Difference Painlevé equations from 5D gauge theories

M. Bershtein

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Introduction

$$\begin{array}{ccc} \text{4d } \mathcal{N}=2 \text{ SUSY gauge theories} & \xrightarrow{\text{[AGT 2009]}} & \text{2d CFT} \\ \big\downarrow \text{[GKMMM 95]} & & \big\downarrow \text{[GIL 12]} \\ \text{Integrable systems} & \dashrightarrow & \text{Isomonodromy deformations} \end{array}$$

- In [Gorsky Krichever Marshakov Mironov Morozov 95] the exact Seiberg-Witten (SW) description of the light sector in the $\mathcal{N}=2$ SUSY 4d Yang-Mills theory is reformulated in terms of integrable systems.
- [Alday Gaiotto Tachikawa 09] relation states that (in paticular) Nekrasov partition functions are equal to conformal blocks.
- [Gamayun Iorgov Lisovyy 12] relation states that isomonodromic tau function is equal to certain series of conformal blocks.
  It was known only for special central charges ($\epsilon_1+\epsilon_2=0$ in Nekrasov terms), at the end of the talk we remove this constraint.
- Today I mainly talk about dashed line — *deautonomization*.

Introduction 2

These objects and relations among them exist also when have been raised from original setup to "5d – relativistic – q-deformed" framework, moreover the objects and relations acquire some new and nice properties.

- Integrable systems, becomes *relativistic* [Nekrasov 96]. This relativization can be more generally formulated in terms of *cluster integrable systems* [Goncharov Kenyon 11], [Fock Marshakov 14].
- 5d Nekrasov partition functions are closely related to the *topological strings partition functions*. Also 5d Nekrasov partition functions can be defined as *indices* (see 1 lectures by [Kim])
- Conformal symmetry becomes q-deformed, and the q-deformed W-algebras do have unified description by generators and relations as a quotient of certain quantum group – the *Ding-Iohara-Miki algebra (quantum toroidal* $\mathfrak{gl}(1)$*)*

$$\begin{array}{ccc} \text{5d SUSY gauge theories} & \longrightarrow & q\text{-deformed algebras} \\ \downarrow & & \downarrow \\ \text{Relativistic integrable systems} & \dashrightarrow & q\text{-isomonodromy deformations} \end{array}$$

Main Example

5d SUSY pure gauge theory ⟶ Whittaker (Gaiotto) limit

↓ ↓

Relativistic Toda (two particles) ⇢ q-Painlev'e equation $A_7^{(1)'}$

- q-Painlevé classification [Sakai 01].
- q-deformation of AGT relation [Awata Yamada 09], [Yanagida 14], [Negut 17]
- q-deformation of GIL relation [MB Shchechkin 16], [Jimbo Nagoya Sakai 17].

Deautonomization of cluster integrable system is done by switching off one of the basic constraints in their construction. We consider all cluster integrable systems with two-dimensional phase space, and show that their deautonomizations are q-difference Painlevé equations (all except two).
The mutations of the quiver (supplemented by permutations of its vertices) generate the q-Painlevé dynamics, as well as the automorphisms of the system.

Integrable systems on cluster varieties

- A *lattice* polygon Δ is a polygon in the plane $\mathbb{R}^2$ with all vertices in $\mathbb{Z}^2 \subset \mathbb{R}^2$. There is an action of the group $SA(2,\mathbb{Z}) = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ on the set of such polygons, which preserves the area and the number of interior points.
- Any convex polygon Δ can be considered as a Newton polygon of polynomial $f_\Delta(\lambda,\mu)$, and equation

$$f_\Delta(\lambda,\mu) = \sum\nolimits_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0. \qquad (1)$$

  defines a plane (noncompact) spectral curve. In general position, the genus g of this curve is equal to the number of integral points inside the polygon Δ.
- According to [Goncharov Kenyon 11], [Fock Marshakov 14]. a convex Newton polygon Δ defines a *cluster integrable system*.
  The phase space is an X-cluster Poisson variety $\mathcal{X}$, of dimension $\dim_{\mathcal{X}} = 2S$, where S is an area of the polygon Δ. The Poisson structure in cluster variables is encoded by the quiver $\mathcal{Q}$ with $2S$ vertices. Let ϵ_{ij} be the number of arrows from i-th to j-th vertex ($\epsilon_{ji} = -\epsilon_{ij}$) of $\mathcal{Q}$, then Poisson bracket has the form $\{y_i, y_j\} = \epsilon_{ij} y_i y_j$. The rank of the Possin form is equal to the number of interior points in Δ.
- The product of all cluster variables $\prod_i y_i$ is a Casimir and set to be 1.

Newton polygons

It is well-known that any convex lattice polygon with the only lattice point in the interior is equivalent by $SA(2,\mathbb{Z})$ to one of the 16 polygons from Fig 1.

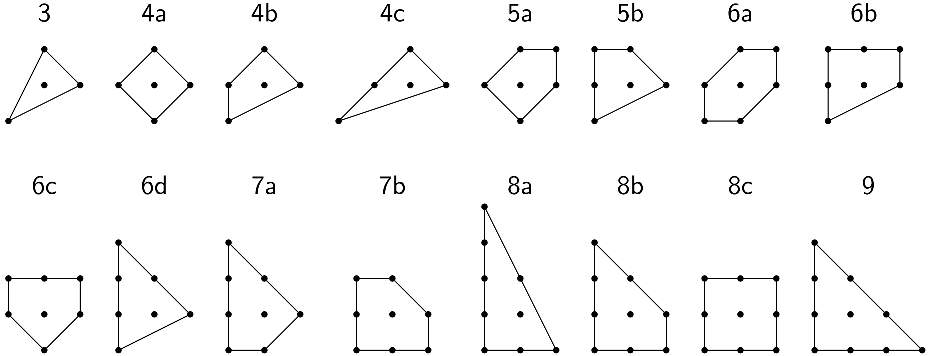


Figure: Polygons with a single internal point and $3 \leq B \leq 9$ boundary points.

We label them Bx, where B is a number of the boundary points, and letter x distinguished their types, if there are several for given B.

Quivers

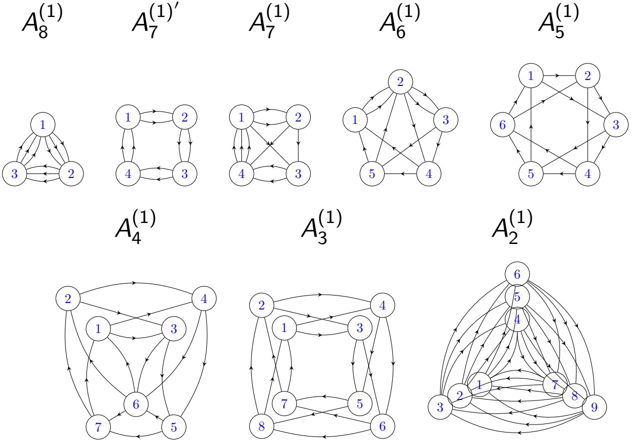


Figure: polygon with B boundary points corresponds to the only quiver with B vertices. The only exceptional case is $B = 4$, where the polygons $4a$, $4c$ correspond to the $A_7^{(1)'}$ quiver, and the polygon $4b$ corresponds to different $A_7^{(1)}$ quiver. The labeling here is consistent with Sakai classification

Poisson maps and discrete flows

- *Permutation of the vertices* of a quiver, together with the cluster variables $\{y_i\}$ assigned to the vertices, complemented with corresponding permutations of the edges.
- *Cluster mutations.* A mutation can be performed at any vertex. Denote by μ_j the mutation at j-th vertex. It acts as

$$\mu_j: \quad y_j \mapsto y_j^{-1}, \qquad y_i \mapsto y_i \left(1 + y_j^{\mathrm{sgn}(\epsilon_{ij})}\right)^{\epsilon_{ij}}, \quad i \neq j, \tag{2}$$

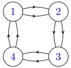
supplemented by transformation of the quiver $\mathcal{Q}$ itself, so that

$$\epsilon_{ik} \mapsto \epsilon_{ik} + \frac{\epsilon_{ij}|\epsilon_{jk}| + \epsilon_{jk}|\epsilon_{ij}|}{2}. \tag{3}$$

- *Inversion* ς, the transformation which reverses orientations of all edges and maps all $\varsigma: \{y_i\} \mapsto \{y_i^{-1}\}$. Note, that ς changes the sign of the Poisson structure, this is natural since it reverses the "time direction".
- Denote by $\mathcal{G}_\mathcal{Q}$ the stabilizer of the quiver $\mathcal{Q}$. Such transformations nevertheless generate nontrivial rational (positive) transformation of the cluster variables $\{y_i\}$.
- This group (up to some details) preserves Goncharov-Kenyon integrable system. This group can be called the group of dicrete flows.

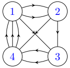
Examples

$\mathbf{A}_7^{(1)'}$. The group $\mathcal{G}_{\mathcal{Q}}$ contains nontrivial element $T = (1,2)(3,4) \circ \mu_1 \circ \mu_3$
Denote $x = y_1,\ y = y_2,\ Z = y_1 y_3$, then $\{x, y\} = 2xy$ and Z is the Casimir function. Transformation T acts as $(x, y) \mapsto (y\frac{(x+Z)^2}{(x+1)^2}, x^{-1})$.
The Hamiltonian, invariant under this transformation, has the form



$$H = \sqrt{xy} + \sqrt{xy^{-1}} + \sqrt{x^{-1}y^{-1}} + Z\sqrt{yx^{-1}} \quad (4)$$

This is the Hamiltonian of relativistic two-particle affine Toda chain.
$\mathbf{A}_7^{(1)}$. The group $\mathcal{G}_{\mathcal{Q}}$ contains element $T = (1324) \circ \mu_3$.
Denote $x = y_3,\ y = y_4,\ Z = y_2\sqrt{\frac{y_4}{y_3}}$, then $\{x, y\} = 2xy$ and Z is the Casimir function. The transformation T acts as $(x, y) \mapsto (\frac{1}{Z\sqrt{x^3 y}}(1+x), Z\frac{\sqrt{x}}{\sqrt{y}}(1+x))$.
The Hamiltonian, invariant under such transformation, has the form



$$H = \sqrt{xy} + \sqrt{xy^{-1}} + \sqrt{x^{-1}y^{-1}} + Zx^{-1} \quad (5)$$

This Hamiltonian is different from (4), though it has the same limit at $Z \to 0$.
A different affinization of two-particle relativistic Toda.

Deautonomization

In the deautomization we set $q = \prod_i y_i \neq 1$.

Theorem

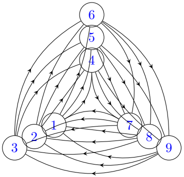
For each quiver from Fig. 2 the group $\mathcal{G}_\mathcal{Q}$ contains subgroup isomorphic to the symmetry group of the corresponding q-Painlevé equation and its action on variables y_i is equivalent to q-Painlevé dynamics.

$\mathbb{A}_2^1$ The group $\mathcal{G}_\mathcal{Q}$ contains elements

$$s_1=(2,3),\;\; s_2=(1,2),\;\; s_4=(4,5),\;\; s_5=(5,6),\;\; s_6=(7,8),\;\; s_0=(8,9),$$
$$s_3=(4,7)\circ\mu_1\circ\mu_4\circ\mu_7\circ\mu_1,\;\; \pi=(1,4,7)(2,5,8)(3,6,9),\; \sigma=(1,7)(2,8)(3,9)\circ\varsigma.$$

The only tricky element is the reflections s_3

$$s_3:(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8,y_9)\mapsto\Big(\frac{y_1}{y_4y_7}\frac{1+y_4+y_1^{-1}}{1+y_1+y_7^{-1}},y_1y_2\frac{1+y_4+y_1^{-1}}{1+y_1+y_7^{-1}},$$
$$y_1y_3\frac{1+y_4+y_1^{-1}}{1+y_1+y_7^{-1}},\frac{y_4}{y_1y_7}\frac{1+y_7+y_4^{-1}}{1+y_4+y_1^{-1}},y_4y_5\frac{1+y_7+y_4^{-1}}{1+y_4+y_1^{-1}},y_4y_6\frac{1+y_7+y_4^{-1}}{1+y_4+y_1^{-1}},\frac{y_7}{y_1y_4}\frac{1+y_1+y_7^{-1}}{1+y_7+y_4^{-1}},$$
$$y_7y_8\frac{1+y_1+y_7^{-1}}{1+y_7+y_4^{-1}},y_7y_9\frac{1+y_1+y_7^{-1}}{1+y_7+y_4^{-1}}\Big).$$



τ variables

- There is an alternative, or dual to the Poisson X-cluster varieties language, called A-cluster varieties. We call the corresponding A-cluster variables as $\{\tau_I\}$ due to their relation to the tau-functions for q-difference Painlevé equations. Under mutation at j-th vertex these variables are transformed as

$$\mu_j: \quad \tau_j \mapsto \tau_j^{-1}\Big(\prod_{b_{Ij}>0} \tau_I^{b_{Ij}} + \prod_{b_{Ij}<0} \tau_I^{-b_{Ij}}\Big) \qquad \tau_I \mapsto \tau_I, \qquad I \neq j \tag{6}$$

  and antisymmetric matrix $B = \{b_{i,j}\}$ is transformed by the formula (3).
- Generally there are more $\{\tau_I\}$-variables, than their X-cluster $\{y_i\}$-relatives. Mutations are allowed only in the vertices of Γ, other vertices of the extended quiver $\hat{\Gamma}$ are called therefore frozen. A relation between τ_I and y_j is given by the formula $y_j = \prod_{I\in\hat{\Gamma}} \tau_I^{b_{Ij}}$,
- $\mathbf{A_7^{(1')}}$. It is convenient to use Denote the action of T as overline, and action of T^{-1} as underline. Then we have

$$\overline{(\tau_1,\tau_2,\tau_3,\tau_4)} = \left(\tau_2, \tau_1^{-1}\left(\tau_2^2 + q^{1/2}Z^{1/2}\tau_4^2\right), \tau_4, \tau_3^{-1}\left(\tau_4^2 + q^{1/2}Z^{1/2}\tau_2^2\right)\right) \tag{7}$$

  These leads to bilinear equations

$$\underline{\tau_1}\overline{\tau}_1 = \tau_1^2 + Z^{1/2}\tau_3^2, \quad \underline{\tau_3}\overline{\tau}_3 = \tau_3^2 + Z^{1/2}\tau_1^2. \tag{8}$$

Solution

- Bilinear relations

$$\underline{\tau_1}\overline{\tau}_1 = \tau_1^2 + Z^{1/2}\tau_3^2, \quad \underline{\tau_3}\overline{\tau}_3 = \tau_3^2 + Z^{1/2}\tau_1^2. \tag{9}$$

- One can consider the $\overline{\tau_i} = \tau_i(qZ)$, $\underline{\tau_i} = \tau_i(q^{-1}Z)$, then the equations (9) become *q-difference bilinear equations*. These equations can be called the bilinear form of the q-Painlevé equation (of the surface type $A_7^{(1)'}$).
- The formal solution of these equations was proposed in [MB Shchechkin 2016], namely $\tau_1 = \mathcal{T}(u, s; q|Z)$, $\tau_3 = is^{1/2}\mathcal{T}(uq, s; q|Z)$, where

$$\mathcal{T}(u, s; q|Z) = \sum_{m\in\mathbb{Z}} s^m \mathrm{F}(uq^{2m}; q, q^{-1}|Z). \tag{10}$$

  Here $\mathrm{F}(u; q, q^{-1}; Z)$ is a properly normalized 5d Nekrasov partition function for pure $SU(2)$ gauge theory.
- There is a similar conjecture for any Newton polygon Δ— deautonomization of Goncharov Kenyon integrable system corresponding to Δ can be solved in terms of topological strings partition functions corresponding to Δ.
  This is equivalent to bilinear relation on partition functons simialar to blowup equations on $\mathbb{C}^2/\mathbb{Z}_2$ (c.f. yesterday talk [Sun]).

Quantization

- We denote the multiplicative quantization parameter as p in order to distinguish it from the parameter q in difference equations. We do not impose any relation on p, q, at the end of the day it will be convenient to express them $p = q_1^2 q_2^2, q = q_2^2$ in terms of Nekrasov background parameters q_1, q_2.
- The quantization of the quadratic Poisson bracket $\{y_i, y_j\} = \epsilon_{ij} y_i y_j$ has the form

$$y_i y_j = p^{-2\epsilon_{ij}} y_j y_i \tag{11}$$

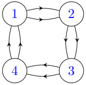
- Quantum mutations μ_j for these generators are given by (compare to (2))

$$\mu_j: \quad y_j \mapsto y_j^{-1}, \quad y_i^{1/|\epsilon_{ij}|} \mapsto y_i^{1/|\epsilon_{ij}|} \left(1 + p y_j^{\operatorname{sgn} \epsilon_{ij}}\right)^{\operatorname{sgn} \epsilon_{ij}}, \; i \neq j \tag{12}$$

  and the same formula (3) for the exchange matrix ϵ. One can check that mutations of $\{y_i\}$ and ϵ preserve the relations (11).

Example $A_7^{(1')}$

- Again we have two central or Casimir elements $Z = y_1 y_3$, $q = y_1 y_3 y_2 y_4$



- Similarly to classical case consider the discrete flow $T = (1,2)(3,4) \circ \mu_1 \circ \mu_3$. In quantum case it reads

$$\left(y_1^{1/2}, y_2^{1/2}, y_3^{1/2}, y_4^{1/2}\right) \mapsto \left(y_2^{1/2} \frac{1+py_3}{1+py_1^{-1}},\ y_1^{-1/2},\ y_4^{1/2} \frac{1+py_1}{1+py_3^{-1}},\ y_3^{-1/2}\right).$$

  where the ratios in the r.h.s. are well-defined, since y_1 and y_3 commute with each other

- For $q = 1$ we have an invariant Hamiltonian

$$H = y_2^{1/2} y_1^{1/2} + y_1^{1/2} y_2^{-1/2} + y_2^{-1/2} y_1^{-1/2} + Z y_1^{-1/2} y_2^{1/2} \tag{13}$$

  This is a Hamiltonian of quantum relativistic two-particle affine Toda chain.

Quantum τ

- There are quantum analogs of the A-cluster τ-variables. Following [Berenstein Zelevinsky 04] we quantize $\{\tau_I\}$-variables and consider them as elements of the quantum cluster algebra with relations $\tau_I\tau_J = p^{\Lambda_{IJ}/2}\tau_J\tau_I$, where $I, J = 1, \dots, 6$ and the matrix Λ is

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{14}$$

- For quantum $\{\tau_I\}$-variables we now fix the notations $\tau_1 = \mathcal{T}_1$, $\tau_2 = \mathcal{T}_2$, $\tau_3 = \mathcal{T}_3$, $\tau_4 = \mathcal{T}_4$, so that first four will be quantum $\mathcal{T}$-functions. Two last are $\tau_5 = q^{1/4}$, $\tau_6 = Z^{1/4}$, they are still generally noncommutative with $\mathcal{T}_i$.
- We now define the discrete dynamics of the quantum $\mathcal{T}$-functions by

$$\overline{(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, Z, q)} = (\mathcal{T}_2, \mathcal{T}_1^{-1}(\mathcal{T}_2^2 + p^2(qZ)^{1/2}\mathcal{T}_4^2), \mathcal{T}_4, \mathcal{T}_3^{-1}(\mathcal{T}_4^2 + p^2(qZ)^{1/2}\mathcal{T}_2^2), Zq, q), \tag{15}$$

$$\underline{(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, Z, q)} = ((\mathcal{T}_1^2 + p^2 Z^{1/2}\mathcal{T}_3^2)\mathcal{T}_2^{-1}, \mathcal{T}_1, (\mathcal{T}_3^2 + p^2 Z^{1/2}\mathcal{T}_1^2)\mathcal{T}_4^{-1}, \mathcal{T}_3, Zq^{-1}, q). \tag{16}$$

 It is straightforward to check that this dynamics preserves commutation relations
- We have a bilinear relations $\underline{\mathcal{T}_1}\overline{\mathcal{T}_1} = \mathcal{T}_1^2 + p^2 Z^{1/2}\mathcal{T}_3^2$, $\underline{\mathcal{T}_3}\overline{\mathcal{T}_3} = \mathcal{T}_3^2 + p^2 Z^{1/2}\mathcal{T}_1^2$,

Solution of the quantization

- We have a bilinear relations

$$\underline{\mathcal{T}_1}\overline{\mathcal{T}_1} = \mathcal{T}_1^2 + p^2 Z^{1/2}\mathcal{T}_3^2, \quad \underline{\mathcal{T}_3}\overline{\mathcal{T}_3} = \mathcal{T}_3^2 + p^2 Z^{1/2}\mathcal{T}_1^2, \qquad (17)$$

- Now we want to present explicit formula for the $\mathcal{T}_i$. Denote $q_2 = q^{1/2}$, $q_1 = q_2^{-1}p^2$. The solution will be the function depending on variables q_1, q_2, u, s, Z, a, bm with nontrivial commutation relations:

$$q_2^2 a = p^{-2} a q_2^2, \quad q_1 q_2^{-1} a = p^2 a q_1 q_2^{-1}, \quad us = p^4 su, \quad Zb = p^2 bZ. \qquad (18)$$

- The discrete flow of this set of quantum variables is

$$\overline{(q_1, q_2, u, s, Z, a, b)} = (q_1, q_2, u, s, q_2^2 Z, ab, b)$$

It is easy to check, that this discrete flow preserves the commutation relations

Conjecture

Bilinear equations (17) *are solved in terms of 5d Nekrasov functions:*

$$\mathcal{T}_1 = a\sum\nolimits_{m\in\mathbb{Z}} s^m \mathrm{F}(uq_2^{4m}; q_1q_2^{-1}, q_2^2|Z),\ \mathcal{T}_2 = ab\sum\nolimits_{m\in\mathbb{Z}} s^m \mathrm{F}(uq_2^{4m}; q_1q_2^{-1}, q_2^2|q_2^2 Z),$$

$$\mathcal{T}_3 = ia\sum_{m\in\mathbb{Z}+1/2} s^m \mathrm{F}(uq_2^{4m}; q_1q_2^{-1}, q_2^2|Z),\ \mathcal{T}_4 = iab\sum_{m\in\mathbb{Z}+1/2} s^m \mathrm{F}(uq_2^{4m}; q_1q_2^{-1}, q_2^2|q_2^2 Z).$$

Abstract

We discuss the relation between the cluster integrable systems and q-difference Painlevé equations. The Newton polygons corresponding to these integrable systems are all 16 convex polygons with a single interior point. The Painlevé dynamics is interpreted as deautonomization of the discrete flows, generated by a sequence of the cluster quiver mutations, supplemented by permutations of quiver vertices.

We also define quantum q-Painlevé systems by quantization of the corresponding cluster variety. We present formal solution of these equations for the case of pure gauge theory using q-deformed conformal blocks or 5-dimensional Nekrasov functions. We propose, that quantum cluster structure of the Painlevé system provides generalization of the isomonodromy/CFT correspondence for arbitrary central charge.

Thank you for your attention!

Discussion

- Extended global symmetries [Seiberg 96] and others.
- Quivers by [Ceccotti Vafa 11].
- Topological strings – spectral theory duality [Grassi Marino Hatsuda 11], [Bonelli Grassi Tanzini 17]
- Different approaches to quantization [Kuroki 08], [Hasegawa 07], [Nagoya, Yamada 12]