cases}0, & \text{for } \mathbf{r}\in\mathcal{S}_{\mathrm{vanish}},\\ \Lambda(\epsilon_1,\epsilon_2;\mathbf{m},\mathbf{r})\widehat{Z}_{\mathrm{ref}}\left(\epsilon_1,\epsilon_2;\mathbf{t}\right), & \text{for } \mathbf{r}\in\mathcal{S}_{\mathrm{unity}}.\end{cases}$$

where $|\mathbf{n}|=\sum_{i=1}^g n_i$, $\mathbf{R}=\mathbf{C}\cdot\mathbf{n}+\mathbf{r}/2$ and Λ is a simple factor purely determined by the polynomial part of the refined free energy.

Generalized K-theoretic blowup equations

In addition, all the vector **r** are the representatives of the **B** field of X, which means for all triples of degree **d**, spin j_L and j_R such that the refined BPS invariants $N^{\mathbf{d}}_{j_L,j_R}(X)$ is non-vanishing, they must satisfy

$$(-1)^{2j_L+2j_R-1} = (-1)^{\mathbf{r}\cdot\mathbf{d}}.$$

Besides, both sets $\mathcal{S}_{\text{vanish}}$ and $\mathcal{S}_{\text{unity}}$ are finite under the quotient of shift $2\mathbf{C}\cdot\mathbf{n}$ symmetry.

We furthur conjecture that with the classical information of an arbitrary local Calabi-Yau threefold, the blowup equations combined together can uniquely determine its refined partition function, in particular all the refined BPS invariants.

Remarks

- checked for local $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$, $\mathfrak{B}_3$, resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold, $SU(3)$ geometries, half K3
- generalization of Göttsche-Nakajima-Yoshioka K-theoretic blowup equations to all local Calabi-Yau
- NS limits of vanishing blowup equations give the compatibility formula (the previous identities)
- imply the Constrain on $(j_L, j_R, \mathbf{d})$ of refined BPS invariants
- can be used to determine the refined BPS invariants of local Calabi-Yau to arbitrary degree and genus!

Resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\mapsto\mathbb{P}^1$

- mirror curve genus zero
- defining equation $xy - zw = 0$
- a single Kähler parameter t measuring the size of base $\mathbb{P}^1$
- the only non-vanishing refined BPS invariant is $n^1_{0,0} = 1$
- B field is 1.
- refined partition function was computed with the refined topological vertex as

$$Z(q,t,Q) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{Q^n}{n(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})} \right\}, \quad (0.13)$$

  where $q = e^{\epsilon_1}$, $t = e^{-\epsilon_2}$ and $Q = e^{-t}$.
- It is easy to check that

$$Z(q, qt, \frac{1}{\sqrt{q}} Q) Z(qt, t, \sqrt{t} Q) = Z(q, t, Q) \quad (0.14)$$

- the unity blowup equation holds for $r = 1$
- also holds for $r = -1$

Local $\mathbb{P}^2$

- Local $\mathbb{P}^2$ is a geometry of line bundle $\mathcal{O}(-3) \to \mathbb{P}^2$
- mirror curve $1 + x + y + \frac{z}{xy} = 0$ is an ellitpic curve
- $B = 1,\ C = 3.$
- genus zero free energy is

$$F_0 = -\frac{1}{18}t^3 + \frac{1}{12}t^2 + \frac{1}{12}t + 3Q - \frac{45}{4}Q^2 + \cdots, \qquad (0.15)$$

where $Q = e^t$.

- Define modular parameter $2\pi i\tau = 3\frac{\partial^2}{\partial t^2}F_0$ of the mirror cyrve, the modular group of local $\mathbb{P}^2$ is $\Gamma(3) \in SL(2,\mathbb{Z})$. It has generators

$$a := \theta^3\begin{bmatrix}\frac{1}{6}\\ \frac{1}{6}\end{bmatrix}, \quad b := \theta^3\begin{bmatrix}\frac{1}{6}\\ \frac{1}{2}\end{bmatrix}, \quad c := \theta^3\begin{bmatrix}\frac{1}{6}\\ \frac{5}{6}\end{bmatrix}, \quad d := \theta^3\begin{bmatrix}\frac{1}{2}\\ \frac{1}{6}\end{bmatrix}, \qquad (0.16)$$

all have weight $3/2$.

- The Dedekind η function satisfies the identity $\eta^{12} = \frac{i}{3^{3/2}}abcd$

Local $\mathbb{P}^2$

The genus one free energy can be compute from holomorphic anomaly equation:

$$F^{(0,1)} = -\frac{1}{6}\log(d\eta^3), \quad F^{(1,0)} = \frac{1}{6}\log(\eta^3/d), \tag{0.17}$$

then

$$F^{(0,1)} - F^{(1,0)} = \log(\eta(\tau)). \tag{0.18}$$

The r fields of local $\mathbb{P}^2$ are

$$\mathcal{S}_{\text{vanish}} = \{\dots,-9,-3,3,9,\dots\}$$
$$\mathcal{S}_{\text{unity}} = \{\dots,-7,-5,-1,1,5,7,\dots\}$$

For unity case, $R = 3n + 1/2$ and the leading order of unity blowup equation gives

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-)^k x^{k(3k-1)/2}.$$

Euler's Pentagonal number theorem! Higher orders give infinite identities among modular form of $\Gamma(3)$!

E-strings

- M2-brane stretched between M9 and M5-branes $\to$ E-string
- the simplest 6d $(1,0)$ SCFT
- correspond to refined topological string on local half K3, which is $\mathfrak{B}_9(\mathbb{P}^2)$
- elliptic genus of n E-strings $Z_n(\epsilon_1,\epsilon_2)$
- many methods to compute Z_n for small n
- but no concise formula for general n
- *vanishing blowup equations for E-strings*

$$\sum_{i=0}^{n} Z_{n-i}(\epsilon_1,\epsilon_2-\epsilon_1)Z_i(\epsilon_1-\epsilon_2,\epsilon_2)\theta_1((n-i)\epsilon_1+i\epsilon_2)=0.$$

- checked up to $n=3$, extremly complicated identities among Jacobi forms, highly nontrivial!
- same index quadratic form for each term in the summand!

Questions beg to be answered

- How general is the blowup equations?
  - Not known
  - Many generalization in progress
- How to prove the blowup equations physically and mathematically?
  - Can one prove its equivalence with refined holomorphic anomaly equations?
  - This requires a non-holomorphic version of the blowup equations
- Is the refined partition function uniquely determined by the blowup equations? How to prove?
  - Nekrasov partition function are indeed uniquely determined by GNY blowup equations
  - True for all local CYs reduced from $SU(N)$ geometries
  - We also proved this is true for resolved conifold

Future Works

- extend to 6d SCFT (Gu, Haghighat, KS, Wang, in progress)
- Dijkgraaf-Vafa geometries? (KS, Wang, in progress)
- blowup equations on $\mathbb{C}^2/\mathbb{Z}_2$ (Bonelli, KS, Tanzini, in progress)
- incorporate knot/links?
- compact elliptic Calabi-Yau?
- non-holomorphic blowup equations?
- relation with holomorphic anomaly equations?
- any physical setting of unity blowup equations?
- quest for a rigorous proof?