Kontsevich integral in topological models of gravity

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3d gravity as gauge theory

Main feature of 3d gravity: there are no dynamical degrees of freedom.

$$\text{3d-Einstein theory} \leftrightarrow \text{Chern-Simons theory} \quad (\text{Witten}, 88)$$

Connection 1-form includes spin connection ω and frame field e

$$\mathrm{A} = \omega + \mathrm{e} = \omega_{\mathrm{ab}} \mathrm{J}^{\mathrm{ab}} + \mathrm{e}_{\mathrm{a}} \mathrm{P}^{\mathrm{a}}.$$

Problem: how to describe Chern-Simons theory with non-compact gauge group?

Chern-Simons theory

Connection 1-form

$$A = A_\mu dx^\mu = A_\mu^a(x) dx^\mu \otimes T^a.$$

Chern-Simons theory

$$S_{CS}(A) = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

Equations of motion

$$F = dA + A \wedge A = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c) dx^\mu \wedge dx^\nu \otimes T^a = 0.$$

Chern-Simons theory: quantization in holomorphic gauge

Let decompose $\mathbb{R}^3$ as

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

$$(x_0, x_1, x_2) \mapsto (t = x_0,\ z = x_1 + ix_2,\ \bar{z} = x_1 - ix_2).$$

$$(A_0^a, A_1^a, A_2^a) \mapsto (A_t^a = A_0^a,\ A_z^a = A_1^a + iA_2^a,\ A_{\bar{z}}^a = A_1^a - iA_2^a)$$

Holomorphic gauge

$$A_{\bar{z}}^a = 0$$

Then

$$A \wedge A \wedge A \mid_{A_{\bar{z}}=0} = 0$$

Chern-Simons theory: quantization in holomorphic gauge

Then the effective lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \frac{i}{2}\varepsilon^{\alpha\beta\nu}[A^a_\alpha \partial_\beta A^a_\nu + \frac{g}{3} f^{abc} A^a_\alpha A^b_\beta A^c_\nu] + B^a n^\mu A^a_\mu + \bar{c}^a \left(\delta^{ad} n^\mu \partial_\mu + g f^{abd} n^\mu A^b_\mu\right) c^d$$

in holomorphic gauge takes the form

$$\mathcal{L}_{\mathrm{eff}} = \delta^{ab}\varepsilon^{mn} A^a_m \partial_{\bar{z}} A^b_n - B^a \partial_{\bar{z}} c^a = A^a_t \partial_{\bar{z}} A^a_z - A^a_z \partial_{\bar{z}} A^a_t - B^a \partial_{\bar{z}} c^a$$

Since last term doesn't contain gauge field, it can be ignored. We see that obtained lagrangian describes free theory

$$\mathcal{L}_{\mathrm{eff}} = A^a_t \partial_{\bar{z}} A^a_z - A^a_z \partial_{\bar{z}} A^a_t$$

Wilson loops

Consider Wilson loop operator

$$W_\gamma^R(A) = \mathrm{Tr}_R \mathcal{P} \exp\{i \int_\gamma A\}$$

In holomorphic gauge if can be evaluated explicitly

$$\langle W_R(C, A) \rangle = \sum_{n=0}^{\infty} \frac{1}{(2\pi i)^n} \int_{\Delta} \sum_{p \in P_{2n}} (-1)^{p\downarrow} \bigwedge_{k=1}^{n} d \log(z_{i_k} - z_{j_k}) G_p,$$

where

$$G_p = \mathrm{Tr}_R \big(T^{a_{\sigma_p(1)}} T^{a_{\sigma_p(2)}} \dots T^{a_{\sigma_p(2n)}} \big) \text{ called group factors.}$$

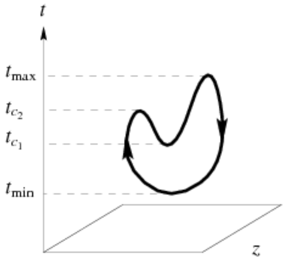
For more details see (Kauffman, Knot and Physics).

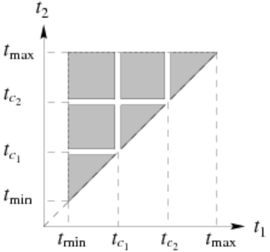
More about Kontsevich integral

$$\langle \mathrm{W_R(C,A)} \rangle = \sum_{\mathrm{n}=0}^{\infty} \frac{1}{(2\pi \mathrm{i})^{\mathrm{n}}} \int_{\mathrm{o(z_1)<o(z_2)<\cdots<o(z_{2n})}} \times$$

$$\times \sum_{\mathrm{p}=\{(\mathrm{i_1,j_1}),(\mathrm{i_2,j_2}),\ldots,(\mathrm{i_n,j_n})\}\in \mathrm{P_{2n}}} (-1)^{\mathrm{p}\downarrow} \bigwedge_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{dz_{i_k}} - \mathrm{dz_{j_k}}}{\mathrm{z_{i_k}} - \mathrm{z_{j_k}}} \mathrm{G_p}$$

Domain of integration



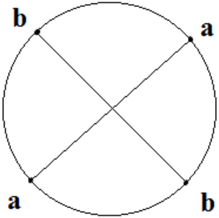


More about Kontsevich integral

Group factors

$$G_p = \mathrm{Tr}_R\left(T^{a_{\sigma_p(1)}} T^{a_{\sigma_p(2)}} \dots T^{a_{\sigma_p(2n)}}\right)$$

can be represented as chord diagrams.
For example consider $\mathrm{Tr}(T^a T^b T^a T^b)$. Corresponding chord diagram is



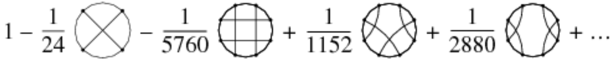
More about Kontsevich integral

(Kontsevich, 92) Every term in perturbation series is finite.

$$\langle W_R(C,A)\rangle = \sum_{n=0}^{\infty} \frac{1}{(2\pi i)^n} \int_{o(z_1)<o(z_2)<\cdots<o(z_{2n})} \times$$

$$\times \sum_{p=\{(i_1,j_1),(i_2,j_2),\ldots,(i_n,j_n)\}\in P_{2n}} (-1)^{p\downarrow} \bigwedge_{k=1}^{n} \frac{dz_{i_k}-dz_{j_k}}{z_{i_k}-z_{j_k}} G_p$$

For unknot Kontsevich integral is



Every coefficient in this series is knot invariant.

Wigner-Inönü contraction

SO(4) generators in fundamental representation

$$(\mathrm{T}_{\mathrm{ab}})_{\mathrm{rs}} = \mathrm{i}\ (\delta_{\mathrm{as}}\ \delta_{\mathrm{br}} - \delta_{\mathrm{ar}}\ \delta_{\mathrm{bs}}), \quad \mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d} \in \{1, 4\}.$$

$$\mathrm{L}_1 \equiv \mathrm{T}_{23},\ \mathrm{L}_2 \equiv \mathrm{T}_{13},\ \mathrm{L}_3 \equiv \mathrm{T}_{12},\ \mathrm{K}_1 \equiv \mathrm{T}_{14},\ \mathrm{K}_2 \equiv \mathrm{T}_{24},\ \mathrm{K}_3 \equiv \mathrm{T}_{34}.$$

$$\mathrm{SO}(4) \to \mathrm{SO}(3) \ltimes \mathrm{T}(3)$$

Wigner-Inönü contraction

$$\mathrm{SO}(4) \to \mathrm{SO}(3) \ltimes \mathrm{T}(3)$$

Now we include parameter of the contraction

$$\mathrm{L}_\mathrm{i}^\mathrm{R} \equiv \mathrm{L}_\mathrm{i}, \ \mathrm{i} \in \{1, 3\},$$

$$\mathrm{K}_1^\mathrm{R} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\mathrm{i}}{\mathrm{R}} & 0 & 0 & 0 \end{pmatrix}, \ \mathrm{K}_2^\mathrm{R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mathrm{i}}{\mathrm{R}} & 0 & 0 \end{pmatrix},$$

$$\mathrm{K}_3^\mathrm{R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \frac{\mathrm{i}}{\mathrm{R}} & 0 \end{pmatrix}.$$

Wigner-Inönü contraction

Now we can identify

$$\mathrm{L}_i^{\mathrm{R}} \to \mathrm{L}_i^{\mathrm{SO(3)}}, \ \ \mathrm{K}_i^{\mathrm{R}} \to \mathrm{P}_i^{\mathrm{T(3)}}, \ \mathrm{R} \to \infty, \ \mathrm{i} \in \{1, 3\}.$$

Traces (in fundamental representation) which contain generators of translations vanish in limit $\mathrm{R} \to \infty$

$$\mathrm{Tr}_{\mathrm{F}}\left((\mathrm{K}_j^{\mathrm{R}})^{\mathrm{n}}\right) = \left((-1)^{\mathrm{n}} + 1\right) \mathrm{R}^{-\mathrm{n}/2} \longrightarrow 0 \ \text{ as } \mathrm{R} \to \infty,$$

$$\mathrm{Tr}_{\mathrm{F}}\left((\mathrm{L}_i^{\mathrm{R}})^{\mathrm{n}}(\mathrm{K}_j^{\mathrm{R}})^{\mathrm{m}}\right) = \frac{1}{4}\left((-1)^{\mathrm{m}} + 1\right)\left((-1)^{\mathrm{n}} + 1\right)\left(1 - \delta_{\mathrm{ij}}\right) \mathrm{R}^{-\mathrm{m}/2} \longrightarrow 0 \ \text{as } \mathrm{R} \to \infty$$

Only traces of SO(3) generators are still non-zero.

Kontsevich integral after contraction

After contraction $SO(4) \to SO(3) \ltimes T(3)$ we obtain Kontsevich integral for maximal compact subgroup $SO(3)$

$$\langle W_R(C, A)\rangle_{SO(4)} \to \langle W_R(C, A)\rangle_{SO(3)}$$

$$\langle W_R(C, A)\rangle = \sum_{n=0}^{\infty} \frac{1}{(2\pi i)^n} \int_{\Delta} \sum_{p \in P_{2n}} (-1)^{p\downarrow} \bigwedge_{k=1}^{n} d\log(z_{i_k} - z_{j_k}) G_p,$$

where

$$G_p = \mathrm{Tr}_F\big(T^{a_{\sigma_p(1)}} T^{a_{\sigma_p(2)}} \dots T^{a_{\sigma_p(2n)}}\big),\ T^a - \text{generators of } SO(3)$$

Discussion

Kontsevich integral for SO(4) group reduces after contraction to integral for SO(3) - maximal compact subgroup of $SO(3) \ltimes T(3)$.

From perturbative point of view we can't see effects from non-compactness of gauge group.

Another possible contraction

$$SO(3) \to SO(2) \ltimes T(2) \cong U(1) \times \mathbb{R}^2 \cong \text{Heisenberg algebra}$$

Theta representation of Heisenberg algebra $\leftrightarrow$

$\leftrightarrow$ Appearance of integrals of modular forms in perturbative series.

Discussion

Thank you!

References

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