Lecture 2: Instantons in 5d QFTs

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"Partition functions and automorphic forms"

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Plan

- Lecture 2:

- 5d instantons for 5d SCFTs: dualities, UV symmetry enhancements

[Seiberg] [Morrison, Seiberg] (1996), [Intriligator, Morrison, Seiberg] (1997) [Aharony, Kol, Hanany]
[H.-C.Kim, S.-S. Kim, K. Lee] [Hwang, J. Kim, SK, Park] [Hayashi, S.-S.Kim, K. Lee, Taki, Yagi] ……

- 5d instantons for 6d SCFTs

[Aharony, Berkooz, Seiberg] [H.-C. Kim, SK, Koh, K. Lee, S. Lee] [Haghighat, Iqbal, Kozcaz, Lockhart, Vafa]
[Aharony, Berkooz, Kachru, Silverstein] [J. Kim, SK, K. Lee, Park, Vafa] [Hayashi, S.-S.Kim, K. Lee, Taki, Yagi] ……

- Can't cover in detail here: many other developments (defects, dualities, topological strings/vertices, etc.)

[Gaiotto, H.-C. Kim], [Iqbal, Marino, Vafa] [Huang, Klemm, et.al.] [Kim, Hayashi, Nishinaka]
[H.-C.Kim] [Hayashi, Ohmori] ……

Examples: SU(2) theories w/ N_f matters

- 4d: $N_f = 0,1,2,3,4$. Yields 4d QFTs which are well-defined at short distance.
- No Landau poles, $\beta(g_{YM}^2) \leq 0$. Asymptotically free ($N_f < 4$) or conformal ($N_f = 4$)

- 5d: $N_f = 0, 1, \cdots, 7$. Related to 5d SCFTs at short distance. (next slides)

- 5d: $N_f = 8$. Related to a 6d SCFT compactified on S^1. (later in this talk)

- 5d: $N_f > 8$. Too many matters. No known relation to sensible quantum systems.

- Meanings of "instantons" are all different in these examples.
  - 4d: vacuum tunneling
  - 5d: particles which become massless ($4\pi^2/g_{YM}^2 \to 0$) at strong coupling.
  - 6d: Exotic roles, as infinite tower of Kaluza-Klein particles (inspired by M-theory)
- Thus, the ways of using the instanton partition functions differ, as well.

5d SCFTs

- 5d Yang-Mills theories are non-renormalizable, inconsistent by themselves.

- $[g_{YM}^2] = M^{-1}$, perturbative corrections cannot be unambiguously cured within this QFT

- Should be used as effective field theory (EFT) of bigger systems, e.g. D-branes in strings theory, for limited computations.
- Seiberg (1996) argued that certain 5d SUSY Yang-Mills are mass deformations of 5d superconformal field theory, by $g_{YM}^{-2} \sim M_{inst}$, and EFT valid at $E \ll M_{inst}$

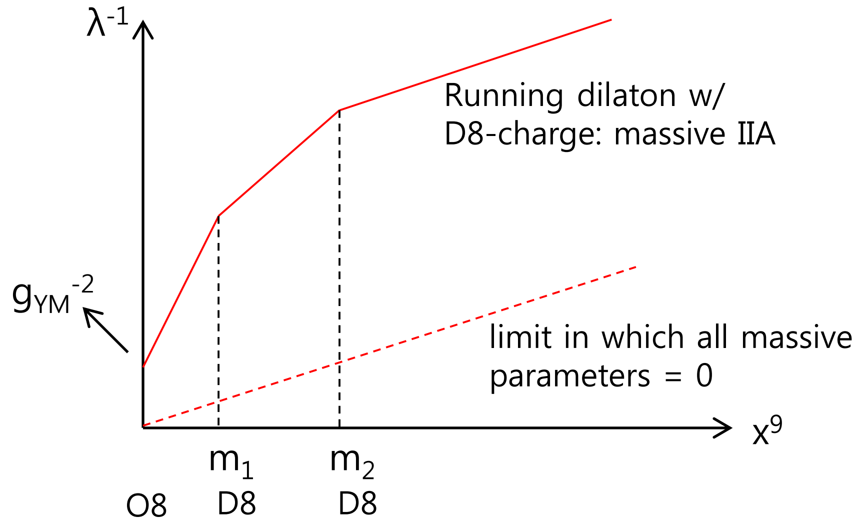
  ($M_{inst} = \frac{4\pi^2}{g_{YM}^2}$ denotes the mass of unit instanton)

- Consistent 5d QFT (SCFT) lives at the infinite coupling point of Yang-Mills.
- Such QFT's do not admit standard Lagrangian descriptions, so defined rather indirectly/abstractly.
- Let me briefly explain how we got to such conclusions in string theory.

SU(2) w/ $N_f \leq 7$ fundamental hypers

- Engineered on 1 D4 probing Nf D8 + O8 [Seiberg] 1996

	0	1	2	3	4	5	6	7	8	9
D4	•	•	•	•	•					v
D8/O8	•	•	•	•	•	•	•	•	•	m_i



- D8/O8 source the inverse-coupling to (linearly) run.
- String coupling constant at $x^9 = 0$ sets the 5d gauge coupling.
- For $N_f \leq 7$, this defines scale-free system (∞ coupling) at $x^9 = 0$ on D4-D8-O8.

- After YM deformation:

- $Sp(1) \sim SU(2)$ gauge theory on D4.
- N_f fundamental hypers from D4-D8 open strings
- $SO(2N_f)$ global symmetry at finite coupling, which rotates N_f quarks

Enhanced symmetries & "dualities"

- Question: If one can reach infinite coupling SCFT point,

1) Symmetries at infinite coupling, non-perturbative in Yang-Mills description?
2) What stays beyond it? New phase?

- String duality predicts interesting symmetry enhancements at infinite coupling

- E.g. our previous models: $SU(2)$ w/ $N_f \leq 7$ quarks.
- Duality to $E_8 \times E_8$ heterotic strings predicts $SO(2N_f) \to E_{N_f+1}$ symmetry enhancement
- non-perturbative mechanisms of realizing exceptional symmetries using D-branes

- Beyond strong coupling: new mass deformations, new 5d Yang-Mills description

- Trivial in our previous models. But many other models run into "dual" phases.
- Somewhat similar to "Seiberg dualities" in 4d, 3d: IR dualities of two different UV theories
- 5d: Two different mass deformations of same UV SCFT. "UV duality…" ?

Symmetry enhancement & instantons

- Instantons are responsible for the symmetry enhancements.
- On spatial R^4 slice of $R^{4,1}$, $\int_{R^4} \mathrm{tr}(F \wedge F) \propto$ instanton particle number
- Particle number ~ conserved charge: topological $U(1)_I$ conserved current in 5d

$$J_\mu = \star_5 \mathrm{tr}(F \wedge F)_\mu$$

$$\partial_\mu J^\mu = 0 \ \text{ from } \ d\ \mathrm{tr}(F \wedge F) = 2\mathrm{tr}(F \wedge DF) = 0$$

- String theory predicts: $SO(2N_f) \times U(1)_I \to E_{N_f+1}$. Encoded in instanton ptn. ftn.

- In Witten index Z_k, extra chemical potentials: m_a for $U(1)^{N_f} \subset SO(2N_f)$

$$Z_k(\epsilon_{1,2}, a_i, m_a) = \mathrm{Tr}_k \left[(-1)^F e^{-\epsilon_1(J_1+J_R)-\epsilon_2(J_2+J_R)} e^{-a^i q_i} e^{-m_a F_a} \right]$$

- Trace of $SO(2N_f)$: Expanding in $e^{-\epsilon_{1,2}}, e^{-a_i}$, coefficients are characters of $SO(2N_f)$ irreps.

- Grand partition function $Z(q, \epsilon_{1,2}, a_i, m_a) = Z_{\mathrm{pert}}(\epsilon_{1,2}, a, m) \sum_{k=0}^{\infty} Z_k(\epsilon_{1,2}, a, m) q^k$

- Coefficients should be characters of E_{N_f+1}, with parameters e^{-m_a}, q.

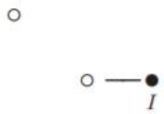
Predicted enhancement patterns

$SO(2N_f) \times U(1)_I \to E_{N_f+1}$

$N_f = 2: \quad E_3 = SU(3) \times SU(2) \supset SO(4) \times U(1)_I$

$SU(3) \supset SU(2) \times U(1)_I$

$\mathbf{8} = \mathbf{1}_0 + \mathbf{3}_0 + \mathbf{2}_1 + \mathbf{2}_{-1}.$

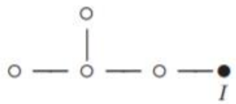


$N_f = 3: \quad E_4 = SU(5) \supset SU(4) \times U(1)_I$

$\mathbf{24} = \mathbf{1}_0 + \mathbf{15}_0 + \mathbf{4}_1 + \overline{\mathbf{4}}_{-1}$

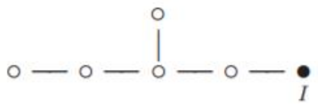
$N_f = 4: \quad E_5 = SO(10) \supset SO(8) \times U(1)$

$\mathbf{45} = \mathbf{1}_0 + \mathbf{28}_0 + \mathbf{8}_{-1} + \mathbf{8}_1$



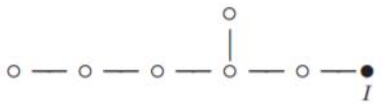
$N_f = 5: \quad E_6 \supset SO(10) \times U(1)$

$\mathbf{78} = \mathbf{1}_0 + \mathbf{45}_0 + \mathbf{16}_{-1} + \overline{\mathbf{16}}_1$



$N_f = 6: \quad E_7 \supset SO(12) \times U(1)_I$

$\mathbf{133} = \mathbf{1}_0 + \mathbf{66}_0 + \mathbf{32}_1 + \mathbf{32}_{-1} + \mathbf{1}_2 + \mathbf{1}_{-2}$



$N_f = 7: \quad E_8 \supset SO(14) \times U(1)_I$

$\mathbf{248} = \mathbf{1}_0 + \mathbf{91}_0 + \mathbf{64}_1 + \overline{\mathbf{64}}_{-1} + \mathbf{14}_2 + \mathbf{14}_{-2}$

Calculus at $N_f \leq 4$

- Extra 1d matters from N_f hypers: N_f $U(k)$ fundamental Fermi multiplets.
- Integrand/residues: $$Z_{\text{1-loop}} = \frac{\prod_{I\neq J} 2\sinh\frac{\phi_{IJ}}{2} \cdot \prod_{I,J=1}^{k} 2\sinh\frac{\phi_{IJ}+2\epsilon_+}{2}}{\prod_{I=1}^{k}\prod_{i=1}^{2} 2\sinh\frac{\epsilon_+ \pm(\phi_I - a_i)}{2} \prod_{I,J=1}^{k} 2\sinh\frac{\phi_{IJ}+\epsilon_1}{2}\cdot 2\sinh\frac{\phi_{IJ}+\epsilon_2}{2}} \cdot \prod_{I=1}^{k}\prod_{a=1}^{N_f} 2\sinh\frac{\phi_I + m_a}{2}$$

$$Z_k = \sum_{\sum_i |Y_i| = k} \prod_{i=1}^{2}\prod_{s\in Y_i} \frac{\prod_{a=1}^{N_f} 2\sinh\frac{\phi(s)+m_a}{2}}{\prod_{j=1}^{2} 2\sinh\frac{E_{ij}(s)}{2}\cdot 2\sinh\frac{E_{ij}(s)-2\epsilon_+}{2}}$$

- Should restrict to $N_f \leq 4$. Otherwise, there appear new poles at $|\phi_I| = \infty$

- residues at finite ϕ_I: states made w/ the corresponding matter scalar fields.
- residues at ∞: states made w/ UV artifact fields $\phi = \varphi + iA_\tau$ in 1d vector multiplet
- At $N_f > 4$, don't know how to project out contributions from the last spurious states.

- E.g. partition function at $N_f = 0$: $SO(0) \times U(1)_I \rightarrow E_1 = SU(2)$ predicted.

- W/ renormalized Coulomb VEV $A^4 = e^{-4a}q$ [Mitev, Pomoni, Taki, Yagi] (2014)

$$Z(q, A, \epsilon_{1,2})^{N_f=0} = 1 + \frac{\mathfrak{t}+\mathfrak{q}}{(1-\mathfrak{t})(1-\mathfrak{q})}\chi_2^{E_1}(q)A^2 + \left[\frac{(\mathfrak{q}^2+\mathfrak{t}^2)(\mathfrak{q}+\mathfrak{t}+\mathfrak{q}^2+\mathfrak{t}^2+\mathfrak{q}\mathfrak{t}(1_{\mathfrak{q}}+\mathfrak{t}))}{\mathfrak{q}\mathfrak{t}(1-\mathfrak{q}^2)(1-\mathfrak{t}^2)} + \frac{(\mathfrak{q}+\mathfrak{t}+\mathfrak{q}^2+\mathfrak{t}^2+\mathfrak{q}\mathfrak{t}(1+\mathfrak{q}+\mathfrak{t}))}{(1-\mathfrak{q})(1-\mathfrak{q}^2)(1-\mathfrak{t})(1-\mathfrak{t}^2)}\chi_3^{E_1}(q)\right]A^4 + \mathcal{O}(A^6)$$

$$\mathfrak{q} = e^{-\epsilon_1}\ ,\quad \mathfrak{t} = e^{\epsilon_2}$$

[In this case, this simply tests $q \rightarrow q^{-1}$ invariance]

5d SU(2) at $N_f = 5,6,7$

- SU(2) ADHM construction, w/ U(k) 1d gauge symmetry, fails.
- Bad UV completion: extra UV d.o.f. messes up spectrum. Hard to disentangle.
- However, see topological vertex analyses, which directly uses these brane webs for the SU(2) systems [Mitev, Pomoni, Taki, Yagi] (2014), [S.-S. Kim, Taki, Yagi] (2015)

- Use coincidence $SU(2) \sim Sp(1)$: $Sp(N)$ ADHM w/ 1d $O(k)$ gauge symmetry.
- Upshot: Very carefully follow what string theory demands you do to. (More subtle stories. Please refer to [Hwang, J.Kim, SK. Park] (2014) for details.)
- E.g., at $N_f = 5$: can check $SO(10) \times U(1)_I \to E_6$ [Mitev, Pomoni, Taki, Yagi]

$$Z^{N_f=5} = 1 - \frac{\mathfrak{q}^{1/2}\mathfrak{t}^{1/2}}{(1-\mathfrak{q})(1-\mathfrak{t})}\chi^{E_6}_{\mathbf{27}}A + \left[\frac{\mathfrak{q}+\mathfrak{t}}{(1-\mathfrak{q})(1-\mathfrak{t})}\chi^{E_6}_{\mathbf{27}} + \frac{\mathfrak{q}\mathfrak{t}}{(1-\mathfrak{q}^2)(1-\mathfrak{t}^2)}\chi^{E_6}_{\mathbf{351}} + \frac{\mathfrak{q}\mathfrak{t}(\mathfrak{q}+\mathfrak{t})}{(1-\mathfrak{q})(1-\mathfrak{q}^2)(1-\mathfrak{t})(1-\mathfrak{t}^2)}(\chi^{E_6}_{\mathbf{27}})^2\right]A^2 + \cdots$$

$$\mathbf{27} \to \mathbf{1}_{-4} + \mathbf{10}_2 + \mathbf{16}_{-1}$$

$$\mathbf{351} \to \mathbf{10}_2 + \overline{\mathbf{16}}_5 + \mathbf{16}_{-1} + \mathbf{45}_{-4} + \mathbf{120}_2 + \mathbf{144}_{-1}$$

"Dualities"

- For our $SU(2)$ examples w/ quarks, going beyond infinite coupling point is trivial.
- $U(1)_I \to SU(2)$: $q = e^{-4\pi^2/g_{YM}^2} \to q^{-1}$ is its Weyl symmetry. Instantons ↔ anti-instantons

- In asymptotically free 4d theories, or 3d theories, couplings are weak in UV.
- Two QFT's with different elementary fields are different in UV
- May describe same system in IR, at strong coupling (IR duality, Seiberg duality)

- In 5d gauge theories, couplings are weak in IR, strong in UV (nonrenormalizable)
- Different gauge theories in IR may have same UV origin from a strong coupling SCFT
- 5d SCFT = phase transition, $1/g_{YM}^2 =$ mass deformation [Witten] (1996)
- Some examples are [Gaiotto, H.-C. Kim] (2015)

$Sp(N)$ w/ N_f fundamental hypers ↔ $SU(N+1)$ w/ N_f hyper at CS level $\kappa = N + 3 - \frac{N_f}{2}$

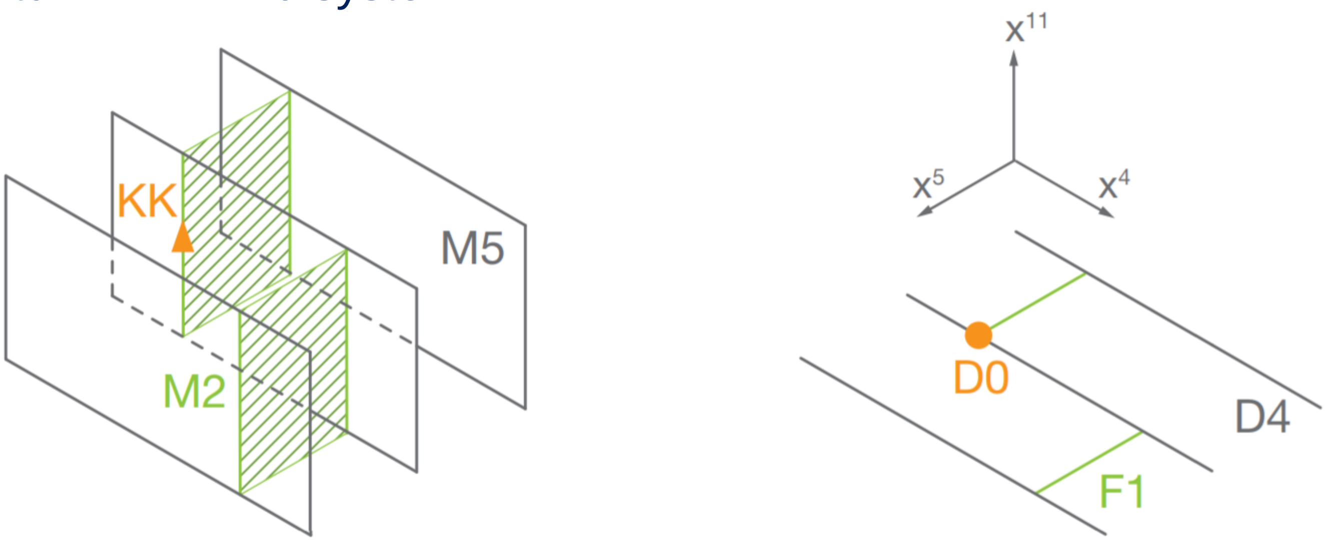
- The "duality" between these theories lead to nontrivial relations between the instanton partition function, given by the so-called elliptic Fourier transformation [Spiridonov, Warnaar]

5d QFT for 6d SCFT on S^1

- Now consider 5d SU(N) or U(N) SYM, with a hypermultiplet in adjoint rep.

  $A_\mu \ (\mu = 0, \cdots, 4)$, Φ : real scalar , fermions $\qquad q_A \sim (q, \tilde{q}^\dagger)$: two complex scalars , fermions

- Maximally supersymmetric Yang-Mills in 5d (w/ mass deformation. 5d $N = 1^*$)
- Internal symmetry (R-symmetry): $SU(2)_R \to SO(5)$, or $SU(2)_R \times SU(2)_L$ in Coulomb branch
- M5 on S^1 w/ momentum = D4-D0 system



- Instanton solitons here form ∞ tower of Kaluza-Klein field/particles of 6d QFT.
- Therefore, by studying instanton partition functions, one constructs 6d physics.
- It is crucial to have finite 5d coupling, $1/g_{YM}^2 \sim 1/R$, at least to start with.

Instanton partition functions for 5d maximal SYM

- (0,4) QM → (4,4) SUSY QM: more matters added to ADHM fields

$$\begin{aligned} \text{chiral} + \text{Fermi} &: (q,\psi),\ (\psi')_m \in (\mathbf{k},\overline{\mathbf{N}}),\ (\tilde{q},\tilde{\psi}),\ (\tilde{\psi}')_m \in (\overline{\mathbf{k}},\mathbf{N}),\ (a,\Psi),\ (\Psi')_m,\ (\tilde{a},\tilde{\Psi}),\ (\tilde{\Psi}')_m \in (\mathbf{adj},\mathbf{1}) \\ \text{vector} + \text{chiral} &: (A_t,\varphi,\lambda_0),\ (\lambda),\ (\phi,\chi)_m,\ (\tilde{\phi},\tilde{\chi})_m \in (\mathbf{adj},\mathbf{1}) \end{aligned}$$

- Index: $Z_k(\epsilon_{1,2},a_i,m) = \mathrm{Tr}_k\left[(-1)^F e^{-\epsilon_1(J_1+J_R)-\epsilon_2(J_2+J_R)} e^{-a^i q_i} e^{-2mJ_L}\right]$

- U(N) result: extra chiral multiplets' JK-Res turn out to be 0 [Hwang, Kim, SK, Park]

$$Z_k = \sum_{\sum_i |Y_i|=k} \prod_{i=1}^{N} \prod_{s\in Y_i} \prod_{j=1}^{N} \frac{2\sinh\frac{E_{ij}(s)+m-\epsilon_+}{2} \cdot 2\sinh\frac{E_{ij}-m-\epsilon_+}{2}}{2\sinh\frac{E_{ij}(s)}{2} \cdot 2\sinh\frac{E_{ij}(s)-2\epsilon_+}{2}}$$

$$E_{ij}(s) = a_i - a_j - \epsilon_1 h_i(s) + \epsilon_2(v_j(s)+1)$$

- Other gauge groups: more involved contour integrals [Hwang, S. Kim, SK]

5d or 6d?

- Apparently a 5d observable on $R^4 \times S^1$. But M-theory predicts that, summing over k, it is a 6d observable, on $R^4 \times T^2$. Can we confirm this?
- Start from $N = 1$: D0-branes bound to single D4. 6d QFT on single M5…?

- One can keep showing the identities: [H.-C. Kim, SK, Koh, K.Lee, S.Lee]

$$Z_1(\epsilon_{1,2}, m) = \frac{\sinh\frac{m+\epsilon_-}{2}\sinh\frac{m-\epsilon_-}{2}}{\sinh\frac{\epsilon_1}{2}\sinh\frac{\epsilon_2}{2}}$$

a ½-BPS superparticle on R^4

$$Z_2 = \frac{Z_1^2 + Z_1(2\epsilon_{1,2}, 2m)}{2} + Z_1$$

new (unique) bound state at k=2, 3

$$Z_3 = \frac{Z_1^3 + 3Z_1 Z_1(2\epsilon_{1,2}, 2m) + 2Z_1(3\epsilon_{1,2}, 3m)}{6} + Z_1^2 + Z_1$$

identical 2 or 3 particles, w/ Bose/Fermi statistics

2 particle states: instanton at k=1 + new bound state at k=2

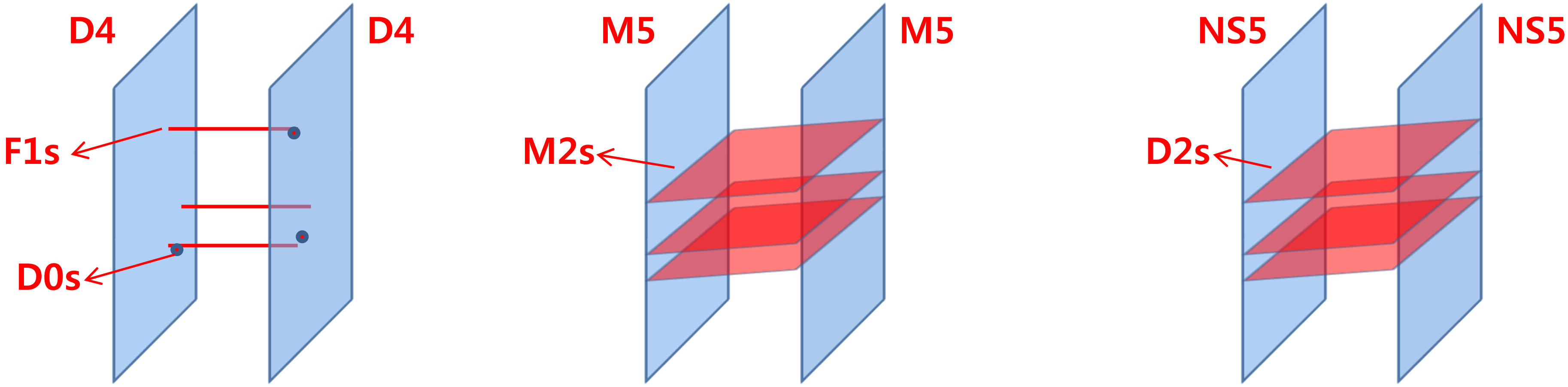
- In fact, [Iqbal, Kozcaz, Shabbir] showed, using topological vertex techniques, that

$$Z(q, \epsilon_{1,2}, m) = PE\left[Z_1(\epsilon_{1,2}, m)\frac{q}{1-q}\right] \equiv \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}Z_1(n\epsilon_{1,2}, nm)\frac{q^n}{1-q^n}\right]$$

- Expression inside "PE" is the index over single particle states ~ "5d fields"
- Same field content appears at all q^k order: comes from a free 6d field.

More systematic studies: M-strings

- $N \geq 2$: Electric charge ~ strings between D4 ~ M2 between M5's, wrapping S^1
- F1-D0 ~ M2-momentum.
- Alternative approach: For simplicity, let us only consider the case with $N = 2$.



- Previously, we used the QM (left figure) at fixed q^k order, to study $Z_k(a, \epsilon_{1,2}, m)$.
- First expand $Z(q, a, \epsilon_{1,2}, m)$ in $w = e^{-a} = e^{-(a_i - a_{i+1})}$, w/ q dependence kept

$$Z(q, a, \epsilon_{1,2}, m) = Z_{N=1}(q, \epsilon_{1,2}, m)^2 \sum_{n=0}^{\infty} e^{-na} Z_n(q, \epsilon_{1,2}, m)$$

- 2d gauge theory on D2's computes $Z_n(q, \epsilon_{1,2}, m)$ exactly in q: elliptic genus

M-strings

- 2d QFT at $N=2$: (0,4) SUSY, $U(n)$ gauge symmetry, w/ following fields

$$
\begin{aligned}
(A_\mu, \lambda_0, \lambda) &: \text{ vector mutiplet } \in \mathbf{adj} \\
q_{\dot{\alpha}} = (q, \tilde{q}^\dagger) &: \text{ hypermultiplet } \in \mathbf{k} \\
a_{\alpha\dot{\beta}} \sim (a, \tilde{a}^\dagger) &: \text{ hypermultiplet } \in \mathbf{adj} \\
\Psi_a &: \text{ Fermi multiplets } \in \mathbf{k} \quad (a = 1, 2 \text{ for } SU(2)_L)
\end{aligned}
$$

- Elliptic genus Z_n computed using completely same techniques of lecture 1:

$$Z_n = (-1)^n \sum_{|Y|=n} \prod_{s \in Y} \frac{\theta(m+\phi(s))\theta(m-\phi(s))}{\theta(E(s))\theta(E-2\epsilon_+)}$$

$$E(s) = -\epsilon_1 h(s) + \epsilon_2(v(s)+1)$$

$$\theta(z) \equiv \frac{i\theta_1(\tau|\frac{z}{2\pi i})}{\eta(\tau)}$$

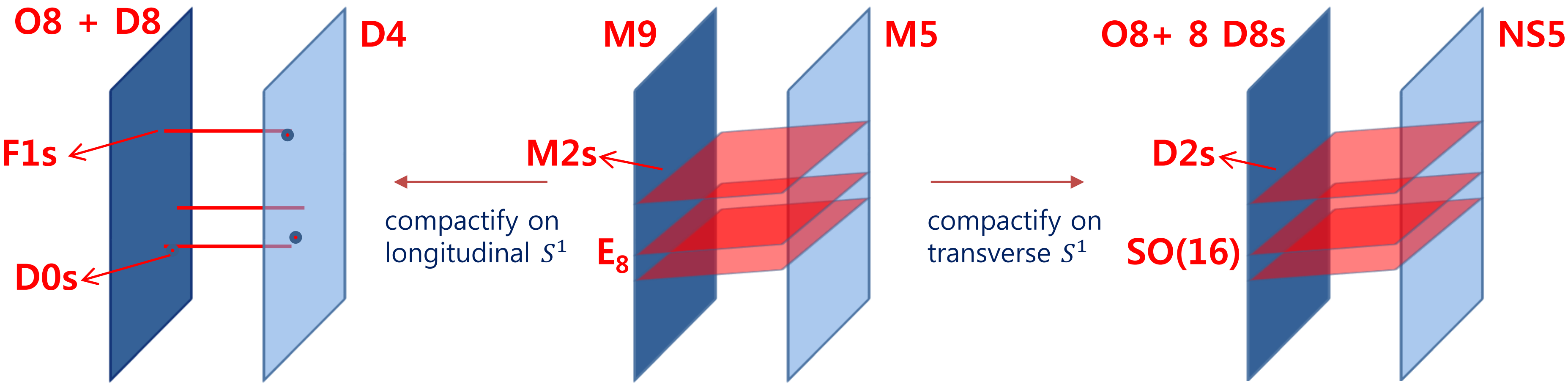
- String theory asserts that

$$Z_{\text{pert}}(a, \epsilon_{1,2}, m) \sum_{k=0}^{\infty} q^k Z_k(a, \epsilon_{1,2}, m) = Z_{N=1}(q, \epsilon_{1,2}, m)^2 \sum_{n=0}^{\infty} e^{-na} Z_n(q, \epsilon_{1,2}, m)$$

- Checked by double-expanding both sides in high orders of $q = e^{2\pi i\tau}, e^{-a}$.
- Implies that the infinite towers of instantons arrange themselves into elliptic genera, which manifestly sees the 6th circle direction. Also, a kind of duality.

5d SU(2) at $N_f = 8$ & E-strings

- Engineered on a D4-brane probing an O8$^-$ plane & $N_f = 8$ D8-branes.
- Their D8-brane charges cancel, not causing the coupling constant to run.
- Have finite coupling const. $1/g_s \sim 1/R$ of S^1/Z_2 for M-theory



	0	1	2	3	4	5	6	7	8	9
NS5	•	•		•	•	•	•			
D8-O8	•	•		•	•	•	•	•	•	•
D2	•	•	•							

- 2d gauge theories for E-strings constructed.
- The 5d instanton partition functions sum to yield elliptic genera of E-strings.
- Both 2d gauge theory & 1d ADHM only see $SO(16)$, but indices show enhanced E_8.

Generalizations and perspectives

- Many 5d models which describe 6d SCFTs on S^1. Just for instance…

- 5d SU(N+1), $N_f = 2N + 6$ $\leftrightarrow$ 5d Sp(N+1), $N_f = 2N + 8$ $\leftrightarrow$ 6d Sp(N), $N_f = 2N + 8$ [Gaiotto, H.-C.Kim,] [Hayashi, S.-S.Kim, K. Lee, Taki] [Yun] ……
- SU(3) w/ no matters & Chern-Simons level $\kappa = 9$ $\leftrightarrow$ outer automorphism twist reduction of 6d SU(3) w/ no matters [H.-C. Kim, Jefferson, Vafa, Zafrir] [H.-C. Kim, Jefferson, Katz, Vafa]

- These approaches provide useful ways of studying challenging 6d SCFTs (some examples in lecture 3,4)

- We still do not have full controls over $Z_{R^4 \times T^2}(\tau, a, \cdots)$ as exact function of τ, a.
- This often makes studies of SCFTs in the interesting regimes highly limited, which sets a major technical hurdle at the moment (despite some studies that I shall explain later).