Supersymmetric indices elliptic hypergeometric functions and integrability

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Outline

- Part I: Superconformal index and SUSY duality
- Part II: Integrable models on the square lattice
- Part III: Integrable models from SUSY computations

Integrable models from Supersymmetric gauge theories

There are several connections of integrable models to supersymmetric gauge theories.

One of such connections is a correspondence between supersymmetric quiver gauge theories with four supercharges and integrable lattice models of statistical mechanics such that the two-dimensional spin lattice is the quiver diagram, the partition function of the lattice model is the partition function of the gauge theory and the Yang-Baxter equation expresses the identity of partition functions for dual pairs.

[Spiridonov 1011.3798]

[Benini, Nishioka, Yamazaki 1109.0283]

[Terashima, Yamazaki 1203.5792], [Yamazaki 1203.5784]

The idea of the correspondence allows one to obtain

solutions to the quantum Yang-Baxter equation

from supersymmetric gauge theory calculations.

Superconformal index

[talks by Kim and Spiridonov]

Example: $4d$ $\mathcal{N} = 1$ theory

Consider the $\mathcal{N} = 1$ **superconformal theory** on $S^3 \times S^1$.

The symmetry group of this theory is $SU(2,2|1)$.

which has the following generators

$$J_i,\ \bar{J}_i \text{— Lorentz rotations}$$

$$P_\mu,\ Q_\alpha,\ \bar{Q}_{\dot{\alpha}} \text{— Supertranslations}$$

As in any conformal invariant field theory, we have superconformal generators

$$K_\mu,\ S_\alpha,\ S_{\dot{\alpha}} \text{— Special superconformal transformation}$$

$$H \text{— Dilatations}$$

The action of a supersymmetric theory should also invariant under R–symmetry

$$R \text{— } U(1)_R \text{ rotations.}$$

$3d$ $\mathcal{N}=2$ index

In $4d$ SCFT there exists a supercharge operator Q, such that

$$\frac{1}{2}\{Q,S\} = \Delta - \frac{3}{2}R - 2\bar{J}_3$$

with Δ the Hamiltonian in the radial quantization.

The superconformal index of four-dimensional $\mathcal{N}=1$ superconformal field theory is a twisted partition function defined on $S^3 \times S^1$ as follows

$$I(q,\{t_i\}) = \mathrm{Tr}\left[(-1)^{\mathrm{F}} e^{-\beta\{Q,Q^\dagger\}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} \prod_i t_i^{F_i}\right]$$

- the trace is taken over the Hilbert space of the theory
- F is the fermion number which takes value zero on bosons and one on fermions
- F_i is the charge of global symmetry with fugacity t_i

Matrix integral

Römelsberger introduced a simple procedure for an explicit computation of the superconformal index.

[Romelsberger 0707.3702]

First compute a single letter index which one obtains by summing over all the fields contributing to the index

$$\text{ind}(p, q, \underline{z}, \underline{t}) = \frac{2pq - p - q}{(1-p)(1-q)} \chi_{adj}(\underline{z}) \\ + \sum_j \frac{(pq)^{R_j/2} \chi_{R_F,j}(\underline{t}) \chi_{R_G,j}(\underline{z}) - (pq)^{1-R_j/2} \chi_{\bar{R}_F,j}(\underline{t}) \chi_{\bar{R}_G,j}(\underline{z})}{(1-p)(1-q)}.$$

The full index is formed by summing over multiparticle states, i.e. by inserting the single letter index into the "plethystic" exponential PE[.] and integrating over the gauge group in order to get gauge–invariant quantity

$$\int_G d\mu(g) \, \mathrm{PE}[ind(\{f_i\})]$$

- $\mu(g)$ is the invariant Haar measure
- the plethystic exponential is defined as

$$\mathrm{PE}[f(x_i)] = \exp\left(\sum_{n=1}^{\infty} \frac{f(x_1^n, x_2^n, \dots)}{n}\right)$$

[Benvenuti, Feng, Hanany and Y.H. He 0608050]

[Feng, Hanany and Y.H. He 0701063]

The group–theoretical data to calculate the superconformal index is contained in such tables:

	$SU(2)$	$SU(8)$	$U(1)_R$
Q	f	f	$\frac{1}{4}$
V	adj	1	$\frac{1}{2}$

Index can be expressed in terms of integrals over elliptic Gamma functions

[Dolan and Osborn 0801.4947]

the elliptic Gamma function is

$$\Gamma(z;p,q) = \prod_{i,j=0}^{\infty} \frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j}, \quad |p|,|q|<1\ .$$

[Ruijsenaars '97]

Example: A chiral multiplet with R–charge r and flavor charge f contributes to the index as a factor:

$$\Gamma((pq)^{r/2}z^f;p,q)$$

Seiberg duality

Different theories describe the same physics in their IR fixed points.

Example:

- **electric theory**: with SU(2) gauge group and quark superfields in the fundamental representation of the SU(6) flavour group.
- **magnetic theory**: does not have gauge degrees of freedom, the matter sector contains meson superfields in 15-dimensional antisymmetric SU(6)-tensor representation of the second rank

[Seiberg 9402044]

$$\frac{(p;p)_\infty (q;q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{j=1}^{6} \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)\Gamma(z^{-2}; p, q)} \frac{dz}{2\pi i z} = \prod_{1 \leq i < j \leq 6} \Gamma(t_i t_j; p, q)$$

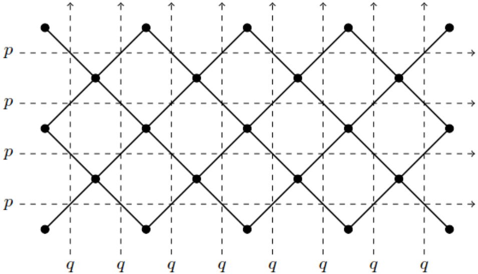
with the balancing condition $\prod_{j=1}^{6} t_j = pq$.

[Dolan and Osborn 0801.4947]

from math [Spiridonov '01]

Integrable models on the square lattice

Integrable models on the square lattice



- Spin variables

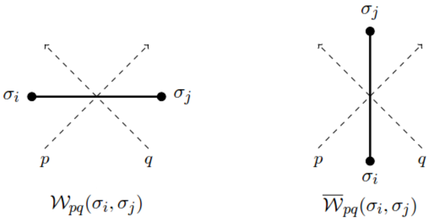
$$\sigma_j = (x_j, m_j)\,, \quad x_j \in \mathbb{R}\,,\; m_j \in \mathbb{Z}\,, \quad j = 1, 2, \ldots, N,$$

  are assigned to each vertex of the square lattice.

- Two spins σ_i, σ_j, interact only if they are at vertices i, j, connected by an edge (ij), of the square lattice.

Boltzmann weights

- Interactions are characterized by the Boltzmann weights



- Two edge Boltzmann weights only depend on the difference of rapidity variables $p - q := \alpha$

$$W_\alpha(\sigma_i, \sigma_j) := W_{pq}(\sigma_i, \sigma_j)$$
$$\bar{W}_\alpha(\sigma_i, \sigma_j) := \bar{W}_{pq}(\sigma_i, \sigma_j)$$

Symmetries of Boltzmann weights

- Two Boltzmann weights are also related by the **crossing symmetry**

$$\bar{W}_\alpha(\sigma_i, \sigma_j) = W_{\eta-\alpha}(\sigma_i, \sigma_j)$$

  where $\eta > 0$ is a real valued, model dependent **crossing parameter**.

- Boltzmann weights are also spin **reflection symmetric**, such that

$$W_\alpha(\sigma_i, \sigma_j) = W_\alpha(\sigma_j, \sigma_i)$$

Partition function

The **partition function** is defined as

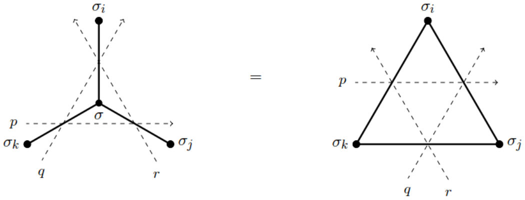
$$\mathcal{Z} = \sum \int \prod_{(ij)} W_\alpha(\sigma_i, \sigma_j) \; \prod_{(kl)} W_{\eta-\alpha}(\sigma_k, \sigma_l) \; \prod_n S(\sigma_n) \, dx_n \, .$$

- integral and sum run over all possible values of internal spins in the lattice
- boundary spins are assigned fixed values.

The goal of statistical mechanics is to evaluate the partition function in the thermodynamic limit, as $N \to \infty$.

Star-triangle relation

An exact evaluation is possible if the Boltzmann weights satisfy the Yang-Baxter equation, which for the model we consider here takes the form of the following **star-triangle relation**



$$\int dx\, S(\sigma) W_{\eta-\alpha_i}(\sigma_i,\sigma) W_{\eta-\alpha_j}(\sigma_j,\sigma) W_{\eta-\alpha_k}(\sigma,\sigma_k)$$
$$= \mathcal{R}(\alpha_i,\alpha_j,\alpha_k)\, W_{\alpha_i}(\sigma_j,\sigma_k) W_{\alpha_j}(\sigma_i,\sigma_k) W_{\alpha_k}(\sigma_j,\sigma_i)$$

Inversion relations

The simple consequence of the star-triangle relation and initial condition gives the unitarity and inversion relations

$$\mathcal{W}_\alpha(\sigma_i,\sigma_j)\mathcal{W}_{-\alpha}(\sigma_i,\sigma_j)=1$$

$$\sum_{\sigma_0} S(\sigma_0)\mathcal{W}_{\eta-\alpha}(\sigma_i,\sigma_0)\mathcal{W}_{\eta+\alpha}(\sigma_0,\sigma_j)=\frac{1}{\mathcal{S}(\sigma_i)}\left(\delta(x_i+x_j)+\delta(x_i-x_j)\right)\,.$$

Bazhanov-Sergeev model

In this integrable lattice model spin variables get continuous values

$$0 \leq \sigma_i < 2\pi,$$

and the Boltzmann weights are expressed in terms of elliptic gamma functions

$$\mathcal{W}_\alpha(\sigma_i, \sigma_j) = \frac{1}{k(\alpha)} \Gamma(e^{\alpha - \eta \pm i\sigma_i \pm i\sigma_j}; p, q),$$

$$\mathcal{S}(\sigma_0) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi} \theta(e^{\pm 2i\sigma_0}; q),$$

where

$$k(\alpha) = \frac{\Gamma(e^{2\alpha}(pq)^2; p, q, (pq)^2)}{\Gamma(e^{2\alpha} pq; p, q, (pq)^2)} \quad \text{and} \quad \Gamma(z; p, q, t) := \prod_{i,j,k=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1} t^{k+1}}{1 - z p^i q^j t^k}.$$

[Bazhanov and Sergeev 1006.0651; 1106.5874]

[Spiridonov 1011.3798]

Integrable models from SUSY

Correspondence

By adding a certain superpotential one may break flavor symmetry of dual theories from $SU(6)$ down to $SU(2) \times SU(2) \times SU(2)$.

Then the elliptic beta integral identity gets the form of the **star-triangle relation**

Interpretation in terms of $4d$ $\mathcal{N} = 1$ theory

- spin lattice — quiver theory with $SU(2)$ gauge groups
- Boltzmann weights — contribution of chiral multiplets to PF
- Self interaction — contribution of vector multiplets to PF
- Star-triangle relation — Seiberg duality in terms of PF

Star-star relation

One can also construct an IRF-type integrable models. In this case the identity of partition functions for dual theories can be written as the star-star relation for the IRF-type model.

The R-matrix of the IRF type Bazhanov-Sergeev model is the superconformal index of $SU(2)$ gauge theory with $SU(8)$ flavor symmetry

$$V(t_1,\ldots,t_8;p,q) = \frac{(p;p)_\infty (q;q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{i=1}^{8} \Gamma(t_i z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)} \frac{dz}{2\pi i z}.$$

The $W(E_7)$ symmetry transformation of this integral can be reduced to the star-star relation for integrable models

$$V(t_1,\ldots,t_8;p,q) = \prod_{1\le j<k\le 4} \Gamma(t_j t_k;p,q)\Gamma(t_{j+4}t_{k+4};p,q)\, V(s_1,\ldots,s_8;p,q),$$

where

$$s_j = \rho^{-1} t_j,\ j=1,2,3,4, \quad s_j = \rho t_j,\ j=5,6,7,8,$$
$$\rho = \sqrt{\frac{t_1 t_2 t_3 t_4}{pq}} = \sqrt{\frac{pq}{t_5 t_6 t_7 t_8}}\ .$$

Inversion relation

The inversion relation has an interesting counterpart on supersymmetry side of the correspondence, namely, it is related to the chiral symmetry breaking of the corresponding supersymmetric gauge theory.

Such relation can be derived in many different ways, from supersymmetric gauge theory side one can obtain inversion relation by an accurate limit of parameters in the partition functions of dual theories.

Comments

- The correspondence between susy gauge theories and integrable models has led to the construction of several exactly solvable models of statistical mechanics. [Spiridonov 1011.3798], [Yamazaki 1307.1128], [Kels 1504.07074], [IG, Kels 1610.09229], [Yagi 1504.04055], [Kels 1504.07074], [IG, Spiridonov 1505.00765] [Yamazaki, Yan 1504.05540], [IG, Jafarzade 1712.09651]
- Given a supersymmetric duality with a gauge and matter multiplets in some representation of the gauge and flavor groups, what is the corresponding integrable lattice model?
- The R matrix is dictated by some quantum group. One needs to elucidate the origin of new solutions in the framework of the representation theory of quantum group. [Chicherin, Derkachov 1412.3383, 1411.7595], [Chicherin, Spiridonov 1511.00131]
- V-functions is the τ-functions of the discrete Painleve system of type $E_8^{(1)}$. Is there a new relation between two types of integrability? [talk by Yamada]
- There are a lot of attempts to extend the idea of integrability to three-dimensional lattice models. The Yang-Baxter equation in this case takes the form of the so-called tetrahedron equation by Zamolodchikov. It would be interesting to extend the relationship between supersymmetric dualities and integrable models and find a solution of the tetrahedron equation in this context.