# Grasmannians and form factors

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# Outline

- Formfactors $\Leftrightarrow \langle 0|\mathcal{O}(x)|1, ..., n\rangle$
- On-shell diagram $\Leftrightarrow$ integral over Grassmannian manifold
- Generalization for formfactors
- Gluing operation

# Preliminaries

Basics of spinor helicity formalism

$$(p_i^\mu)^{a\dot{a}} = \begin{pmatrix} p_i^0 + p_i^3 & p_i^1 - ip_i^2 \\ p_i^1 + ip_i^2 & p_i^0 - p_i^3 \end{pmatrix} \Leftrightarrow p_i^{\dot{a}a} = \lambda_i^a \tilde{\lambda}_i^{\dot{a}}$$

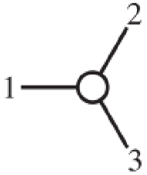
$$\langle ij \rangle := \epsilon_{ab} \lambda_i^a \lambda_j^b, [ij] := \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}}$$

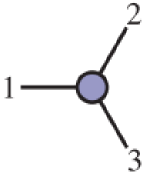
$$\Omega = g^+ + \tilde{\eta}_A \lambda^A - \frac{1}{2!} \tilde{\eta}_A \tilde{\eta}_B S^{AB} - \frac{1}{3!} \tilde{\eta}_A \tilde{\eta}_B \tilde{\eta}_C \lambda^{ABC} + \tilde{\eta}_1 \tilde{\eta}_2 \tilde{\eta}_3 \tilde{\eta}_4 g^-$$

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^4(\sum_{i=1}^n p_i) \delta^8(\sum_{i=1}^n \lambda_i \tilde{\eta}_i)}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle}$$

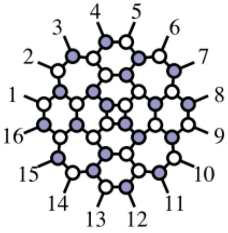
# Preliminaries

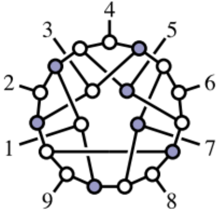
## Novel methods



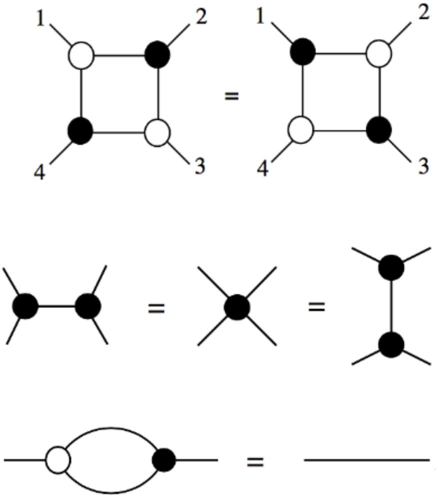

$$\sum_I \int d^4\tilde{\eta}_I \int \frac{d^2\lambda_I d^2\tilde{\lambda}_I}{vol[GL(1)]}$$



# Preliminaries

## Novel methods



- set of all $k$-planes in $n$-dimensional vector space $\Leftrightarrow$ $Gr(k, n)$
- $GL(k)$-transformation $\Leftrightarrow$ dimension $k(n-k)$
- $Gr(k, n)$ is parametrized by $k \times n$ matrix $C$
- orthogonal complement $\Leftrightarrow$ $k \times (n-k)$ matrix $\tilde{C}$: $C\tilde{C}^T = 0$

Chiral part of the stress-tensor supermultiplet:

$$T(x,\theta^+) = \mathrm{tr}(\phi^{++}\phi^{++}) + \ldots + \frac{1}{3}(\phi^{++})^4\mathcal{L}$$

The super formfactor is

$$\mathcal{F}_{k,n} = \int d^4x d^4\theta^+ e^{-iqx - i\theta^{+a}_{\alpha}\gamma^{-\alpha}_{a}} \langle 1, \ldots, n| T(x,\theta^+)|0\rangle$$

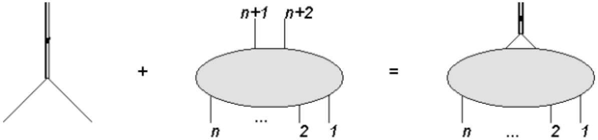
$$\text{Minimal } \mathrm{MHV} \text{ degree} \Leftrightarrow k = 2$$

$$\mathcal{F}_{2,n}(1,\ldots,n,1,\gamma^-) = \frac{\delta^{(4)}(P)\hat{\delta}^{(4)}(Q^+)\hat{\delta}^{(4)}(Q^-)}{\langle 12\rangle\langle 23\rangle \ldots \langle n-1n\rangle\langle n1\rangle}$$

$$P = \sum_{i=1}^{n} \lambda_i\tilde{\lambda}_i - q,\; Q^+ = \lambda_i\tilde{\eta}^+_i,\; Q^- = \lambda_i\tilde{\eta}^-_i - \gamma^-$$

# Preliminaries

## Gluing operation

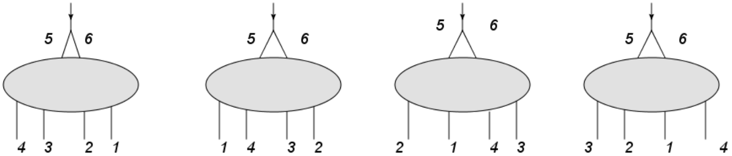


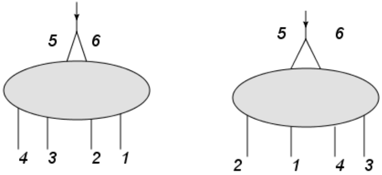

$$\mathcal{F}_{k,n} = \int \prod_{i=n+1}^{n+2} \frac{d^2\lambda_i d^2\tilde{\lambda}_i}{Vol[GL(1)^2]} d^4\tilde{\eta}_i \mathcal{F}_{2,2}(n+1, n+2)\Big|_{\lambda_{n+1,n+2}\to-\lambda_{n+1,n+2}} \mathcal{A}_{k,n+2}$$

$$\mathcal{F}_{k,n} = \langle \xi_A \xi_B \rangle^2 \int \frac{d^{k\times(n+2)}C}{Vol[GL(k)]} \Omega_{k,n} \hat{\delta}^{2\times k}(C\cdot\underline{\underline{\tilde{\lambda}}})\hat{\delta}^{4\times k}(C\cdot\underline{\underline{\tilde{\eta}}})\hat{\delta}^{2\times(n+2-k)}(C^{\perp}\cdot\underline{\underline{\lambda}})$$

$$\mathcal{F}_{3,4} = \langle \xi_A \xi_B \rangle^2 \int \frac{d^{3\times 6} C}{Vol[GL(3)]} \Omega_{3,4} \hat{\delta}^{2\times 3}(C \cdot \underline{\underline{\tilde{\lambda}}}) \hat{\delta}^{4\times 3}(C \cdot \underline{\underline{\tilde{\eta}}}) \hat{\delta}^{2\times 3}(C^{\perp} \cdot \underline{\underline{\lambda}}) +$$

$+$other gluing positions






$$\mathcal{F}_{3,4} = \mathrm{Res}_{41}(123) + \mathrm{Res}_{41}(561) + \mathrm{Res}_{23}(123) + \mathrm{Res}_{23}(561)$$

$$\mathrm{Res}_{41}(123) = \frac{\langle \xi_A \xi_B \rangle^2 \langle 13 \rangle^4 [4\underline{\underline{6}}]^4 \hat{\delta}^{(12)}(C_{(123)} \cdot \underline{\underline{\eta}})}{\langle 12 \rangle \langle 23 \rangle [\underline{\underline{56}}]^2 \langle 3|1+2|4] \langle 1|2+3|4] P_{123}^2}$$

$$\mathrm{Res}_{41}(561) = \frac{\langle \xi_A \xi_B \rangle^2 \langle 1\underline{\underline{5}} \rangle^4 [24]^4 \hat{\delta}^{(12)}(C_{(561)} \cdot \underline{\underline{\eta}})}{[23][34] \langle \underline{\underline{56}} \rangle^2 \langle 1|\underline{\underline{5}}+\underline{\underline{6}}|2] \langle 1|\underline{\underline{5}}+\underline{\underline{6}}|4] P_{234}^2}$$

**Component proportional to $(\tilde{\eta}_1)^2(\tilde{\eta}_2)^2(\tilde{\eta}_3)^4$ from Grassmannian integral:**

$$F(\phi_{12},\phi_{12},g^-,g^+) = c_{16}^2 c_{52}^2 \mathrm{Res}_{41}(561) + c_{36}^2 c_{54}^2 \mathrm{Res}_{23}(123)$$

**Component proportional to $(\tilde{\eta}_1)^2(\tilde{\eta}_2)^2(\tilde{\eta}_3)^4$ from BCFW recursion:**

$$F(\phi_{12},\phi_{12},g^-,g^+) = \frac{1}{\langle 1|q|2]}\left[\frac{[24]^2\langle 1|q|4]}{[23][34]P_{234}^2} + \frac{\langle 13\rangle^2\langle 3|q|2]}{\langle 34\rangle\langle 41\rangle P_{134}^2}\right]$$

- Non-local operator
- From factors of Wilson line operator ⇔ Reggeon amplitudes

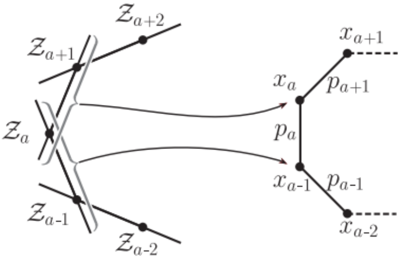
$$\mathcal{W}_p^c(k) = \int d^4x e^{ik\cdot x} \mathrm{Tr}\left\{ \frac{1}{\pi g} t^c \mathcal{P}\exp\left[ \frac{ig}{\sqrt{2}} \int_{-\infty}^{+\infty} dsp \cdot A_b(x+sp) t^b \right] \right\}$$

$$A^*_{m+n}(\Omega_1, ..., \Omega_m, g^*_{m+1}, ..., g^*_{m+n}) = \langle \Omega_1 ... \Omega_m | \prod_{i=1}^{n} \mathcal{W}_{p_{m+i}}^{c_{m+i}}(k_{m+i}) | 0 \rangle$$

# Wilson line operator form factors

## Momentum twistors

- Needed to simplify momentum conservation and 0-mass condition
- Null-rays in space-time ⇔ points in twistor space




**Incidence relations:**

$$\mu_{\dot\alpha} = x_{\alpha\dot\alpha}\lambda^\alpha,\ x_{\alpha\dot\alpha} = (p-q)_\mu\sigma^\mu_{\alpha\dot\alpha} \Rightarrow Z = (\lambda^\alpha, \mu_{\dot\alpha})$$

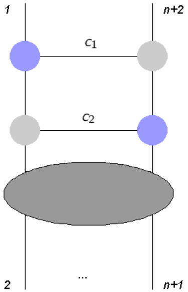
# Wilson line operator form factors

## Gluing operation in momentum twistor space

**Claim:**

Gluing operation in momentum twistor space is represented by attaching two consecutive BCFW bridges times some regulator:

$$\tilde{\mathcal{G}}^{m.tw.}_{i-1,i}[...] = N(\{\lambda\})\mathrm{Br}(\hat{i}, i+1) \circ \mathrm{Br}(\widehat{i+1}, i)[M(\{\lambda\}) \cdot ...]$$



# Wilson line operator form factors

## Gluing operation in momentum twistor space

### Action of BCFW bridge:

$$\mathrm{Br}(\hat{i}, i+1)[Y(1,...,n)] = \int \frac{dc}{c} Y(1,...,\hat{i},...,n),$$

$$\hat{\mathcal{Z}}_i = \mathcal{Z}_i + c\mathcal{Z}_{i+1}$$

- $\mathcal{L}_{k,n+2}$ - Grassmannian integral representation for ratio $\frac{\mathcal{A}_{k,n+2}}{\mathcal{A}_{2,n+2}}$
- $\omega_{k,n+2}$ - Grassmannian integral representation for ratio $\frac{A^*_{k,n+1}}{A^*_{2,n+1}}$ of amplitudes with one leg off-shell

### Fixing M,N:

$$\tilde{\mathcal{G}}^{m.tw.}_{n+1,n+2}[\mathcal{L}_{k,n+2}] = \omega_{k,n+2} \Rightarrow M = N^{-1} = S(i+1,i,i-1) = \frac{\kappa^*_{i-1}\langle i-1 i+1\rangle}{\langle i i+1\rangle\langle i-1 i\rangle}$$

# Wilson line operator form factors

## Examples

$$\mathcal{P}^{4(k-2)}_{n+2}(\mathcal{Z}_1,...,\mathcal{Z}_{n+2}) = \frac{\mathcal{A}^k_{n+2}}{\mathcal{A}^{k=2}_{n+2}}$$

$$\tilde{\mathcal{G}}^{m.tw.}_{i-1,i}[\mathcal{P}^{4(k-2)}_{n+2}] = S^{-1}(i+1,i,i-1)\kappa^*_{i-1}\times$$

$$\times\int\frac{dc_1}{c_1}\frac{dc_2}{c_2}\frac{\langle i-1i+1\rangle + c_1\langle i-1i\rangle + c_1c_2\langle i-1i+1\rangle}{\langle ii+1\rangle(\langle i-1i\rangle + c_2\langle i-1i+1\rangle)}\times$$

$$\times\mathcal{P}^{4(k-2)}_{n+2}(...,\mathcal{Z}_i + c_2\mathcal{Z}_{i+1}, \mathcal{Z}_{i+1} + c_1\mathcal{Z}_i + c_1c_2\mathcal{Z}_{i+1},...)$$

### Corollary

In momentum twistor space gluing operation amounts to shifting $i$-th twistor:

$$\tilde{\mathcal{G}}^{m.tw.}_{i-1,i}[\mathcal{P}^{4(k-2)}_{n+2}] = \mathcal{P}^{4(k-2)}_{n+2}(...,\mathcal{Z}_i - \frac{\langle i-1i\rangle}{\langle i-1i+1\rangle}\mathcal{Z}_{i+1},\mathcal{Z}_{i+1},...)$$

# Wilson line operator form factors

## Examples

$$\mathcal{P}_6^4 = [12345] + [13456] + [12356]$$

$$[ijklm] = \frac{\hat{\delta}^{(4)}(\langle ijkl\rangle\eta_m + \text{cyclic})}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}$$

$$\tilde{\mathcal{G}}_{5,6}^{m.tw.}[\mathcal{P}_6^4] = [12345] + \frac{[13456]}{1 + \frac{\langle p_5\xi_5\rangle}{\langle p_5 1\rangle}\cdot\frac{\langle 1345\rangle}{\langle 3456\rangle}} + \frac{[12356]}{1 + \frac{\langle p_5\xi_5\rangle}{\langle p_5 1\rangle}\cdot\frac{\langle 1235\rangle}{\langle 2356\rangle}}$$

# Summary

- A method for deriving Grassmannian integral representation is obtained
- The new method is valid for different representations of external data
- Results derived by means of new and conventional methods agree

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