Lorentzian Kac–Moody algebras with Weyl groups of 2-reflections

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Abstract: The talk follows to the recent preprint with the same name by Valery Gritsenko and V. Nikulin,
arXiv:1602.08359, February 2016, 75 pages.

One can find some details there.

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1 Lorentzian Kac–Moody algebras: general definition

Lorentzian Kac–Moody algebras which we consider are given by data (I) — (V) below. We follow the general theory of Lorentzian Kac–Moody algebras from our papers 1994 – 2003 where we used ideas and results by Kac–Moody and Borcherds

(I) The datum (I) is given by a *hyperbolic lattice* S of the rank $\mathrm{rk}\, S \geq 3$.

We recall that a lattice (equivalently, a non-degenerate symmetric bilinear form over $\mathbb{Z}$) M means that M is a free $\mathbb{Z}$-module M of a finite rank with symmetric $\mathbb{Z}$-bilinear non-degenerate pairing $(x, y) \in \mathbb{Z}$ for $x, y \in M$. A lattice S is hyperbolic if the corresponding symmetric bilinear form $S \otimes \mathbb{R}$ over $\mathbb{R}$ has signature $(n, 1)$ where $\mathrm{rk}\, S = n + 1$.

(II) This datum is given by a *Weyl group* which is a reflection subgroup $W \subset O(S)$ of the hyperbolic lattice S from (I). It is generated by reflections in roots of S.

We recall that an element α of a lattice M is called *root* if $\alpha^2 = (\alpha,\, \alpha) > 0$ and $\alpha^2\ |2(\alpha, M)$ that is $\alpha^2|2(\alpha, x)$ for any $x \in M$. A root $\alpha \in M$ defines the reflection

$$s_\alpha : x \to x - \frac{2(x, \alpha)}{\alpha^2}\alpha, \quad \forall x \in M \tag{1.1}$$

which belongs to $O(M)$. The reflection s_α is characterized by the properties: $s_\alpha(\alpha) = -\alpha$ and $s_\alpha|(\alpha)^\perp_M$ is identity.

(III) This datum is given by the *set of simple real roots* $P = P(\mathcal{M}) \subset S$ of roots which are perpendicular and directed outwards to the fundamental chamber $\mathcal{M} \subset \mathcal{L}(S)$ of the Weyl group W from the data (II) acting in the hyperbolic space $\mathcal{L}(S)$ defined by S. Each codimension one face of $\mathcal{M}$ must have exactly one element $\alpha \in P(\mathcal{M})$ which is perpendicular to this face and directed outwards. The set $P = P(\mathcal{M})$ must have the *lattice Weyl vector* $\rho \in S \otimes \mathbb{Q}$ which means that

$$(\rho, \alpha) = -\frac{\alpha^2}{2} \quad \forall \alpha \in P = P(\mathcal{M}). \tag{1.2}$$

The fundamental chamber $\mathcal{M}$ must have either a finite volume (then S is called *elliptically reflective*) and then $\rho^2 < 0$ and $P = P(\mathcal{M})$ is finite (*elliptic case*), or almost finite volume (then S is called *parabolically reflective*) and $\rho^2 = 0$, but $\rho \neq 0$ (*parabolic case*). Here almost finite volume means that $\mathcal{M}$ has finite volume in any cone with the vertex $\mathbb{R}_{++}\rho$ at infinity of $\mathcal{M}$.

We recall that, for a hyperbolic lattice S, we can consider the cone

$$V(S) = \{x \in S \otimes \mathbb{R} \mid x^2 < 0\}$$

of S, and its half cone $V^+(S)$. Any two elements $x, y \in V^+(S)$ satisfy $(x, y) < 0$. The half-cone $V^+(S)$ defines *the hyperbolic space of S*,

$$\mathcal{L}(S) = V^+(S)/\mathbb{R}_{++} = \{\mathbb{R}_{++}x \mid x \in V^+(S)\}$$

of the curvature (-1) with the hyperbolic distance

$$\operatorname{ch}\rho(\mathbb{R}_{++}x, \mathbb{R}_{++}y) = \frac{-(x,y)}{\sqrt{x^2y^2}}, \quad x, y \in V^+(S).$$

Here $\mathbb{R}_{++}$ is the set of all positive real numbers, and $\mathbb{R}_+$ is the set of all non-negative real numbers. Any $\delta \in S \otimes \mathbb{R}$ with $\delta^2 > 0$ defines *a half-space*

$$\mathcal{H}_\delta^+ = \{\mathbb{R}_{++}x \in \mathcal{L}^+(S) \mid (x,\delta) \le 0\}$$

of $\mathcal{L}(S)$ bounded by the *hyperplane*

$$\mathcal{H}_\delta = \{\mathbb{R}_{++}x \in \mathcal{L}^+(S) \mid (x,\delta) = 0\}\,.$$

The δ is called orthogonal to the half-space $\mathcal{H}_\delta^+$ and the hyperplane $\mathcal{H}_\delta$, and it is defined uniquely if $\delta^2 > 0$ is fixed. For a root $\alpha \in S$, the reflection s_α gives the reflection of $\mathcal{L}^+(S)$ with respect to the hyperplane $\mathcal{H}_\alpha$, that is s_α is identity on $\mathcal{H}_\alpha$, and $s_\alpha(\mathcal{H}_\alpha^+) = \mathcal{H}_{-\alpha}^+$. It is well-known that the group

$$O^+(S) = \{\phi \in O(S) \mid \phi(V^+(S)) = V^+(S)\}$$

is discrete in $\mathcal{L}^+(S)$ and has a fundamental domain of finite volume. The subgroup $W \subset O(S)$ is its subgroup generated by reflections in hyperplanes of $\mathcal{L}^+(S)$. It has the fundamental chamber

$$\mathcal{M} = \{\mathbb{R}_{++}x \in \mathcal{L}^+(S) \mid (P(\mathcal{M}), x) \le 0\}.$$

The main invariant of the data (I) — (III) is *the generalized Cartan matrix*

$$A = \left(\frac{2(\alpha,\alpha')}{(\alpha,\alpha)}\right) = \quad \alpha,\alpha' \in P = P(\mathcal{M}). \tag{1.3}$$

It defines the corresponding *hyperbolic Kac–Mody algebra* $\mathfrak{g}(A)$, by Kac and Moody. It is *graded by the lattice* S. The next data (IV) and (V) give the *automorphic correction* $\mathfrak{g}$ of this algebra.

By my results and by Vinberg, we have finiteness (for elliptic case) and almost finiteness (for parabolic case) for data 1) — 3).

(IV) For this datum, we need an extended lattice $T = U(m) \oplus S$ (*the symmetry lattice of the Lie algebra* $\mathfrak{g}$) where

$$U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{1.4}$$

$M(m)$ for a lattice M and $m \in \mathbb{Q}$ means that we multiply the pairing of M by m, the orthogonal sum of lattices is denoted by $\oplus$. The lattice T defines the Hermitian symmetric domain of the type IV

$$\Omega(T) = \{\mathbb{C}\omega \subset T \otimes \mathbb{C} \mid (\omega,\omega) = 0,\ (\omega,\overline{\omega}) < 0\}^{+} \tag{1.5}$$

where + means a choice of one (from two) connected components. The domain $\Omega(T)$ can be identified with the complexified cone

$\Omega(V^+(S)) = S\otimes \mathbb{R} + iV^+(S)$ as follows: for the basis e_1, e_2 of the lattice U with the matrix (1.4), we identify $z\in \Omega(V^+(S))$ with $\mathbb{C}\omega_z\in\Omega(T)$ where $\omega_z=(z,z)e_1/2+e_2/m\oplus z\in\Omega(T)^\bullet$ (the corresponding affine cone over $\Omega(T)$). The main datum in (IV) is a *holomorphic automorphic form* $\Phi(z)$, $z\in\Omega(V^+(S))=\Omega(T)$ of some weight $k\in\mathbb{Z}/2$ on the Hermitian symmetric domain $\Omega(V^+(S))=\Omega(T)$ of the type IV with respect to a subgroup $G\subset O^+(T)$ of a finite index (*the symmetry group of the Lie algebra* $\mathfrak{g}$. Here $O^+(T)$ is the index two subgroup of $O(T)$ which preserves $\Omega(T)$.

The automorphic form $\Phi(z)$ must have Fourier expension which gives the denominator identity for the Lie algebra $\mathfrak{g}$:

$$\Phi(z)=\sum_{w\in W}\det(w)(exp(-2\pi i(w(\rho),z))-$$
$$-\sum_{a\in S\cap\mathbb{R}_{++}\overline{\mathcal{M}}} m(a)exp(-2\pi i(w(\rho+a),z))), \qquad (1.6)$$

where all coefficients $m(a)$ must be integral. It also would be nice to calculate the *infinite product expension (the Borcherds product) for the denominator identity of the Lie algebra* $\mathfrak{g}$

$$\Phi(z)=exp(-2\pi i(\rho,z))\prod_{\alpha\in\Delta_+}(1-exp(-2\pi i(\alpha,z)))^{mult(\alpha)}\ , \quad (1.7)$$

which gives *multiplicities* $mult(\alpha)$ of roots of the Lie algebra $\mathfrak{g}$. Here $\Delta_+\subset S$ (see below).

We need finiteness (or almost finiteness) of volume of $\mathcal{M}$ for existence of such automorphic form.

(V) The automorphic form $\Phi(z)$ in $\Omega(V^+(S)) = \Omega(T)$ must be *reflective.* It means that the divisor (of zeros) of $\Phi(z)$ is union of rational quadratic divisors which are orthogonal to roots of T. Here, for $\beta \in T$ with $\beta^2 > 0$ *the rational quadratic divisor which is orthogonal to* β*,* is equal to

$$D_\beta = \{\mathbb{C}\omega \in \Omega(T) \mid (\omega, \beta) = 0\}.$$

The property (V) is valid in a neighbourhood of the cusp of $\Omega(T)$ where the infinite product (1.7) converges, but we want to have it globally.

We believe that with this property the set of data (IV), (V) is finite. This property satisfies in all known interesting cases.

Lorentzian Kac–Moody superalgebra $\mathfrak{g}$ *corresponding to data (I) — (V),* which is a Kac–Moody–Borcherds superablebra or an *automorphic correction* given by $\Phi(z)$ of the Kac–Moody algebra $\mathfrak{g}(A)$ given by the generalized Cartan matrix (1.3) above, is defined by the sequence $P' \subset S$ of *simple roots.* It is divided to the set ${P'}^{re}$ of *simple real root* (all of them are even) and the set ${P'}^{im}_{\bar{0}}$ *of even simple imaginary roots* and the set ${P'}^{im}_{\bar{1}}$ *of odd imaginary roots.* Thus, $P' = {P'}^{re} \cup {P'}^{im}_{\bar{0}} \cup {P'}^{im}_{\bar{1}}$.

For a primitive $a \in S \cap \mathbb{R}_{++}\mathcal{M}$ with $(a, a) = 0$ one should find

$\tau(na) \in \mathbb{Z}$, $n \in \mathbb{N}$, from the indentity with the formal variable t:

$$1 - \sum_{k\in\mathbb{N}} m(ka)t^k = \prod_{n\in\mathbb{N}} (1-t^n)^{\tau(na)}.$$

The set ${P'}^{re} = P$ where P is defined in (III). The set ${P'}^{re}$ is even: ${P'}^{re} = {P'}^{re}_{\overline{0}}$, ${P'}^{re}_{\overline{1}} = \emptyset$. The set

$$
\begin{aligned}
{P'}^{im}_{\overline{0}} = \{m(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M},\ (a,a) < 0 \text{ and } m(a) > 0\} \cup \\
\{\tau(a)a \mid a \in S \cap \mathbb{R}_{++}\overline{\mathcal{M}},\ (a,a) = 0 \text{ and } \tau(a) > 0\}; \qquad (1.8)
\end{aligned}
$$

$$
\begin{aligned}
{P'}^{im}_{\overline{1}} = \{-m(a)a \mid a \in S \cap \mathbb{R}_{++}\mathcal{M},\ (a,a) < 0 \text{ and } m(a) < 0\} \cup \\
\{-\tau(a)a \mid a \in S \cap \mathbb{R}_{++}\overline{\mathcal{M}},\ (a,a) = 0 \text{ and } \tau(a) < 0\} \qquad (1.9)
\end{aligned}
$$

Here, ka for $k \in \mathbb{N}$ means that we repeat a exactly k times in the sequence.

The generalized Kac–Moody superalgebra $\mathfrak{g}$ is the Lie superalgebra with generators h_r, e_r, f_r where $r \in P'$. All generators h_r are even, generators e_r, f_r are even (respectively odd) if r is even (respectively odd).

They have defining relations 1) — 5) of $\mathfrak{g}$ which are given below.

1) The map $r \to h_r$ for $r \in P'$ gives an embedding $S \otimes \mathbb{C}$ to $\mathfrak{g}$ as Abelian subalgebra (it is even).

2) $[h_r, e_{r'}] = (r, r')e_{r'}$ and $[h_r, f_{r'}] = -(r, r')f_{r'}$.

3) $[e_r, f_{r'}] = h_r$ if $r = r'$, and it is 0, if $r \neq r'$.

4) $(ad\ e_r)^{1-2(r,r')/(r,r)}e_{r'} = (ad\ f_r)^{1-2(r,r')/(r,r)}f_{r'} = 0$, if $r \neq r'$ и $(r,r) > 0$ (equivalently, $r \in P'^{\ re}$).

5) If $(r,r') = 0$, then $[e_r, e_{r'}] = [f_r, f_{r'}] = 0$.

The algebra $\mathfrak{g}$ is graded by the lattice S where the generators h_r, e_r and f_r have weights 0, $r \in S$ and $-r \in S$ respectively. We have

$$\mathfrak{g} = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha = \mathfrak{g}_0 \bigoplus \left(\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \right) \bigoplus \left(\bigoplus_{\alpha \in -\Delta_+} \mathfrak{g}_\alpha \right), \qquad (1.10)$$

where $\mathfrak{g}_0 = S \otimes \mathbb{C}$, and Δ is the set of roots (that is the set of $\alpha \in S$ with $\dim \mathfrak{g}_\alpha \neq 0$). The root α is positive ($\alpha \in \Delta_+$) if $(\alpha, \mathcal{M}) \leq 0$. By definition, the multiplicity of $\alpha \in \Delta$ is equal to $mult(\alpha) = \dim \mathfrak{g}_{\alpha,\bar{0}} - \dim \mathfrak{g}_{\alpha,\bar{1}}$.

For this definition, we use results by Kac–Moody, Borcherds, authors, U. Ray.

The case we consider here.

We consider the case when *lattices S for the data (I)—(III) are even hyperbolic lattices, $W \subset O(S)$ is the full group $W = W^{(2)}(S)$ generated by reflections in all elements of S with square 2. They give roots. As the set $P(\mathcal{M})$ of perpendicular roots to the fundamental chamber $\mathcal{M}$ of $W^{(2)}(S)$, we take roots with square 2.*

In our papers 1995 — 2002 we considered this case and some other cases for the rank 3 case, and we constructed many Lorentzian Lie algebras for $\operatorname{rk} S = 3$. Here we want to extend these results for higher ranks.

First, all even hyperbolic lattices S of rank $\operatorname{rk} S \geq 3$ with $[O(S) : W^{(2)}(S)] < \infty$ (equivalently, they are elliptically reflective for $W^{(2)}(S)$) were classified in my papers for $\operatorname{rk} S \neq 4$, and by Vinberg for $\operatorname{rk} S = 4$ around 1982. They are important for K3 surfaces: *K3 surface X with Picard lattice $S_X = S(-1)$ has finite automorphism group, and only for these Picard lattices if $\operatorname{rk} S_X \geq 3$. For such K3 surfaces, the set $P(\mathcal{M})$ is finite and gives all classes of non-singular rational curves of X.*

Second, we find those of these cases which have the lattice Weyl vector ρ for $P(\mathcal{M})$: Thus, there must exist $\rho \in S \otimes \mathbb{Q}$ such that

$$\rho \cdot \alpha = -1 \quad \forall \alpha \in P(\mathcal{M}).$$

For the corresponding K3 surfaces X, the set $P(\mathcal{M})$ gives classes of all non-singular rational curves. They are lines for the hyperplane section defined by ρ.

Theorem 1. *The following and only the following elliptically 2-reflective even hyperbolic lattices S of $\operatorname{rk} S \geq 3$ have a lattice Weyl vector ρ for $W^{(2)}(S)$ (equivalently, for $P(\mathcal{M}^{(2)}(S))$). We order them by the rank and the absolute value of the determinant.*

Rank 3: $S_{3,2} = U \oplus A_1$, $S_{3,8,a} = \langle -2 \rangle \oplus 2A_1$, $S_{3,8,b} = (\langle -24 \rangle \oplus A_2)[1/3, -1/3, 1/3]$, $S_{3,18} = U(3) \oplus A_1$, $S_{3,32,a} = U(4) \oplus A_1$, $S_{3,32,b} = \langle -8 \rangle \oplus 2A_1$, $S_{3,32,c} = U(8)[1/2, 1/2] \oplus A_1$, $S_{3,72} = \langle -24 \rangle \oplus A_2$, $S_{3,128,a} = U(8) \oplus A_1$, $S_{3,128,b} = \langle -32 \rangle \oplus 2A_1$, $S_{3,288} = U(12) \oplus A_1$,
anisotropic cases: $S_{3,12} = \langle -4 \rangle \oplus A_2$, $S_{3,24} = \langle -6 \rangle \oplus 2A_1$, $S_{3,36} = \langle -12 \rangle \oplus A_2$, $S_{3,108} = \langle -36 \rangle \oplus A_2$. *(15 cases).*

Rank 4: $S_{4,3} = U \oplus A_2$, $S_{4,4} = U \oplus 2A_1$, $S_{4,12} = U(2) \oplus A_2$, $S_{4,16,a} = \langle -2 \rangle \oplus 3A_1$, $S_{4,16,b} = \langle -4 \rangle \oplus A_3$, $S_{4,27,a} = U(3) \oplus A_2$, $S_{4,27,b} = \left\langle \begin{array}{rr} 0 & -3 \\ -3 & 2 \end{array} \right\rangle \oplus A_2$, $S_{4,64,a} = U(4) \oplus 2A_1$, $S_{4,64,b} = \langle -8 \rangle \oplus 3A_1$, $S_{4,108} = U(6) \oplus A_2$,

$$S_{4,28} = \left\langle \begin{array}{rrrr} -2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{array} \right\rangle$$ *(anisotropic case). (11 cases)*

Rank 5: $S_{5,4} = U \oplus A_3,\ S_{5,8} = U \oplus 3A_1,\ S_{5,16} = \langle -4 \rangle \oplus D_4,\ S_{5,32,a} = \langle -2 \rangle \oplus 4A_1,\ S_{5,32,b} = \langle -8 \rangle \oplus D_4,\ S_{5,64} = \langle -16 \rangle \oplus D_4,\ S_{5,128} = U(4) \oplus 3A_1$. *(7 cases)*

Rank 6: $S_{6,4} = U \oplus D_4,\ S_{6,5} = U \oplus A_4,\ S_{6,9} = U \oplus 2A_2,\ S_{6,16,a} = U(2) \oplus D_4,\ S_{6,16,b} = U \oplus 4A_1,\ S_{6,64,a} = \langle -2 \rangle \oplus 5A_1,\ S_{6,64,b} = U(4) \oplus D_4,\ S_{6,81} = U(3) \oplus 2A_2$. *(8 cases)*

Rank 7: $S_{7,4} = U \oplus D_5,\ S_{7,6} = U \oplus A_5,\ S_{7,128} = \langle -2 \rangle \oplus 6A_1$. *(3 cases)*

Rank 8: $S_{8,3} = U \oplus E_6,\ S_{8,4} = U \oplus D_6,\ S_{8,7} = U \oplus A_6,\ S_{8,16} = U \oplus 2A_3,\ S_{8,27} = U \oplus 3A_2,\ S_{8,256} = \langle -2 \rangle \oplus 7A_1$. *(6 cases)*

Rank 9: $S_{9,2} = U \oplus E_7,\ S_{9,4} = U \oplus D_7,\ S_{9,8} = U \oplus A_7,\ S_{9,512} = \langle -2 \rangle \oplus 8A_1$. *(4 cases)*

Rank 10: $S_{10,1} = U \oplus E_8,\ S_{10,4} = U \oplus D_8,\ S_{10,16} = U \oplus 2D_4,\ S_{10,64} = U(2) \oplus 2D_4$. *(4 cases)*

Rank 18: $S_{18,1} = U \oplus 2E_8$. (1 case)

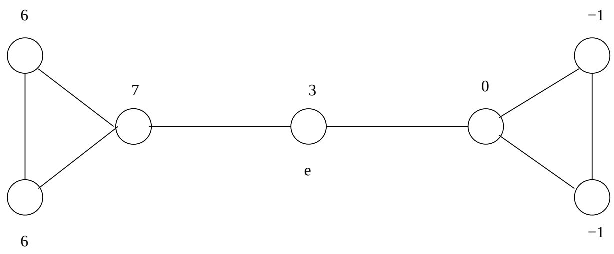

Рис. 1: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U \oplus A_2 \oplus A_2$ is $St(\widetilde{\mathbb{A}_2}, \widetilde{\mathbb{A}_2})$.

For all these cases, we found generalized Cartan matrices A which define hyperbolic Kac–Moody Lie algebras $\mathfrak{g}(A)$. Below, for some of these cases, you can see Dynkin diagrams of elements $P(\mathcal{M})$ which are equivalent to the generalized Cartan matrices A. Equivalently, they describe graphs of all non-singular rational curves on the corresponding K3 surfaces. All of them have the degree 1 for the corresponding lattice Weyl vectors ρ.

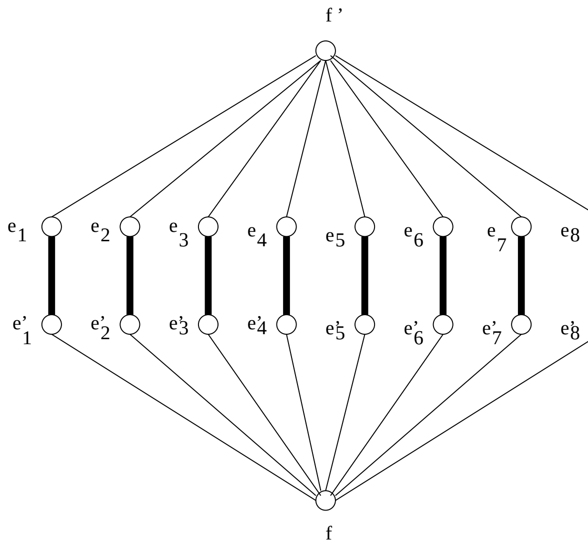

Рис. 2: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(2) \oplus 2D_4$.

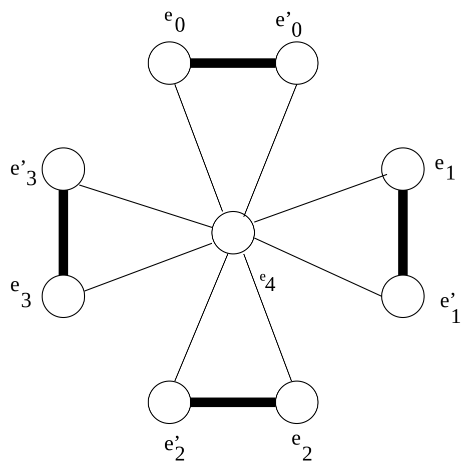

Рис. 3: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(4) \oplus D_4$.

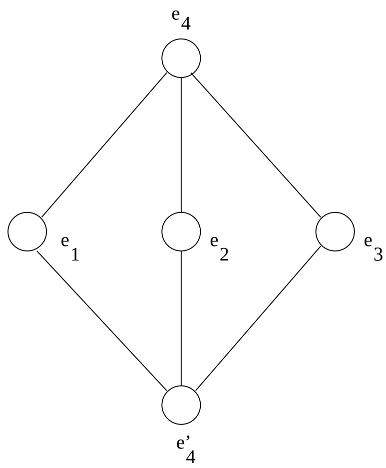

Рис. 4: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -4\rangle \oplus D_4$.

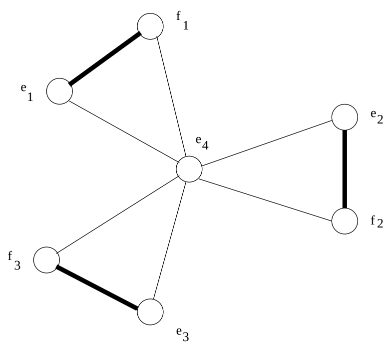

Рис. 5: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -8\rangle \oplus D_4$.

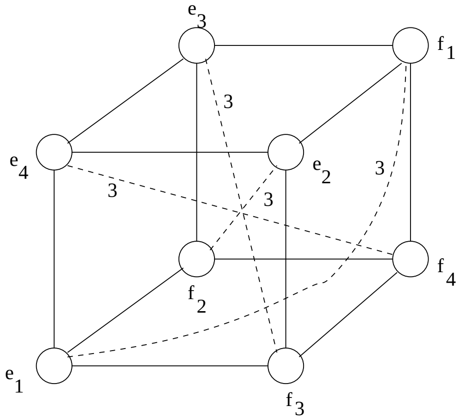

Рис. 6: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -16\rangle \oplus D_4$.

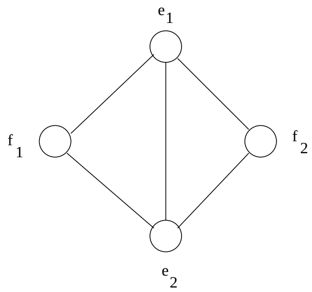

Рис. 7: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(2) \oplus A_2$.

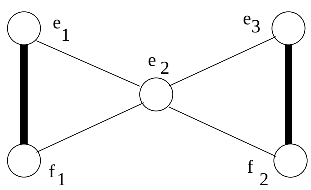

Рис. 8: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $\langle -4\rangle \oplus A_3$.

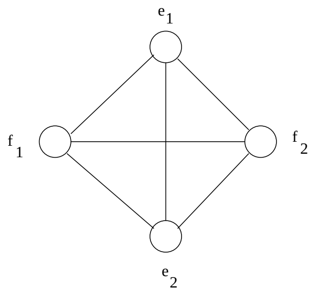

Рис. 9: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U(3) \oplus A_2$.

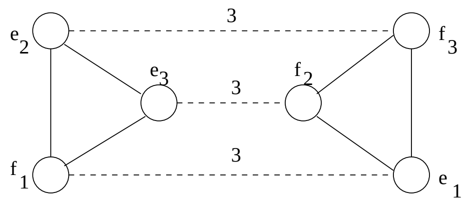

Рис. 10: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $S_{4,27,b}$.

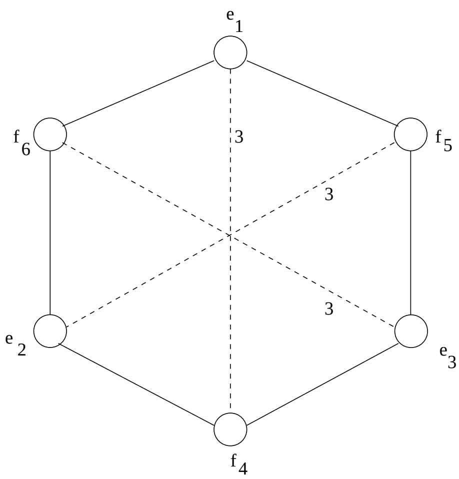

Рис. 11: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $S_{4,28}$.

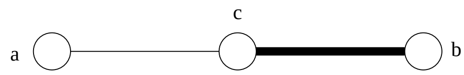

Рис. 12: The graph $\Gamma(P(\mathcal{M}^{(2)}))$ for $U \oplus A_1$.

Almost for all these cases, we found the automorphic form $\Phi(z)$ which gives the automorphic correction and finishes construction of the corresponding Lorentzian Kac–Moody algebra.

We found the automorphic forms $\Phi(z)$ which give automorphic corrections of the corresponding hyperbolic Kac–Moody algebras defined by generalized Cartan matrices of $P(\mathcal{M})$ for $W^{(2)}(S)$ (or by the corresponding Dynkin diagrams) for the following series of hyperbolic lattices S of Theorem 1:

1) For the lattices $U \oplus K$,

$$K = A_1;\ 2A_1, A_2;\ 3A_1,\ A_3;\ 4A_1,\ 2A_2,\ A_4,\ D_4;\ A_5, D_5;$$

$$3A_2,\ 2A_3,\ A_6,\ D_6,\ E_6;\ A_7,\ D_7,\ E_7;\ 2D_4,\ D_8,\ E_8,\ 2E_8$$

and $U(2) \oplus 2D_4$;

2) For the lattices $\langle -2\rangle \oplus kA_1$, $2 \le k \le 9$ (the case $k = 9$ is parabolic).

3) For the lattices $U(4) \oplus kA_1$, $1 \le k \le 4$ (the case $k = 4$ is parabolic).

4) For the lattices $U(3) \oplus kA_2$, $k = 1, 2, 3$ (the last case is parabolic).

5) For the lattices $U(2) \oplus D_4$ and $U(4) \oplus D_4$.

6) For the 2-reflective lattices of parabolic type $U \oplus K$,
$K = A_1(2),\ A_1(3),\ A_1(4),\ D_2(2),\ A_2(2),\ A_2(3),\ A_3(2),\ D_4(2),\ E_8(2)$.

All these automorphic forms $\Phi(z)$ have divisors which are sums of rational quadratic divisors with multiplicity one on the corresponding Hermitian symmetric domains $\Omega(T)$ which are orthogonal to 2-roots of the corresponding lattices $T = U(m) \oplus S$ (for some $m > 0$) of signature $(n, 2)$.

Thus, K3 surfaces with Picard lattice S (*they have finite automorphism group and their non-singular rational curves are lines for the polarization* ρ) are **mirror symmetric** to K3 surfaces with the corresponding transcendental lattice $T = U(m) \oplus S$ (for some $m > 0$) (*their discriminants of moduli are given by zeros of the automorphic forms* $\Phi(z)$ *with irreducible divisors of multiplicity one*).

This mirror symmetry (we call it **arithmetic mirror symmetry**) is given by the automorphic form $\Phi(z)$ which we construct and by the corresponding Lorentzian Kac–Moody algebra with the denominator identity and the root lattice defined by $\Phi(z)$ and S. It can be considered as a kind of Physical evidence of this mirror symmetry.

Almost for all cases, we construct automorphic forms $\Phi(z)$ using quasy pull-back from some automorphic forms of high rank (by embedding $T \subset L$) and by restricting of the automorphic form on $\Omega(L)$ to the subdomain $\Omega(T)$.

For the *Series 1)*, we use Borcherds automorphic form Φ_{12} of weight 12 and character det with respect to $O^+(II_{26,2})$ on $\Omega(II_{26,2})$ where $II_{26,2}$ is even unimodular lattice of signature $(26, 2)$.

For the *Series 2)*, the automorphic correction is defined by $L = U(2) \oplus \langle -2 \rangle \oplus 8\langle 2 \rangle$ and by the modular form

$$\Delta_{5,D_7} = \mathrm{Lift}(\psi_{5,\,D_7}) \in S_5(O^+(U(2) \oplus \langle -2 \rangle \oplus 8\langle 2 \rangle), \chi_2)$$

where

$$\psi_{5,\,D_7}(\tau, \mathfrak{z}) = \eta(\tau)^9\ \vartheta(z_1) \cdot \ldots \cdot \vartheta(z_7) \tag{1.11}$$

is Jacobi form and Lift is arithmetic lifting of Jacobi forms.

For the *Series 3)*, the automorphic correction is given by $L = 2U \oplus 3A_1$ and

$$\Delta_{3,3A_1} = \mathrm{Lift}(\eta(\tau)^9 \vartheta(z_1)\vartheta(z_2)\vartheta(z_3)) \in S_3(O^+(2U \oplus 3A_1)). \tag{1.12}$$

For the *Series 4)*, the automorphic correction is given by $L = 2U(3) \oplus 3A_2$ and

$$\Delta_{3,\,3A_2} \in M_3(O^+(2U(3) \oplus 3A_2), \chi_2)$$

which gives a strongly 2-reflective modular form with the complete 2-divisor where χ_2 is a binary character of the orthogonal group. It is constructed in our preprint.

Series 5): For $S = U(2) \oplus D_4$ we found two automorphic corrections: one with $T = U \oplus U(2) \oplus D_4$ and $\Phi(z)$ of weight 40; another with $T = 2U \oplus D_4$ and $\Phi(z)$ of weight 8.

For $S = U(4) \oplus D_4$, we found automorphic correction with $T = 2U(4) \oplus D_4$ and $\Phi(z)$ of weight 6.

For the *Series 6)*, we also use Borcherds automorphic form Φ_{12} of weight 12 and character det with respect to $O^+(II_{26,2})$ on $\Omega(II_{26,2})$ where $II_{26,2}$ is even unimodular lattice of signature $(26, 2)$.

For the series 1), we write $II_{26,2}$ as $2U \oplus N_j$ where N_j is Niemeier lattice, and we embed $K \subset N_j$. For the series 6), we write $II_{26,2}$ as $2U \oplus Leech$ and we embed $K \subset Leech$.

Look other details in our preprint.

We hope to extend these results to other cases. Because of finiteness results, we have a hope to obtain finite classification finally, and construct a theory of Lorentzian (or hyperbolic automorphic) Lie algebras.