Conformal Lagrangians from (formal) near boundary analysis of AdS gauge fields

Alexander Chekmenev

Lebedev Physical Institute, Moscow

Based on: A.C., Maxim Grigoriev: arXiv:1512.06443 + to appear

Overview

- Standard approach: Lagrangian for conformal fields arise as logarithmically-divergent term in the on-shell action.

$$S[\phi_0] = \int d\rho d^d x \sqrt{g} S^{bulk}[\phi[\phi_0]] \,, \qquad \frac{1}{\rho^{\Delta_-/2}} \phi\Big|_{z=0} = \phi_0$$

  Drawback: need to know the Lagrangian for a field in AdS.
- Alternative approach: conformal equations can be seen as obstructions of extending an off-shell field on the conformal boundary of the AdS to a bulk on-shell field configuration. Can be formulated entirely at the level of EOM.
- That obstructions are found in explicit form (a simple generating procedure is proposed, to be precise) for conformal linear fields being the leading boundary values of AdS massive and unitary-type massless fields and Lagrangians (a generating procedure) for them is presented.

Plan

- Review of the ambient space approach to boundary values
- Mixed symmetry gauge fields on AdS and their boundary values
- Conformal Lagrangians

Ambient space

AdS_{d+1} space can be realized as a quadric in a flat pseudo-Euclidean space $\mathbb{R}^{d,2}$ with Cartesian coordinates $X^A, A = 0, \dots, d+1$ and the metric $\eta_{AB} = \text{diag}(- + \dots + -)$:

$$\eta_{AB} X^A X^B \equiv X \cdot X \equiv X^2 = -1.$$

Pros: $o(d, 2)$ acts linearly.

The conformal boundary $\mathcal{X}$ of AdS_{d+1} can be identified with the quotient of the hypercone $X^2 = 0$ by equivalence relation $X \sim \lambda X, \ \lambda \in \mathbb{R} \backslash 0$.
Can be seen as a surface. E.g. (Minkowski metric):

$$X^2 = 0, \quad X^+ = 1.$$

Ambient scalar

Scalar in $\mathrm{AdS}_{d+1} = \left\{X \in \mathbb{R}^{d,2} \,|\, X^2 = -1\right\}$

$$(\nabla^2 - m^2)\varphi(x) = 0$$

can be equivalently described in terms of $\mathbb{R}^{d,2}$

$$\partial_X \cdot \partial_X \Phi(X) = 0,$$

$$\left(X \cdot \frac{\partial}{\partial X} + \Delta\right)\Phi(X) = 0,$$

$$m^2 = \Delta(\Delta - d).$$

Parent formulation

$$\left(\frac{\partial}{\partial X^A} - \frac{\partial}{\partial Y^A}\right)\Phi = 0,$$

$$\frac{\partial}{\partial Y}\cdot\frac{\partial}{\partial Y}\Phi = 0, \quad \left((X+Y)\cdot\frac{\partial}{\partial Y} + \Delta\right)\Phi = 0,$$

where Φ is now depends on Y.

Interpret the first equation as a covariant constancy condition determined by a particular $iso(d,2)$ connection.

$$\nabla = \mathbf{d} - E^A\frac{\partial}{\partial Y^A} - w^B{}_A Y^A \frac{\partial}{\partial Y^B},$$

namely the one where $E^A = dX^A$, $w^{AB} = 0$.

Parent formulation

$$\nabla\Phi = 0,$$

$$\frac{\partial}{\partial Y}\cdot\frac{\partial}{\partial Y}\Phi = 0, \quad \left((V(X)+Y)\cdot\frac{\partial}{\partial Y}+\Delta\right)\Phi = 0,$$

where $V^A(X)$ are components of the section of the vector bundle s.t. in the suitable local frame coincide with Cartesian coordinates X^A. In particular $V^2 = X^2$.

Compatibility conditions are

$$\mathbf{d}w^A{}_B + w^A{}_C w^C{}_B = 0, \quad \mathbf{d}V^A + w^A{}_B V^B = E^A.$$

Parent formulation

Idea: to use the ambient space construction in the fiber rather in spacetime.

By pulling back the bundle to the submanifold $X^2 = -1$ we get the system defined *explicitly* on $X^2 = -1$.

Identifying the conformal space $\mathcal{X}$ as a submanifold of the hypercone $X^2 = 0$ we arrive to the formulation in terms of fields defined on $\mathcal{X}$.

That formulation can be considered as a generating procedure for the equations satisfied by boundary values.

Parent formulation

Let us pick a local coordinate system x^a on $\mathcal{X}$ and the local frame s.t. the only nonvanishing components of the flat connection w are $w^a{}_+ = dx^a, w^-{}_a = -dx_a$ and $V^+ = 1, V^- = V^a = 0$.

$$\nabla = dx^a \left(\frac{\partial}{\partial x^a} - \left(Y^+ + 1 \right) \frac{\partial}{\partial y^a} + y_a \frac{\partial}{\partial u} \right),$$

where $u \equiv Y^-$.

$$\nabla \Phi = 0,$$

$$\left(\frac{\partial}{\partial Y^+} \frac{\partial}{\partial u} + \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial y} \right) \Phi = 0, \quad \left(\frac{\partial}{\partial Y^+} + Y \cdot \frac{\partial}{\partial Y} + \Delta \right) \Phi = 0,$$

The first and the third equations have a unique solution for a given $\phi(x, u) = \Phi|_{y^a = Y^+ = 0}$. So in terms of ϕ the second implies

$$\frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x} \phi + \frac{\partial}{\partial u} \left(d - 2\Delta - 2u \frac{\partial}{\partial u} \right) \phi = 0$$

Parent formulation: boundary values

$$\frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x}\phi + \frac{\partial}{\partial u}\left(d - 2\Delta - 2u\frac{\partial}{\partial u}\right)\phi = 0$$

This equation does not impose any constraints on $\phi_0(x) = \phi|_{u=0}$ for $\Delta \neq \frac{d}{2} - \ell$ with $\ell \in \mathbb{Z}^{>0}$.

However, if $\Delta = \frac{d}{2} - \ell,\ \ell \in \mathbb{Z}^{>0}$ then ϕ_0 is subject to

$$\Box^{\ell}\phi_0 = 0.$$

In other words the parent system is equivalent through the elimination of auxiliary fields to the system of two scalar fields ϕ_0 subjected to $\Box^{\ell}\phi_0 = 0$ and unconstrained ϕ_ℓ (ℓ-th coefficient in the expansion of ϕ in powers of u).

Ambient description of mixed symmetry fields

This picture can be generalized to the case of gauge fields.

Consider generating functions depending on X^A and P_i^A, $i = 1, \ldots, n-1$

$$\Phi(X, P) \equiv \Phi(X, P_1, \ldots, P_{n-1})$$

$o(d, 2)$ algebra acts by

$$J_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} + \sum_{i=1}^{n-1} \left(P_{iA} \frac{\partial}{\partial P_i^B} - P_{iB} \frac{\partial}{\partial P_i^A} \right)$$

Ambient description of mixed symmetry fields

Configurations of a unitary massless field of spin $\{s_1, s_2, \dots, s_{n-1}\}$ (it is assumed that $s_1 \geq s_2 \geq \dots s_{n-1}$ and $n-1 \leq \left[\frac{d}{2}\right]$) are determined by the following constraints:

Algebraic: $\partial_P^i \cdot \partial_P^j \Phi = 0, \quad P_i \cdot \partial_P^j \Phi = 0 \ \ i < j, \quad (P_i \cdot \partial_P^i - s_i)\Phi = 0,$

Tangent: $X \cdot \partial_P^i \Phi = 0,$

Radial: $(X \cdot \partial_X + \Delta)\Phi = 0 \quad \Delta = 1 + p - s,$

EOM and partial gauge: $\Box\Phi = 0, \quad \partial_P^i \cdot \partial_X \Phi = 0.$

Here p denotes the height of the uppermost block in the Young tableau $(s_1, \dots, s_{n-1})$. I.e. $s_1 = \dots s_p > s_{p+1}$.

Conformal equations

Leading boundary value is determined by boundary data $\phi_{00}(x,p) = \phi_0(x,p,w)|_{w_i=0}$, subjected to

$$(n_i - s_i)\phi_{00} = 0\,, \qquad (\partial_{p_i}\cdot\partial_{p_j})\phi_{00} = 0\,, \qquad (p_i\cdot\partial_{p_j})\phi_{00} = 0 \quad i<j\,.$$

Gauge invariant equations on ϕ_{00}

$$(\tilde{\Box}^{\ell}\phi_0)|_{w_i=0} = 0\,, \qquad \phi_0|_{w_i=0} = \phi_{00}\,,$$

$$(\partial_{p_i}\cdot\partial)\phi_0 + \frac{\partial}{\partial w_i}(d + s_i - \Delta - i - \sum_{j\leq i} n_{w_j})\phi_0 + \sum_{i<j}(p_j\cdot\partial_{p_i})\frac{\partial}{\partial w_j}\phi_0 = 0$$

The last equation fixes the w-dependence.
So there is a bijection $\pi : \phi_0 \mapsto \phi_0|_{w_i=0}$ between solutions $\phi_0(x,p,w)$ and off-shell fields $\phi_{00}(x,p)$.

Conformal equations

That conformal equations above have the form $\mathcal{A}\phi_{00} = 0$ for the operator $\mathcal{A}$ that makes the following diagram commutative

$$\begin{array}{ccc} \Phi_0 & \xrightarrow{\tilde{\Box}^{\ell}} & \Phi_0 \\ {\scriptstyle \pi^{-1}}\uparrow & & \downarrow{\scriptstyle \pi} \\ \Phi_{00} & \overset{\mathcal{A}}{\dashrightarrow} & \Phi_{00} \end{array}$$

where Φ_{00} denotes the space of Lorentz irreducible tensor fields and Φ_0 the space of polynomials in w_i variables with coefficients being smooth functions.
E.g. for $d = 4$, spin 1:

$$\mathcal{A} : \phi_{00} \mapsto (\Box - (p \cdot \partial)(\partial_p \cdot \partial))\phi_{00}$$

In components:

$$\mathcal{A} : p^a \varphi_a \mapsto p^a(\Box\varphi_a - \partial_a \partial^b \varphi_b)$$

Conformal Lagrangians

Let us consider the inner product

$$\langle \phi, \chi \rangle = \int dx^d \langle \phi, \chi \rangle',$$

where $\langle \cdot, \cdot \rangle'$ is the standard inner product on polynomials, defined by the metric η_{ab}.

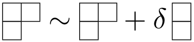
$\mathcal{A}$ acts on the space of Lorentz-irreducible tensor fields and is formally symmetric.

$$f(x)^\dagger = f(x), \qquad \partial_a{}^\dagger = -\partial_a, \qquad p_i^{a\dagger} = \eta^{ab}\frac{\partial}{\partial p_i^b}.$$

(Gauge invariant) equations $\mathcal{A}\phi_{00} = 0$ follow from the (gauge invariant) Lagrangian

$$L = \langle \phi_{00}, \mathcal{A}\phi_{00} \rangle = \langle \phi_{00}, (\tilde{\Box}^\ell \phi_0)|_{w_i=0} \rangle = \langle \phi_0, \tilde{\Box}^\ell \phi_0 \rangle|_{w_i=0}.$$

Example: ''hook''-type field



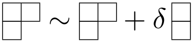
The equations of motion

$$\Box^2\phi_{ab,c} - \Box\partial^e\left(\partial_a\phi_{eb,c} + \partial_b\phi_{ea,c}\right) + \frac{1}{2}\Box\partial^e\left(\partial_a\phi_{bc,e} + \partial_b\phi_{ac,e}\right)$$
$$- 2\Box\partial^e\partial_c\phi_{ab,e} + \frac{1}{2}\left(\eta_{ab}\Box + 2\partial_a\partial_b\right)\partial^e\partial^f\phi_{ef,c}$$
$$- \frac{1}{4}\partial^e\partial^f\left[\left(\eta_{ac}\Box + 2\partial_a\partial_c\right)\phi_{ef,b} + \left(\eta_{bc}\Box + 2\partial_b\partial_c\right)\phi_{ef,a}\right] = 0$$

The gauge transformation

$$\delta\phi_{ab,c} = \partial_a\lambda_{bc} + \partial_b\lambda_{ac} - \frac{1}{3}\partial^e\left(2\eta_{ab}\lambda_{ec} - \eta_{ac}\lambda_{eb} - \eta_{bc}\lambda_{ea}\right)$$

Example: "hook"-type field



$$L = \int d^4x \Big\langle \phi_{00}, \Big(\Box^2 - \Box(p_1 \cdot \partial)(\partial_{p_1} \cdot \partial) - \frac{5}{2}\Box(p_2 \cdot \partial)(\partial_{p_2} \cdot \partial)$$

$$+ \frac{5}{3}(p_1 \cdot \partial)(\partial_{p_1} \cdot \partial)(p_2 \cdot \partial)(\partial_{p_2} \cdot \partial) + \frac{1}{3}(p_1 \cdot \partial)^2(\partial_{p_1} \cdot \partial)^2 \Big) \phi_{00} \Big\rangle$$

In components ($\phi_{00} = p_1^a p_1^b p_2^c \phi_{abc}$, $\phi_{abc} = \phi_{bac}$, $\phi_{abc} + \phi_{acb} + \phi_{bca} = 0$):

$$\frac{1}{2}L = \phi^{abc}\Box^2\phi_{abc} + 2\partial_e\phi^{ebc}\Box\partial^f\phi_{fbc}$$

$$+ \frac{5}{2}\partial_e\phi^{abe}\Box\partial^f\phi_{abf} + \frac{3}{2}\partial_a\partial_b\phi^{abc}\partial^e\partial^f\phi_{efc}$$

Fin

Thanks for attention!