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# Nanomechanics of graphene

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**Gornyi, Kachorovskii, Mirlin,**
**Conductivity of suspended graphene at the Dirac point, PRB (2012)**
**Rippling and crumpling in disordered free-standing graphene, PRB (2015)**
**Anomalous Hooke's law in disordered graphene, 2D Mater. (2017)**
**Burmistrov, Gornyi, Kachorovskii, Katsnelson, Mirlin, Quantum elasticity of graphene: Thermal expansion coefficient and specific heat, PRB (2016)**
**Burmistrov, Gornyi, Kachorovskii, Los, Katsnelson, Mirlin,**
**Stress-controlled Poisson ratio of a crystalline membrane: Application to graphene, PRB (2018)**
**Burmistrov, Kachorovskii, Gornyi, Mirlin, Differential Poisson's ratio of a crystalline 2D membrane, Annals of Physics (2018)**

*"Low Dimensional Materials: Theory, Modeling, Experiment"*
*Dubna, Russia., July 2018*

# Outline

- Introduction. Isolated crystalline membrane. Flexural phonons and ripples

- Phase diagram of clean crystalline membrane. Crumpling and buckling transitions

- Anomalous Hooke's law. Nonlinear scaling of deformation with applied stress

- Disordered membrane. Crumpling transition in the membrane with random curvature

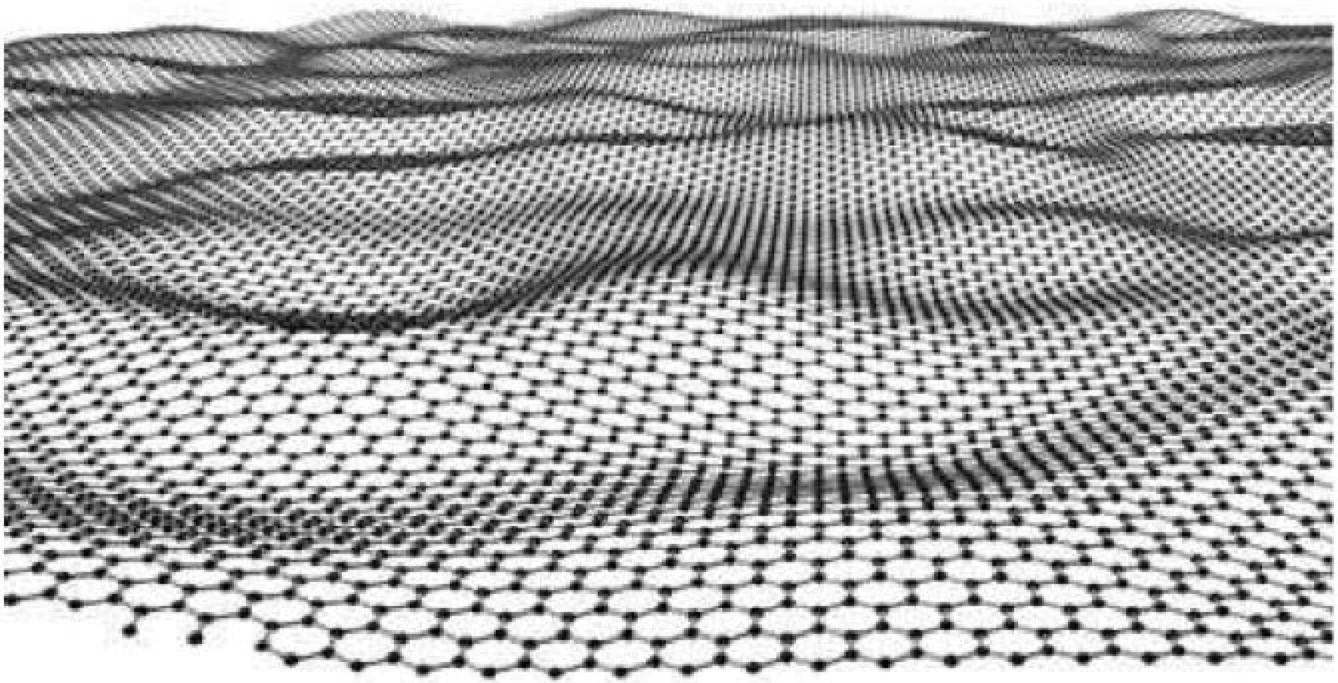
- Thermal expansion . Negative temperature –independent thermal expansion coefficient

- Poisson's ratio. Is graphene an auxetic material?

- Experiment and numerical simulations.

# Isolated crystalline membrane

dynamic out-of-plane deformations (flexural phonons)
+ static frozen-out deformations (ripples)

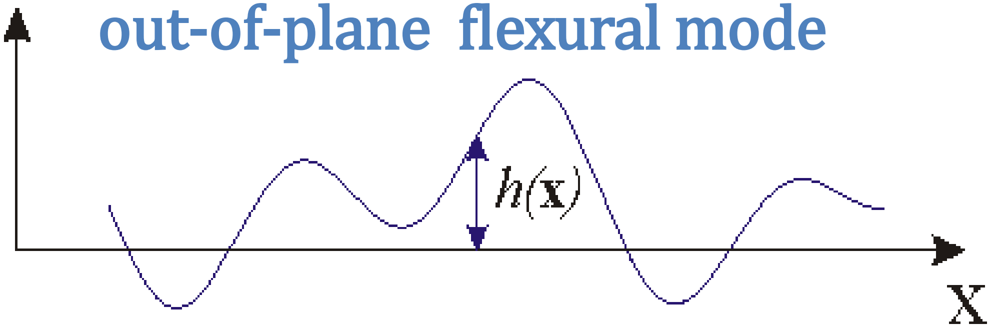


Meyer, Geim, Katsnelson, Novoselov, Booth, Roth, Nature'07

numerical simulations for suspended graphene

# Flexural phonons (FP)

out-of-plane flexural mode



$$E = \frac{1}{2}\int d\mathbf{x}\left[\rho\dot{h}^2 + \varkappa_0(\Delta h)^2\right]$$

$$\varkappa_0 \simeq 1 \text{ eV}$$

bending rigidity

$$\boxed{\omega_q = Dq^2}$$

soft dispersion of FP

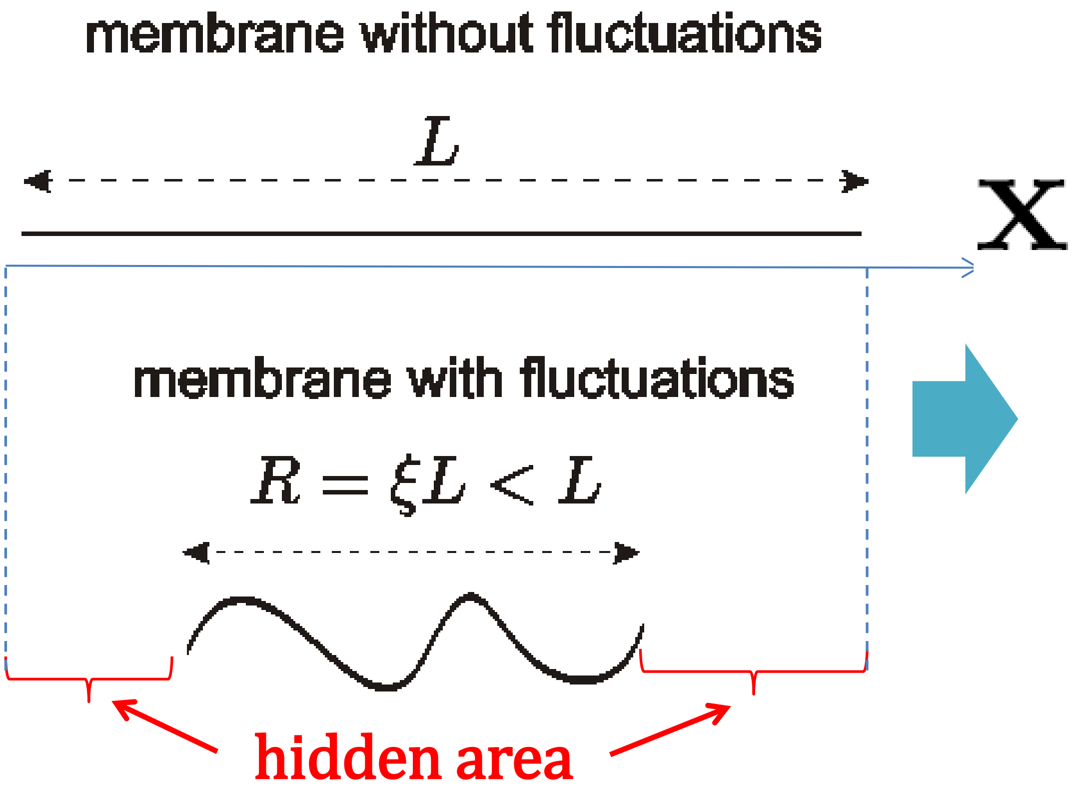
$$D = \sqrt{\varkappa_0/\rho}$$

## In-plane phonons

$$\omega_q^{\perp} = \sqrt{\frac{\mu}{\rho}}\, q, \quad \omega_q^{\parallel} = \sqrt{\frac{2\mu+\lambda}{\rho}}\, q$$

$\mu, \lambda$ in-plane elastic coefficients

# Global shrinking of membrane induced by FP or ripples



**"Membrane effect":** thermal fluctuations in y-direction lead to shrinking in x direction

I.M. Lifshitz, JETP (1952)

$$\boxed{\mathbf{R} = \xi \mathbf{x} + \mathbf{u} + \mathbf{h}}$$

$\xi < 1$ stretching parameter

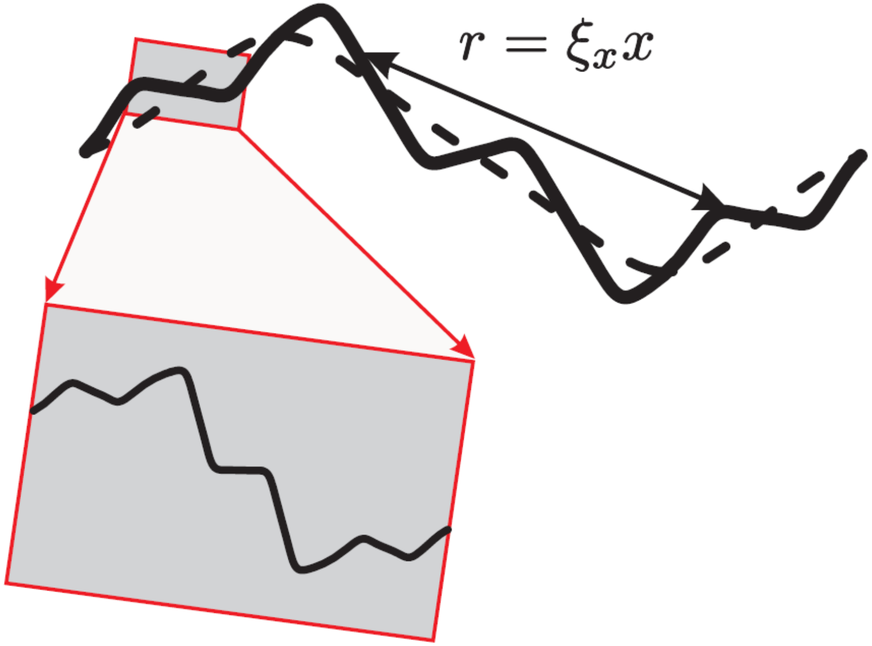
global deformation

in-plane and out-of-plane fluctuations

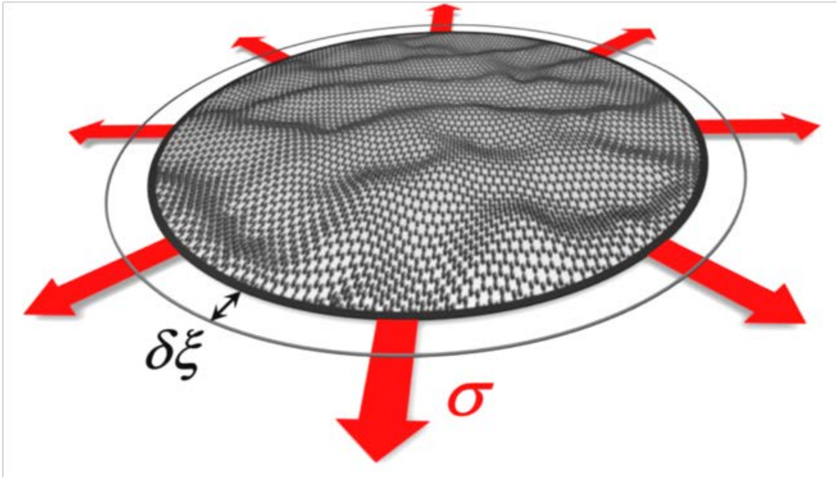
# Geometry of the membrane, effect of the external tension

**fractal geometry**

$$\xi = \xi_L$$



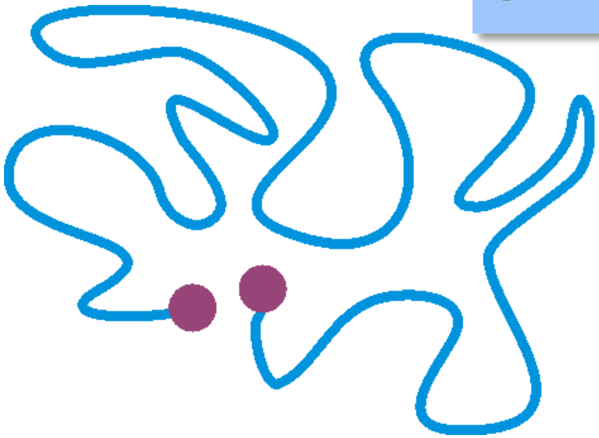
External tension
$\sigma$ "irons" thermal
or static fluctuations



# Crumpling transition

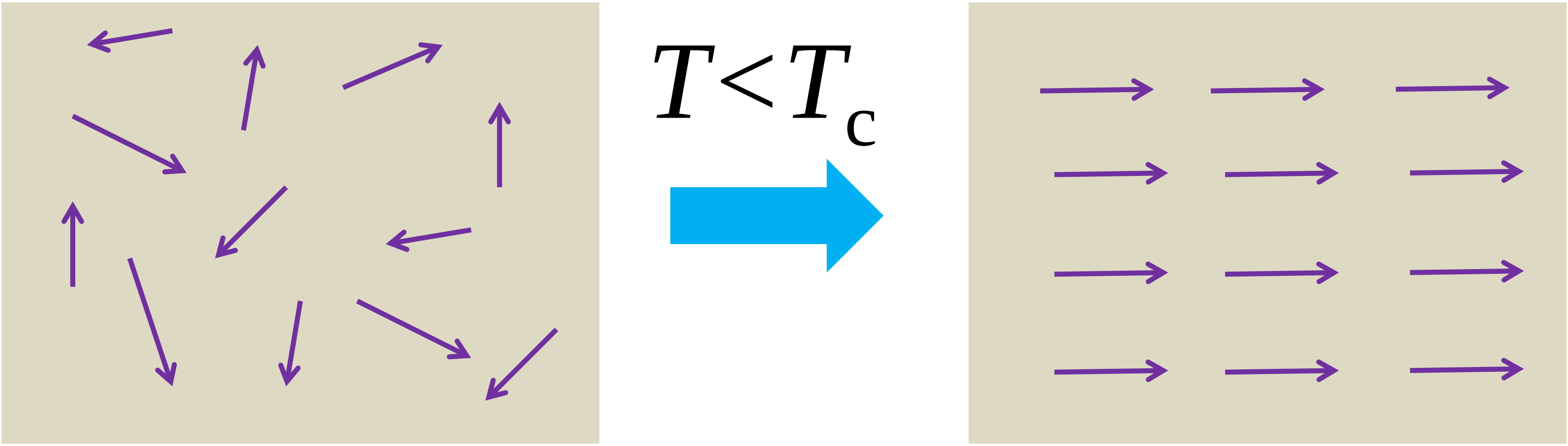
Paczuski, Kardar, Nelson , PRL (1988) ;
David and Guitter, Europhys. Lett. (1988);
Nelson, Piran, Weinberg , *“Statistical Mechanics of Membranes and Surfaces “* (1989);

**Crumpled phase** $\xi^2 \equiv 0$



**Flat phase** $\xi^2 = \xi_T^2 > 0$

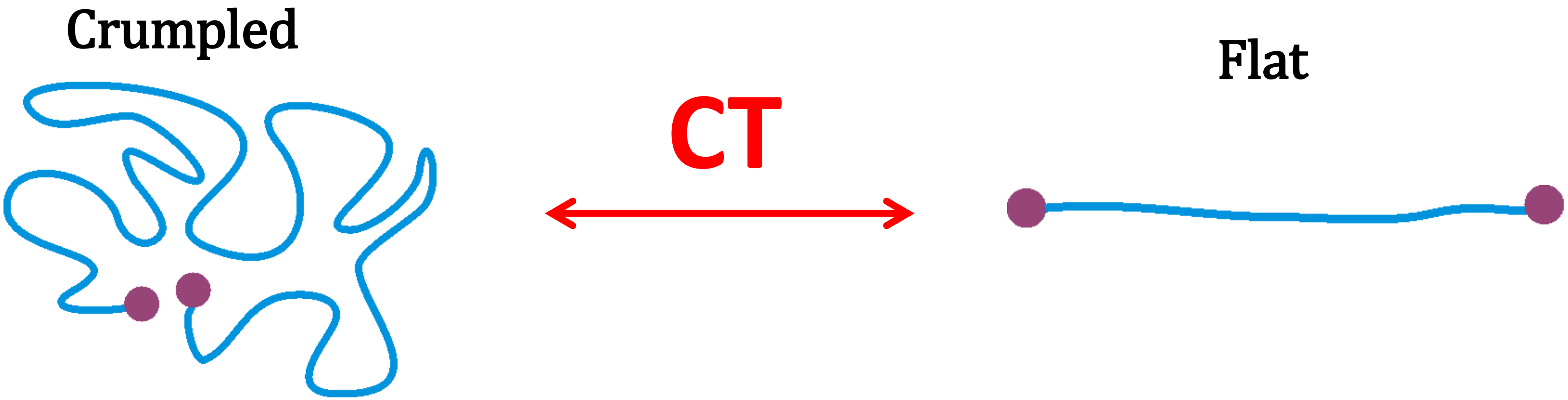
## Analogy with ferromagnetic transition



$T<T_c$

**spontaneous symmetry breaking !!!**

# Physics behind crumpling transition



## Competition between two effects:

1) "membrane effect"
→ shrinking of
membrane due to FP

2) anharmonic coupling between
FP and in-plane modes → infrared
divergence of bending rigidity

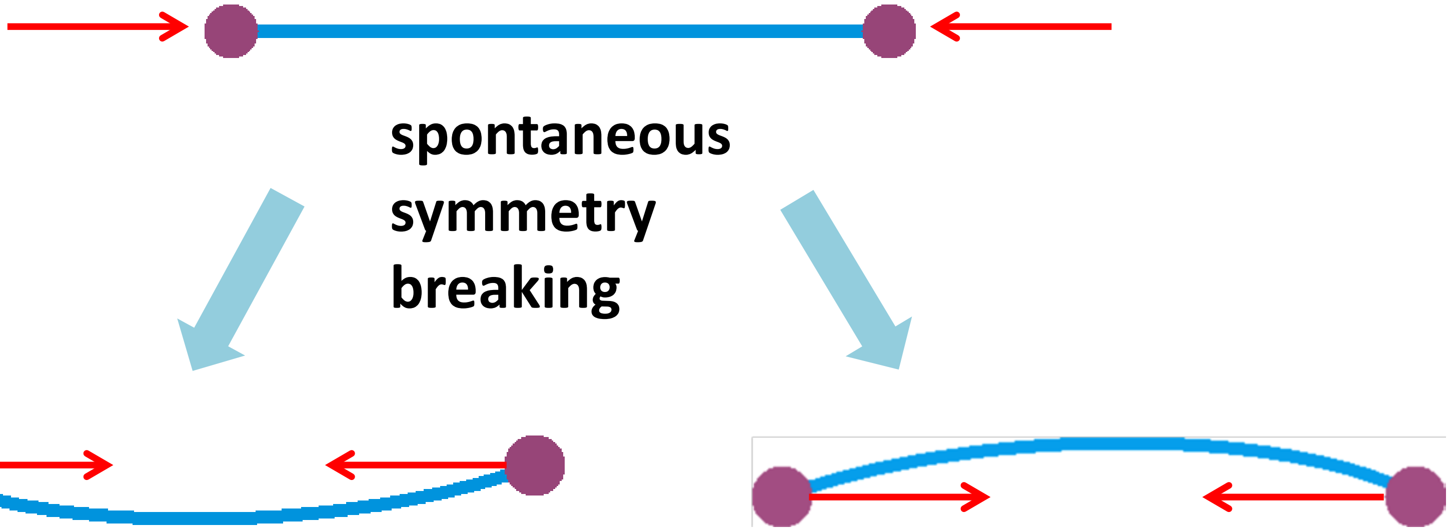
**tendency to crumpling**

**stabilization of the flat phase**

$$\varkappa \propto L^{\eta} \propto \frac{1}{q^{\eta}}$$

$\eta \approx 0.7$ – critical exponent

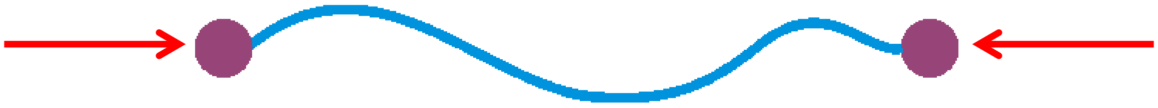
# Buckling transition (BT)



$T = 0$

**spontaneous symmetry breaking**

Membrane with fluctuations, $T \neq 0$



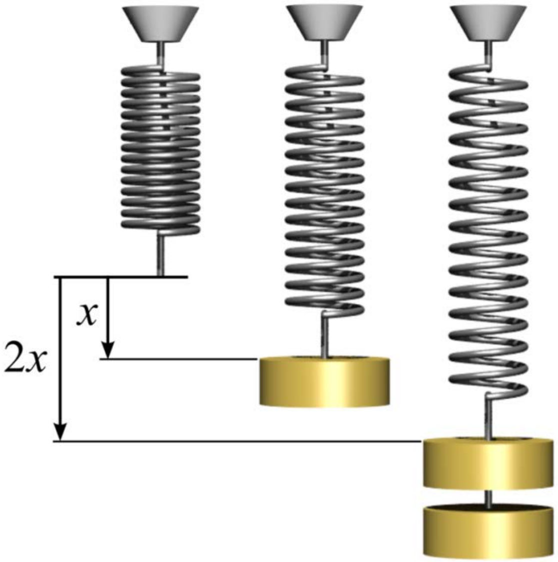
???

# Manifestation of BT: anomalous Hooke's law

$$\boxed{\delta\xi \propto \sigma^{\alpha}}$$

$\alpha$ - critical index of BT

**Hooke's law (1678): α=1**



**Graphene :** $\sigma < \sigma_* \sim \mu \frac{T}{\varkappa_0}$

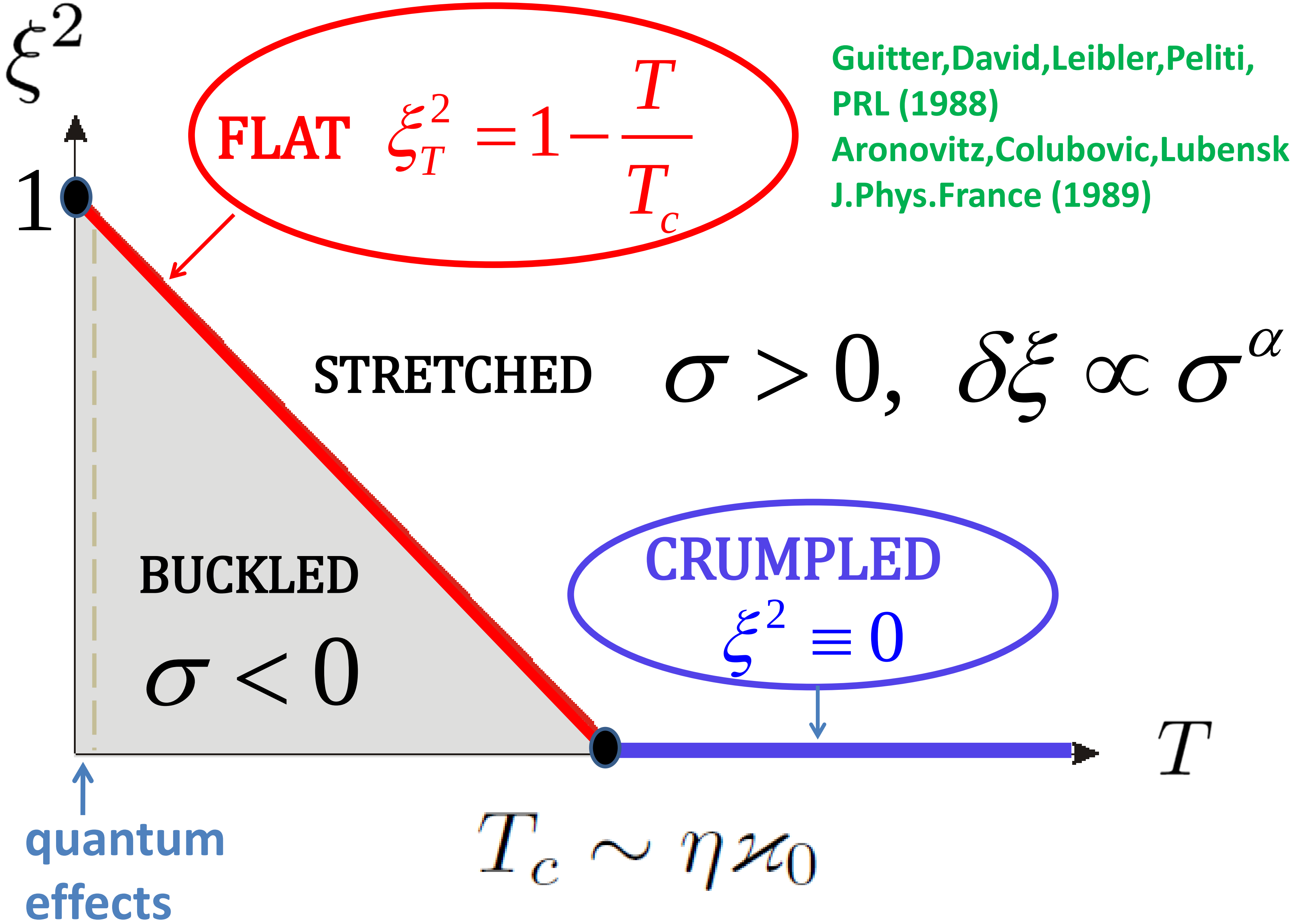
$$1)\ \ \alpha_{\text{clean}} = \frac{\eta}{2-\eta} < 1$$

$$2)\ \ \alpha_{\text{clean}} \neq \alpha_{\text{disordered}}$$

**anomalous Hooke's law at SMALL (!) tension**

1) Guitter, David, Leibler, Peliti, PRL (1988); Aronovitz, Colubovic, Lubensky J.Phys. France (1989)
2) Gornyi, Mirlin, Kachorovskii., 2D Mater. (2017)

# Phase diagram of  clean crystalline membrane



**FLAT** $\xi_T^2 = 1 - \frac{T}{T_c}$

Guitter,David,Leibler,Peliti,
PRL (1988)
Aronovitz,Colubovic,Lubensky
J.Phys.France (1989)

STRETCHED $\sigma > 0, \ \delta\xi \propto \sigma^{\alpha}$

BUCKLED

$\sigma < 0$

CRUMPLED

$\xi^2 \equiv 0$

quantum
effects

$T_c \sim \eta \varkappa_0$

# Experimental evidence of anharmonicity

- huge (unrealistic) theoretical prediction for out-of- plane fluctuations calculated in harmonic approximation
- several order of magnitude discrepancy between theoretical and experimental values of mobility limited by FP
- experimental measurements of anomalous Hooke’s law in suspended graphene

## ▪ Huge out-of-plane fluctuations

$$h(\mathbf{r}) = \sum_{\mathbf{q}} \sqrt{\frac{\hbar}{2\rho\omega_{\mathbf{q}} S}} (b_{\mathbf{q}} + b^{\dagger}_{-\mathbf{q}}) e^{i\mathbf{qr}}$$

$$b_{\mathbf{q}} = \sqrt{N_{\mathbf{q}}} e^{-i\varphi_{\mathbf{q}}}$$

$$N_{\mathbf{q}} \approx \sqrt{T/\hbar\omega_{\mathbf{q}}} \gg 1$$

**Random classical field:**

$$h(\mathbf{r}) = \sum_{\mathbf{q}} \sqrt{\frac{2T}{\varkappa_0 q^4}} \cos(\mathbf{qr} + \varphi_{\mathbf{q}})$$

➡

$$\boxed{\langle h_{\mathbf{q}} h_{-\mathbf{q}} \rangle = \frac{T}{\varkappa_0 q^4}}$$

correlation function of FP

$$\boxed{\sqrt{\langle h^2(\mathbf{r}) \rangle} \propto \sqrt{\frac{T}{\varkappa_0} \int \frac{d^2\mathbf{q}}{q^4}} \propto \sqrt{\frac{T}{\varkappa_0}} L}$$

graphene at T=300 K:

$$\sqrt{T/\varkappa_0} \approx 0.2$$

⬇

**unrealistic (proportional to the system size !!!) thermal out-of-plane fluctuations**

# Scattering off FP in graphene

$$V = g_1 (\nabla h)^2 / 2$$

FP contribution to the deformation potential

$g_1 \simeq 30$ eV deformation coupling constant,

## Theory: Golden-rule calculation

$$\sigma_{\rm ph} = \frac{e^2}{\hbar} \frac{\pi^2 N}{24 g^2 \ln (q_T L)} \approx 10^{-3} \frac{e^2}{h}$$

simple theory yields unrealistic (too small) values of conductivity at the Dirac point

## Experiment:

$$\sigma_{\rm ph} \sim 10 \div 50 \frac{e^2}{h}$$

K. Bolotin *et al*., PRL (2008)

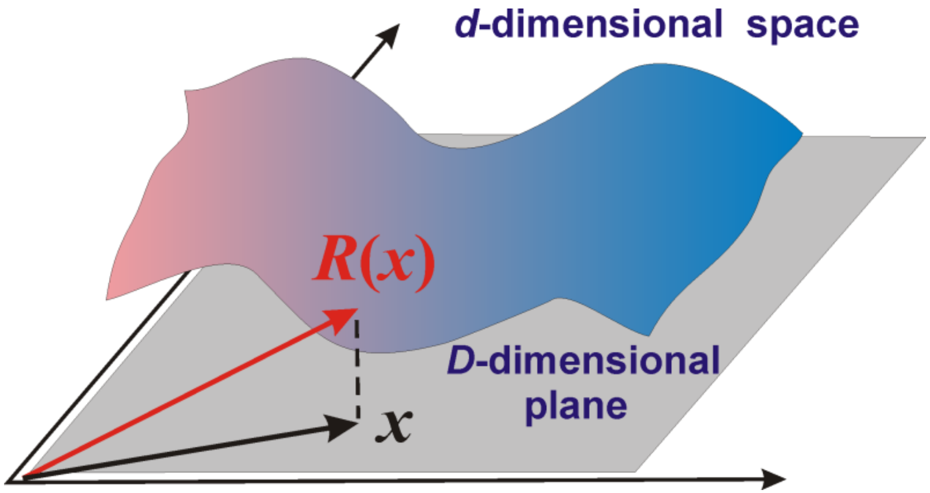
$$g = \frac{g_1}{\sqrt{32 \varkappa_0}} \simeq 5.3$$

dimensionless e-ph coupling constant

$N = 4$ spin×valleys,

$q_T = T/\hbar v$

# Theory of crumpling transition



Paczuski, Kardar, Nelson , PRL (1988) ;
David and Guitter, Europhys. Lett. (1988);
Nelson, Piran, Weinberg , “Statistical Mechanics of Membranes and Surfaces “ (1989);

$$E= \int d^D x \left\{ \frac{\varkappa_0}{2}(\partial_\alpha\partial_\alpha \mathbf{R})^2 - \frac{t}{2}(\partial_\alpha\mathbf{R}\partial_\alpha\mathbf{R}) + u(\partial_\alpha\mathbf{R}\partial_\beta\mathbf{R})^2 + v(\partial_\alpha\mathbf{R}\partial_\alpha\mathbf{R})^2 \right\}$$

$$\alpha, \beta = 1, ..., D$$

$\mathbf{R}(\mathbf{x})$ is d-dimensional vector
$\mathbf{x}$ is D-dimensional vector

For physical membranes d=3, D=2

$$\boxed{\mathbf{R} = \underbrace{\xi\mathbf{x}}_{\text{global deformation}} + \underbrace{\mathbf{u} + \mathbf{h}}_{\text{in-plane and out-of-plane fluctuations}}}$$

# Energy of membrane

stretching energy

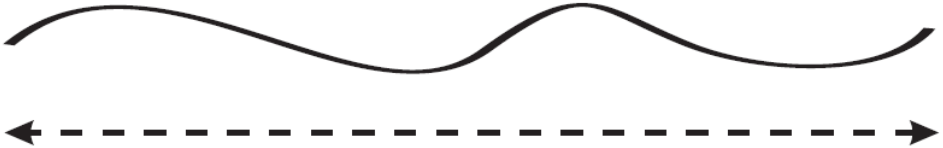
energy of fluctuations

$$E_0 = \frac{L^2(\mu+\lambda)}{2}\left[(\xi^2-1)^2 + \underbrace{(\xi^2-1)\int \frac{d^2\mathbf{x}}{L^2}\,\partial_\alpha \mathbf{h}\partial_\alpha \mathbf{h}}_{\text{coupling between stretching and fluctuations}}\right] + E(\tilde{\mathbf{u}},\mathbf{h})$$

$$\tilde{\mathbf{u}} = \xi\mathbf{u}$$

In the absence of fluctuations $\xi$=1

**Membrane effect:** transverse fluctuations lead to decrease of membrane size in x-direction



$$R = \xi_L L$$

# Energy of fluctuations

$$E = \int d^2\mathbf{x} \left\{ \frac{\varkappa_0}{2} (\Delta \mathbf{h})^2 + \mu u_{ij}^2 + \frac{\lambda}{2} u_{ii}^2 \right\}$$

strong
anharmonicity

$$u_{\alpha\beta} = \frac{1}{2} \left( \partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha \mathbf{h} \partial_\beta \mathbf{h} \right)$$

strain tensor

# Scaling of $\xi$

minimization of energy →

$$\xi^2 = 1 - \frac{\langle \nabla h \nabla h \rangle}{2}$$

$$E_h = \frac{1}{2}\int \varkappa_0 (\Delta h)^2 d^2\mathbf{x}$$

$$\langle \nabla h \nabla h \rangle = \int (\nabla h \nabla h)\ e^{-E_h/T}\,\{dh\}$$

$$\langle h_{\mathbf{q}} h_{-\mathbf{q}} \rangle = \frac{T}{\varkappa_0 q^4}$$

$$\langle \nabla h \nabla h \rangle = \frac{T}{\varkappa_0} \int \frac{d^2\mathbf{q}}{(2\pi)^2 q^2} \qquad \propto \ln L$$

$$\xi^2 = 1 - \frac{\langle \nabla h \nabla h \rangle}{2}$$

logarithmic divergence → scaling with $L$

$$\boxed{\frac{d\xi^2}{d\Lambda} = -\frac{T}{4\pi\varkappa_0}}$$

$\xi \to 0$, for certain value of $L$

⬇

flat phase is destroyed by thermal fluctuations

$$\Lambda = \ln L \longleftrightarrow \ln(1/q)$$

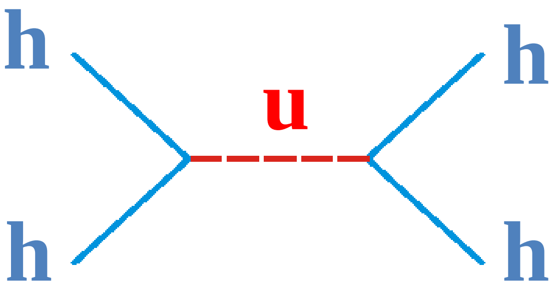
# How to stabilize the flat phase?

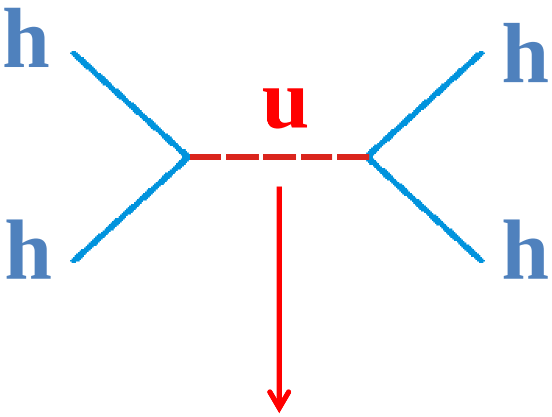
$$\varkappa_0 \longrightarrow \varkappa_q$$

$$\boxed{\frac{d\xi^2}{d\Lambda} = -\frac{T}{4\pi\varkappa_0}} \quad ➡ \quad \boxed{\frac{d\xi^2}{d\Lambda} = -\frac{T}{4\pi\varkappa_q}}$$

## Physical mechanism:

anharmonicity → interaction between h-modes due to the exchange of u-modes





$$E_h = \frac{1}{2}\sum_{\mathbf{q}} \varkappa_0 q^4 h_{\mathbf{q}} h_{-\mathbf{q}} + \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} q^4 R^{\mathbf{q}}_{\mathbf{k},\mathbf{k}'} h_{-\mathbf{k}} h_{\mathbf{k}+\mathbf{q}} h_{\mathbf{k}'} h_{-\mathbf{q}-\mathbf{k}'}$$

**pairing**

**RPA**

$$h_{-\mathbf{k}} h_{-\mathbf{q}-\mathbf{k}'} \rightarrow \langle h_{-\mathbf{k}} h_{-\mathbf{q}-\mathbf{k}'} \rangle$$

$$E_h = \frac{1}{2}\sum_{\mathbf{q}} \boxed{\varkappa_q} q^4 h_{\mathbf{q}} h_{-\mathbf{q}}$$

# Anharmonicity-induced increase of the bending rigidity

$q \ll q_*$ → universal scaling

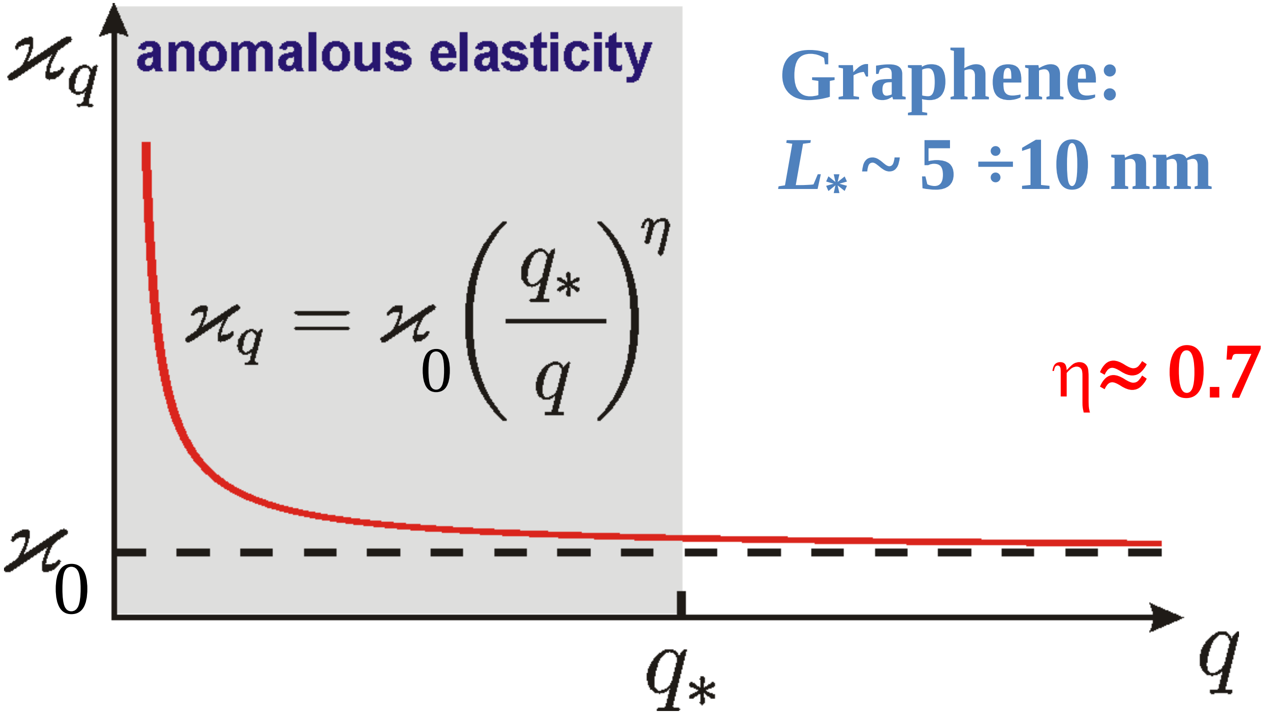
$$q_* \sim \frac{1}{L_*} \sim \sqrt{\frac{\mu T}{\varkappa_0^2}}$$

Ginzburg scale

$$\frac{d\varkappa}{d\Lambda} = \eta \varkappa$$

$$\Lambda = \ln(q^*/q)$$

$$\varkappa \propto L^\eta \propto \frac{1}{q^\eta}$$



**Graphene:**

$L_* \sim 5 \div 10$ nm

η≈ 0.7-0.8 – critical index

David ,Guitter,
Europhys. Lett. (1988);
Guitter, David, Leibler, Peliti,
J. Phys. France (1989);
Le Doussal , Radzihovsky,
PRL (1992)

# Crumpling transition, $\sigma$=0

$$\left\{ \begin{array}{lcl} \dfrac{d\varkappa}{d\Lambda} = \eta\varkappa & \Longrightarrow & \varkappa = \varkappa_0 e^{\eta\Lambda} \\[2ex] \dfrac{d\xi^2}{d\Lambda} = -\dfrac{T}{4\pi\varkappa} & \Longrightarrow & \xi = \xi_{q\to 0} \end{array} \right.$$

$$\boxed{\xi^2 = 1 - \frac{T}{T_c}}$$

**second-order phase transition**

$$\Lambda = \ln(q^*/q)$$

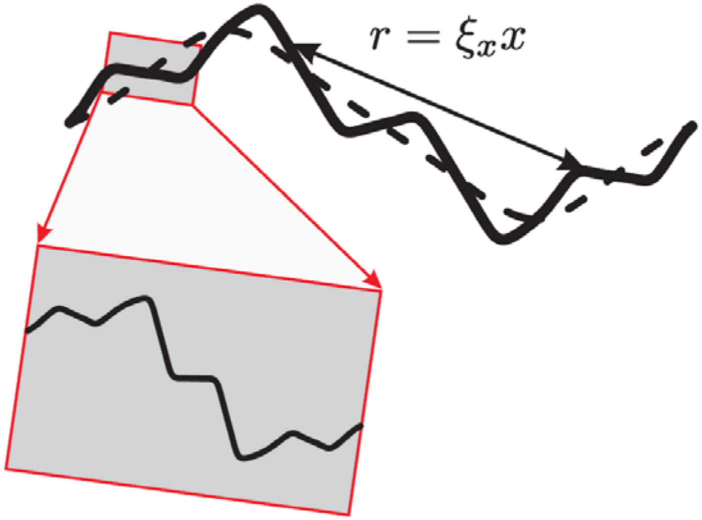
$$\boxed{T_c = 4\pi\eta\varkappa_0}$$

**critical temperature of CT for fixed bending rigidity**

$$\boxed{\varkappa_c = \frac{T}{4\pi\eta}}$$

**critical bending rigidity for fixed temperature**

# Fractal geometry of the membrane



Exactly at the transition point

$$\xi \propto \frac{1}{L^{\eta/2}}$$

$$R = \xi_L L$$

$$R^{D_H} \propto L^2$$

⇒

$$\boxed{D_H = \frac{2}{1-\eta/2}} > 2$$

fractal (Hausdorff) dimension

# Anomalous Hooke's law

External tension → $$E_{\mathbf{h}} = \frac{1}{2}\int \left[\varkappa(\Delta \mathbf{h})^2 + \sigma(\nabla \mathbf{h})^2\right] d^2\mathbf{x}$$

**new scale $q = q_\sigma$ :**

$$\begin{cases} \varkappa_q q^4 \sim \sigma q^2 \\ \varkappa_q = \varkappa_0 \left(\dfrac{q_*}{q}\right)^\eta \end{cases}$$

$$\boxed{q_\sigma = q_* \left(\frac{\sigma}{\sigma_*}\right)^{1/(2-\eta)}}$$

**scaling stops at $q = q_\sigma$**

$$\sigma_* \sim (\mu + \lambda)\frac{T}{\varkappa_0}$$

# Finite tension

**1) T=0, σ≠0 → fluctuations are absent**

$$\frac{\sigma}{\lambda+\mu}=\xi^2-1\approx 2(\xi-1)$$

linear Hooke's law

**2) T≠0 → fluctuations**

$$\frac{\sigma}{\lambda+\mu}=\xi^2-1+\underbrace{\frac{T}{T_\mathrm{c}}}_{\substack{\text{contribution}\\ \text{of fluctuations}\\ \text{at } \sigma=0}}$$

➡ does not take into account suppression of fluctuations by $\sigma$

# 3) Effect of tension on fluctuations

$$\frac{\sigma}{\mu+\lambda} = \xi^2 - 1 + \frac{\langle \nabla h \nabla h \rangle}{2}$$

$$\langle \nabla h \nabla h \rangle = \int_{q_\sigma} \frac{T}{\varkappa_q} \frac{d^2\mathbf{q}}{q^2}$$

$$q_\sigma = q_* \left(\frac{\sigma}{\sigma_*}\right)^{1/(2-\eta)}$$

$$\sigma_* \sim (\mu+\lambda) \frac{T}{\varkappa_0}$$

$$\frac{\sigma}{\mu+\lambda} = \xi^2 - 1 + \underbrace{\frac{T}{T_c}\left[1 - \left(\frac{\sigma}{\sigma_*}\right)^{\alpha}\right]}_{\text{“hidden area”}}$$

**stress suppresses fluctuations !!!**

# Balance equation for membrane under isotropic tension

$$\boxed{\frac{\sigma_*}{\mu+\lambda}\left[\frac{\sigma}{\sigma_*}+\frac{1}{\alpha}\left(\frac{\sigma}{\sigma_*}\right)^{\alpha}\right]=\xi^2-\xi_T^2}$$

↑ normal ↑ anomalous

$$\xi_T^2 = 1 - \frac{T}{T_c}$$

global deformation for $\sigma = 0$

$$\sigma_* \sim (\mu+\lambda)\frac{T}{\varkappa_0}$$

crossover tension

$$\sigma \ll \sigma_* \;\Rightarrow\; \boxed{\delta\xi = \xi - \xi_T \propto \sigma^{\alpha}}$$

$$\boxed{\alpha = \frac{\eta}{2-\eta} < 1}$$

critical index of buckling transition

# Disordered membrane: Random curvature

$$E = \int d^2\mathbf{x} \left\{ \frac{\varkappa_0}{2}[\Delta\mathbf{h} + \boldsymbol{\beta}(\boldsymbol{x})]^2 + \mu u_{ij}^2 + \frac{\lambda}{2} u_{ii}^2 \right\}$$

↑ random field

Radzihovsky, Nelson, PRA (1991);

Other models of disorder, see Le Doussal , Radzihovsky, PRB (1993)

$$P(\boldsymbol{\beta}) = Z_{\boldsymbol{\beta}}^{-1} \exp\left( -\frac{1}{2b} \int \beta^2(\mathbf{x}) d^2\mathbf{x} \right)$$

*b* – strength of the disorder

Similar to dynamical fluctuations

$$\frac{T}{\varkappa} \to b$$

Calculations: RPA + replica trick

# Scaling in disordered graphene

$$\frac{d\xi^2}{d\Lambda} = -\frac{1}{4\pi}\left(\frac{T}{\varkappa} + b\right)$$

$$\frac{df}{d\Lambda} = -\eta\frac{f\,(1+3f)}{(1+2f)^2}$$

$$\frac{d\varkappa}{d\Lambda} = \eta\varkappa\frac{1+3f+f^2}{(1+2f)^2}$$

$$f = \frac{b\varkappa}{T}$$

**key parameter**

$f \gg 1$ → ripples dominate

$f \ll 1$ → thermal fluctuations (FP) dominate

$f \gg 1$

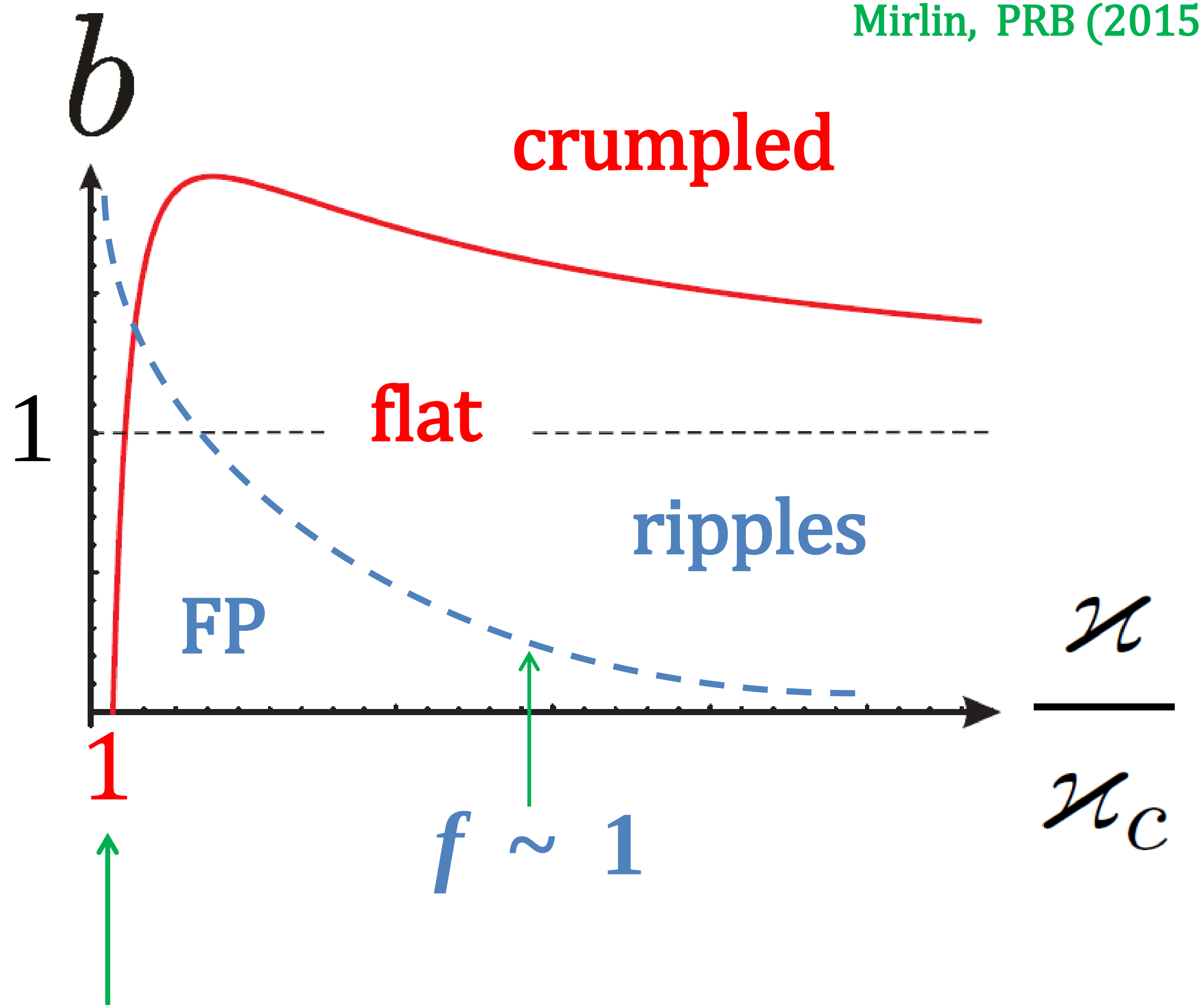
$$\frac{d\varkappa}{d\Lambda} = \frac{\eta}{4}\varkappa$$

strongly disordered case: $\eta \to \frac{\eta}{4}$

# Crumpling transition in disordered membrane, $\sigma = 0$

Gornyi, Kachorovskii, Mirlin, PRB (2015)

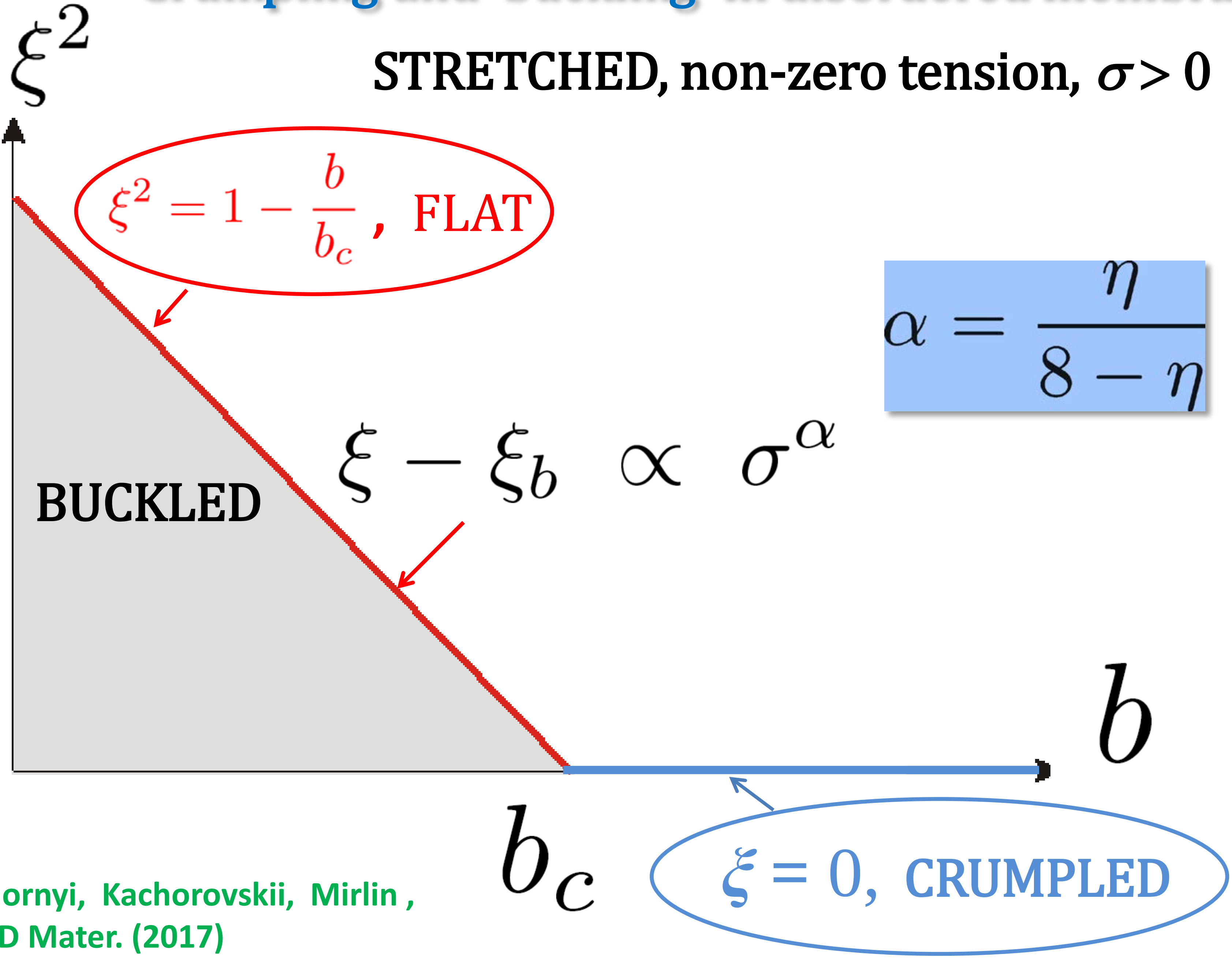
critical disorder strength →



$\varkappa_c = \frac{T}{4\pi\eta}$ - critical bending rigidity for fixed temperature

# Crumpling and buckling in disordered membrane

STRETCHED, non-zero tension, $\sigma > 0$



$\xi^2 = 1 - \frac{b}{b_c}$, FLAT

$$\alpha = \frac{\eta}{8-\eta}$$

$$\xi - \xi_b \propto \sigma^\alpha$$

$\xi = 0$, CRUMPLED

Gornyi, Kachorovskii, Mirlin,
2D Mater. (2017)

# Negative thermal expansion coefficient

$$\xi^2 = 1 - \frac{T}{T_c}$$

→

$$\boxed{\alpha_T = -\frac{1}{T_c}}$$

$$T_c = 4\pi\eta\varkappa_0 \sim \varkappa_0$$

Agrees with experiment:

Chen et al., 2009; Singh et al., 2010;
Bao et al., 2009; Bolotin et al 2014

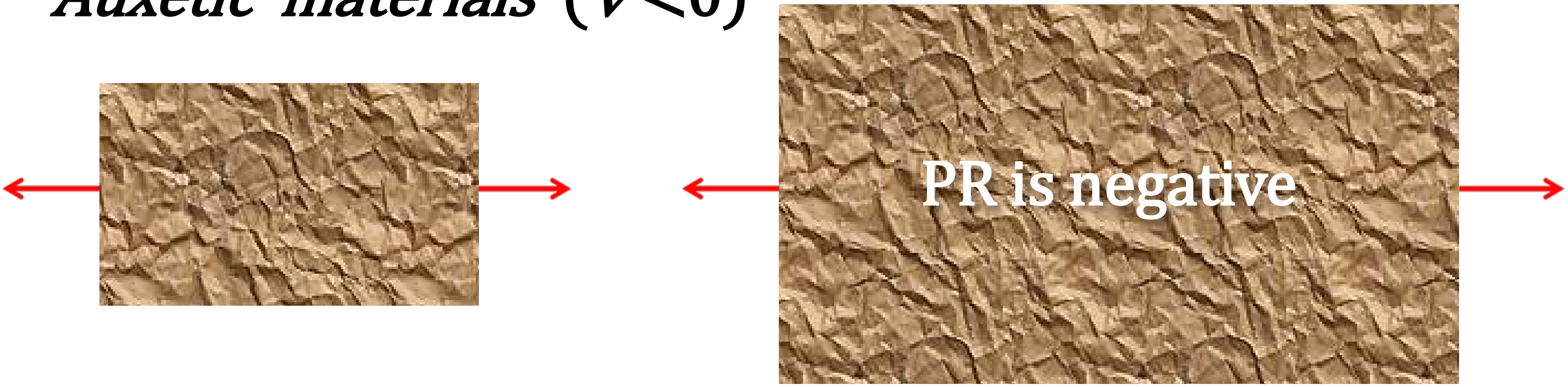
# Uniaxial stress

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

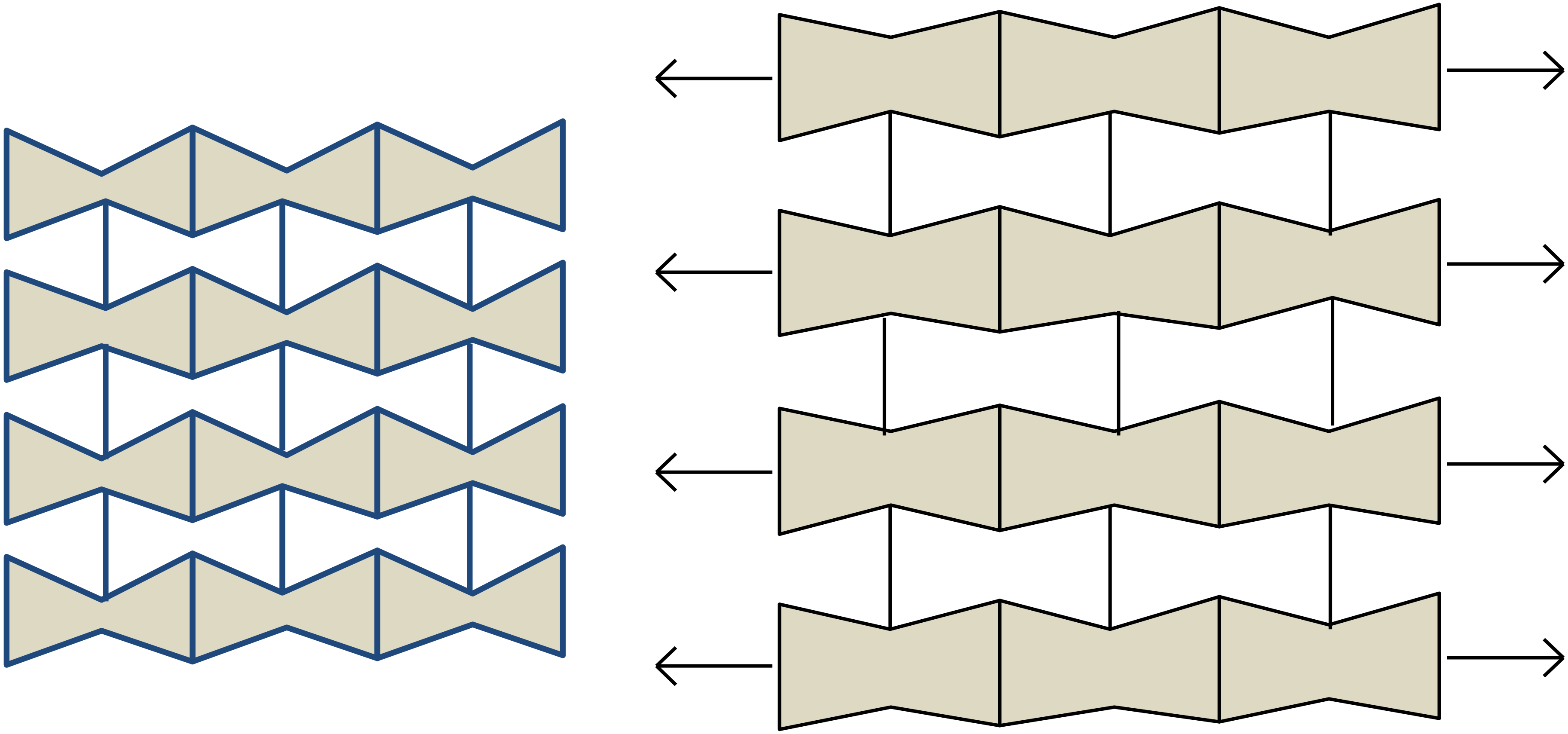
Poisson‘s ratio (PR)

*Conventional materials* ($\nu > 0$)

*Auxetic materials* ($\nu < 0$)



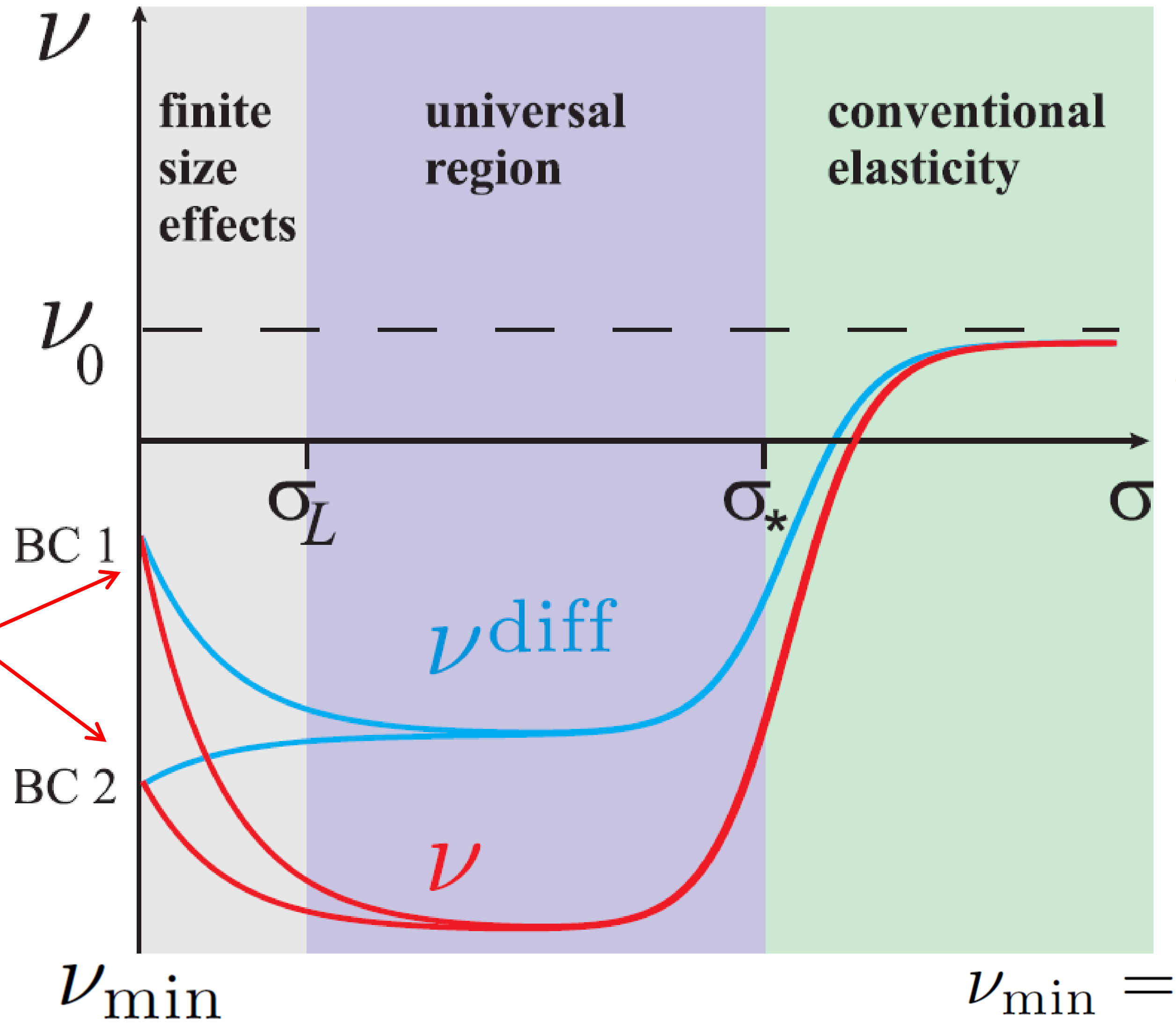
# Example: engineered auxetic structure with PR<0



PR of graphene: negative or positive?

# Poisson ratio of a generic membrane

Burmistrov, Gornyi, Kachorovskii,
Los, Katsnelson, Mirlin PRB (2018)



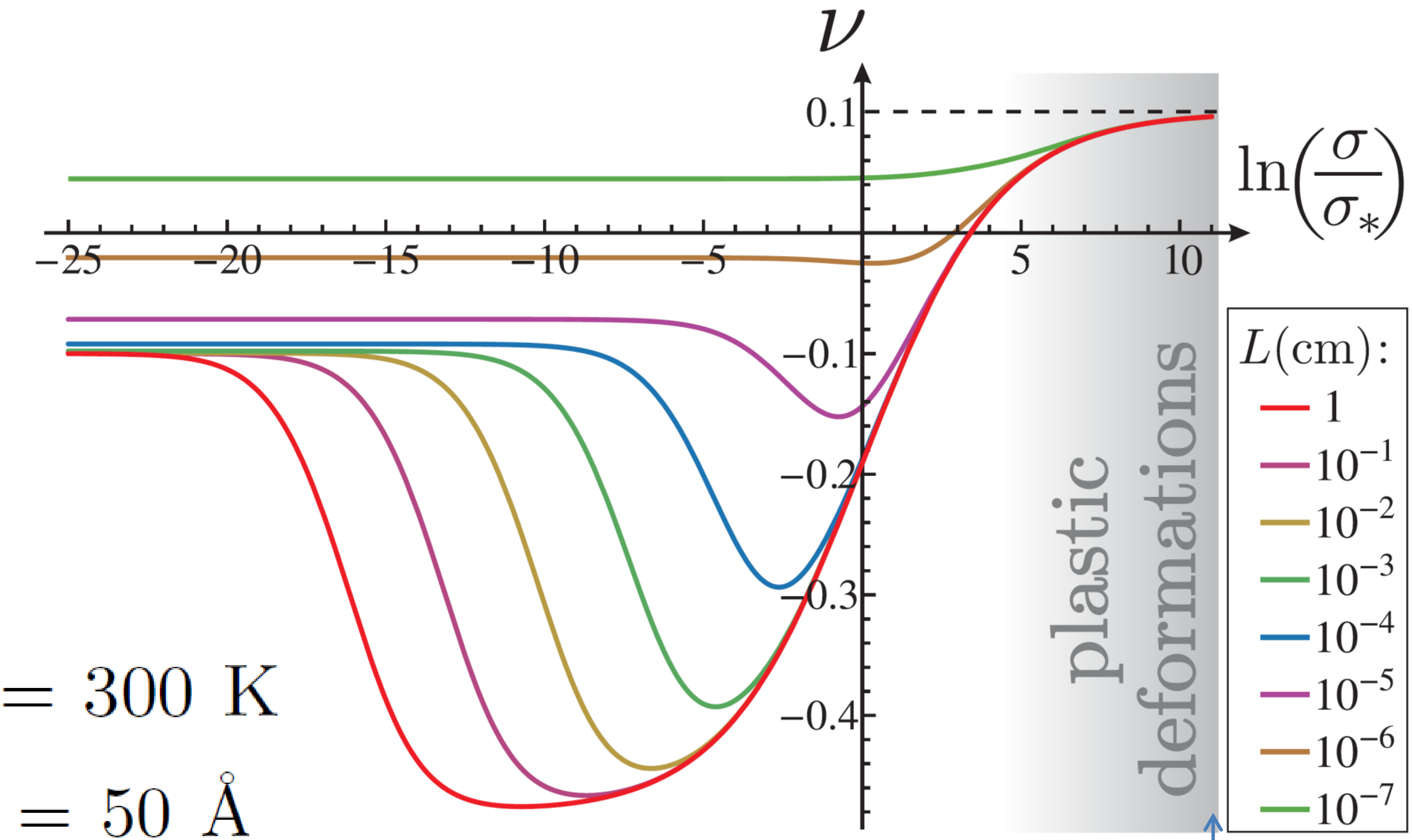
$$\nu_0 = \frac{\lambda}{2\mu + \lambda}$$

depends on boundary conditions

$$\nu_{\rm min} = -\frac{1}{1+2\alpha}$$

# Poisson ratio of graphene

Burmistrov, Gornyi, Kachorovskii,
Los, Katsnelson, Mirlin PRB (2018)



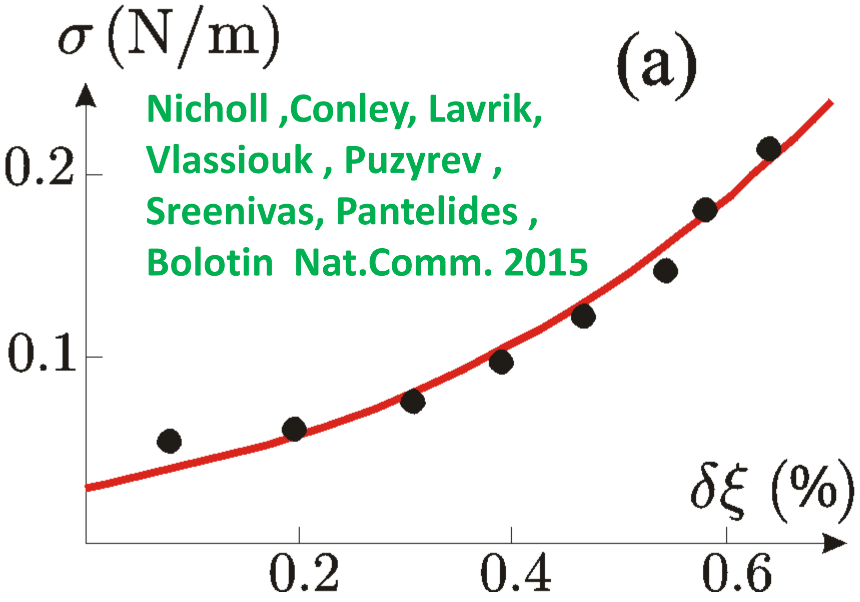
$T = 300$ K

$L_* = 50$ Å

$\sigma_* = 0.1$ N/m

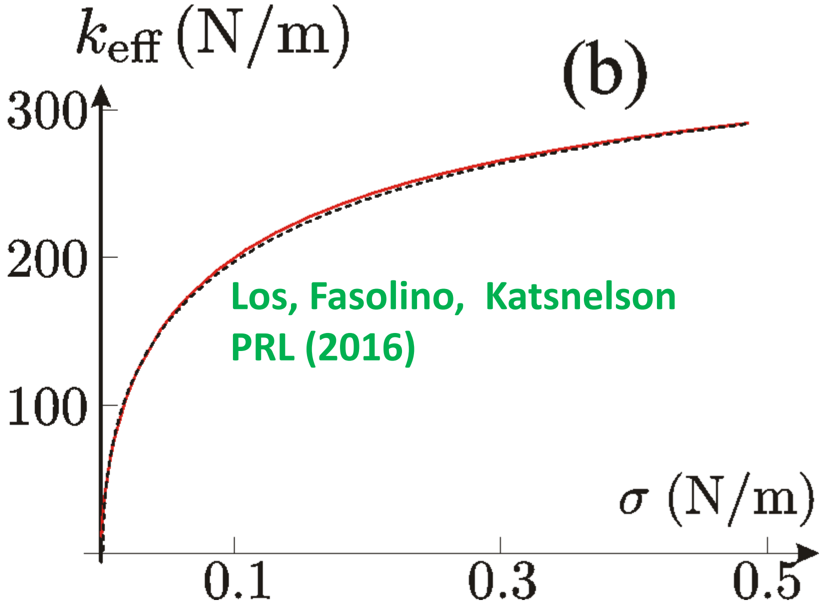
$\sigma > 50$ N/m,

# Anomalous Hooke’s law (experiment+simulation)



**Nicholl ,Conley, Lavrik, Vlassiouk , Puzyrev , Sreenivas, Pantelides , Bolotin Nat.Comm. 2015**

(a) Stress-strain dependence. Dots – experiment (Nicholl et al), red line – theory (Gornyi et al) for strongly disordered case $\alpha = 0.1$ with degree of disorder $B = 0.004$.



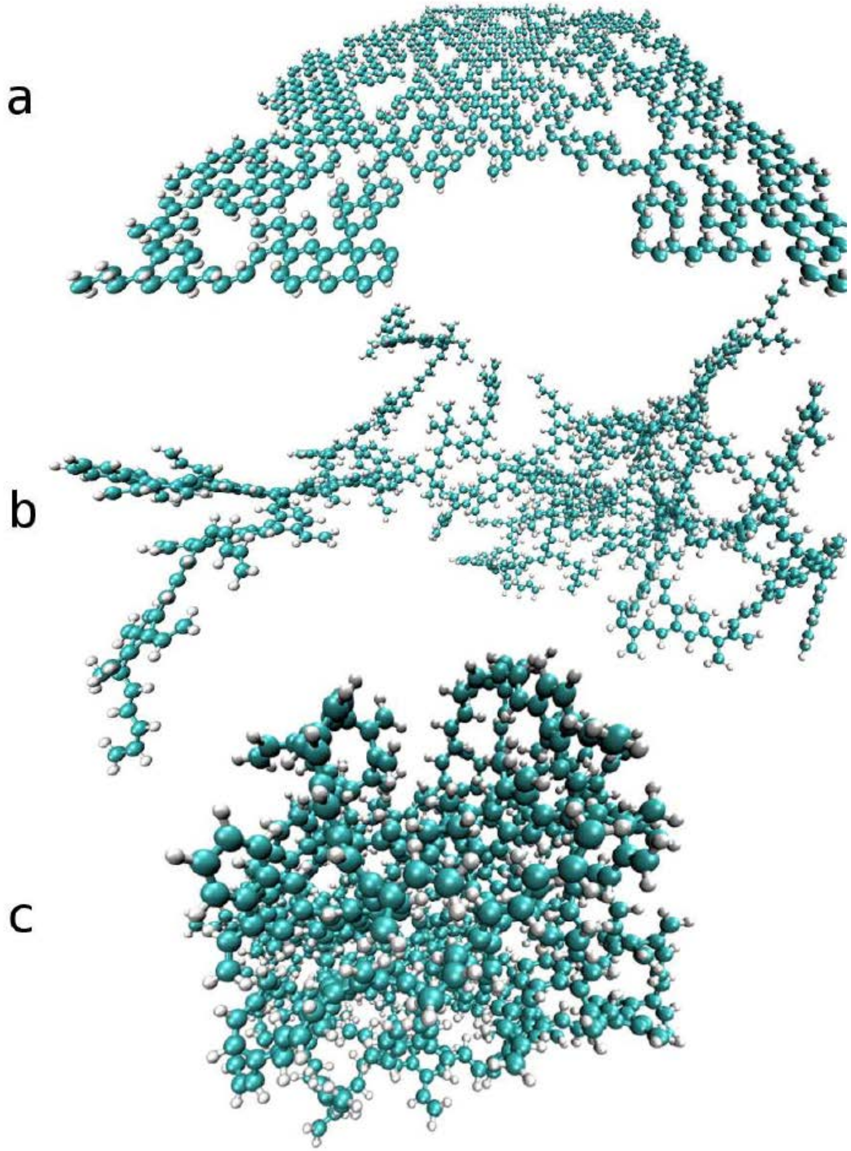
**Los, Fasolino, Katsnelson PRL (2016)**

(b) Effective stiffness $k_{\rm eff}$ *vs.* stress $\sigma$ in clean graphene at $T = 300$K. Dashed line – numerical simulations (Los et al), red line – theory (Gornyi et al) with $\alpha = 0.62$ (i.e., $\eta = 0.765$) and $\sigma_* \simeq 0.1\,$N/m.

$$k_{\rm eff} = \partial\sigma/\partial\xi \simeq k_0 \frac{(\sigma/\sigma_*)^{1-\alpha}}{1+(\sigma/\sigma_*)^{1-\alpha}}$$

**Gornyi, Kachorovskii, Mirlin , 2D Mater. (2017)**

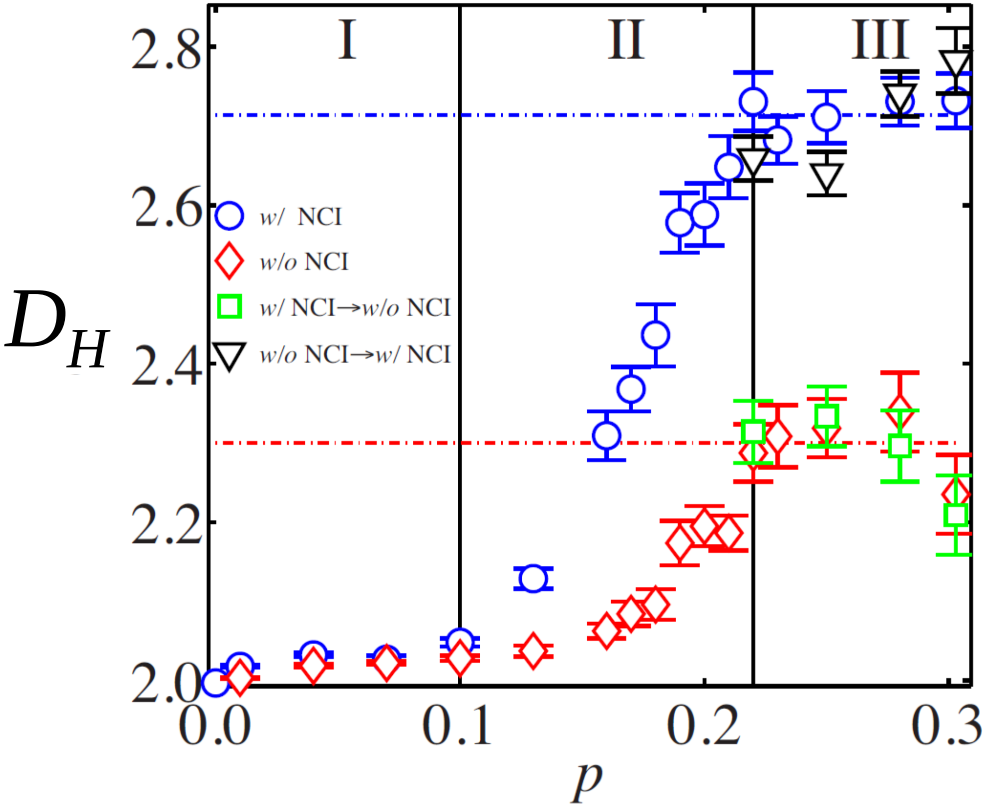
# Disorder-induced crumpling



Giordanelli, Mendoza, Andrade, Gomes, Herrmann, Scientific Reports (2016)

Pristine graphene membranes were damaged by adding random vacancies and carbon-hydrogen bonds.

# Fractal dimension of crumpled graphene



concentration of vacancies

$$D_H^{clean} = \frac{2}{1-\eta/2}$$

$$\downarrow \quad (\eta \to \eta/4)$$

$$D_H^{dis} = \frac{2}{1-\eta/8} \approx 2.2$$

for $\eta \approx 0.8$

# Interesting theoretical problems to be solved:

## 1) Bubbles on the substrate

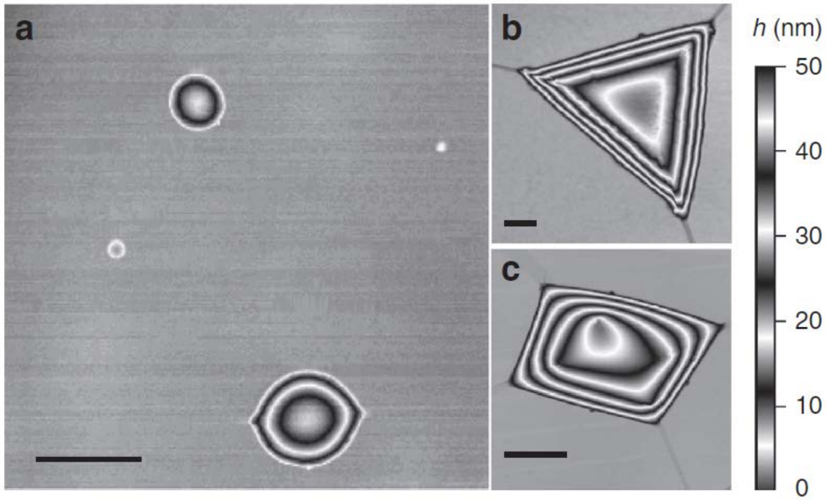


**Figure 1 | Graphene bubbles.** (**a**–**c**) AFM images of graphene bubbles of different shapes. Scale bars, 500 nm (**a**); 100 nm (**b**); 500 nm (**c**). The vertical scale on the right indicates the height of the bubbles.

**Khestanova, Guinea, Fumagalli, Geim, I.V. Grigorieva, Nature Comm. 2016**

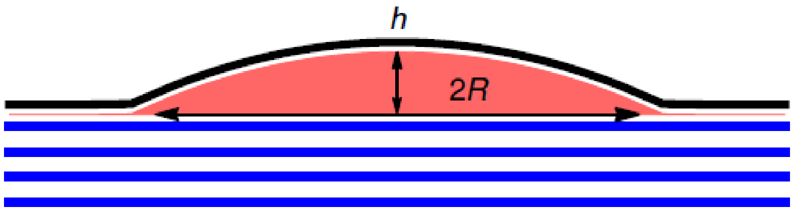
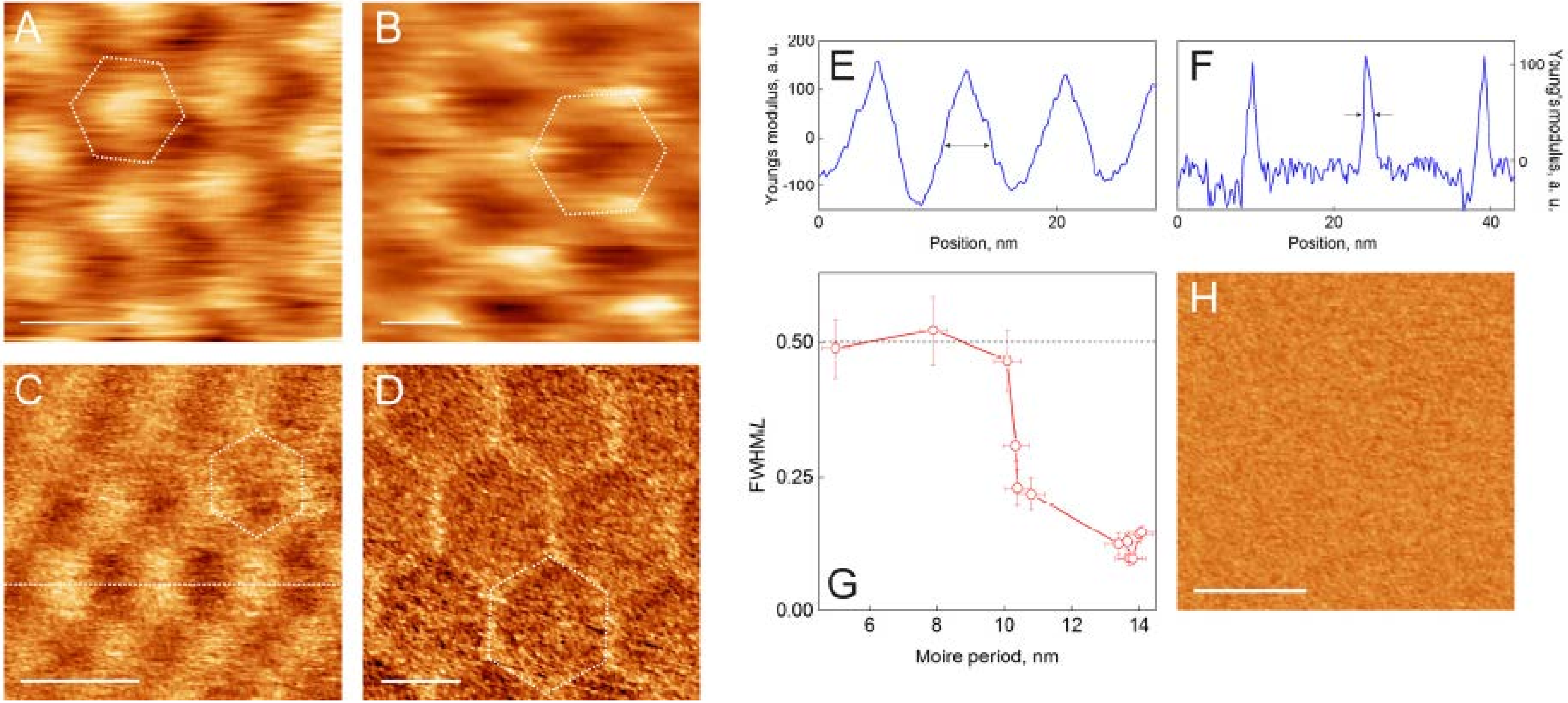


**Figure 5 | Sketch of the bubble considered in our theoretical analysis.** The bubble is formed by material trapped between a substrate and a 2D layer (graphene).

# 2) Commensurate-incommensurate transition in graphene on hBN

C. R. Woods et al *Nature Physics* 2014

## Moiré patterns



# Main results

- Anharmonicity crucially effects elastic properties of graphene → crumpling and buckling transitions
- Stretching of the membrane is non-linear function of tension
- Strong disorder leads to crumpling transition
- Thermal expansion coefficient is negative up to very low temperatures
- Poisson ratio is controlled by applied stress and can change sign