Anomalous Scaling in the Compressible Kazantsev-Kraichnan Model with Spatial Parity Violation

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ÚEF SAV

Kazantsev-Kraichnan Model

- model of kinematic magnetohydrodynamic (MHD) turbulence
- solenoidal magnetic field $\mathbf{b}(t,\mathbf{x})$ is considered as a passive vector admixture described by the stochastic equation
$$\partial_t \mathbf{b} = \nu_0 \Delta \mathbf{b} - (\mathbf{v}\cdot\partial)\,\mathbf{b} + (\mathbf{b}\cdot\partial)\,\mathbf{v} + \mathbf{f}, \tag{1}$$
where $\partial_t \equiv \frac{\partial}{\partial t}, \Delta \equiv \partial^2$ is the Laplace operator, $\nu_0 = \dfrac{c^2}{4\pi\sigma_0}$ is the magnetic diffusivity with magnetic conductivity σ_0
- $\mathbf{f}(t,\mathbf{x})$ represents a transverse Gaussian random noise with zero mean and the correlation function
$$D^b_{ij}(t,\mathbf{x};t',\mathbf{x}') \equiv \left\langle f_i(t,\mathbf{x}) f_j(t',\mathbf{x}')\right\rangle = \delta(t-t') C_{ij}(\left|\mathbf{x}-\mathbf{x}'\right|/L)$$
- exact form of function $C_{ij}(|\mathbf{x}-\mathbf{x}'|\,/L)$ is unimportant

$$\partial_t \mathbf{b} = \nu_0 \Delta \mathbf{b} - (\mathbf{v} \cdot \partial)\, \mathbf{b} + (\mathbf{b} \cdot \partial)\, \mathbf{v} + \mathbf{f}$$

- $\mathbf{v}(t, \mathbf{x})$ is random compressible $(\partial \cdot \mathbf{v} \neq 0)$ velocity field, which obeys Gaussian statistics $(\langle \mathbf{v}(t, \mathbf{x}) \rangle = 0)$ with the pair correlation function

$$D_{ij}(x; x') \equiv \left\langle v_i(x) v_j(x') \right\rangle = \delta(t - t') D_0 \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{R_{ij}(\mathbf{k})}{k^{d+\varepsilon}} e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')},$$

  where d denotes the spatial dimension of the system and $D_0 \equiv g_0 \nu_0$ is positive amplitute

- $R_{ij}(\mathbf{k})$ represents a projector defined as

$$R_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2} + \alpha \frac{k_i k_j}{k^2} + \mathrm{i}\varepsilon_{ijs}\rho \frac{k_s}{|\mathbf{k}|},$$

  where $0 < \alpha < \infty$ is the compressibility parameter and $0 < |\rho| < 1$ determines the amount of helicity in the system

Field Theoretic Model

Theorem

DeDominicis-Janssen theorem states that stochastic problem (1) is equivalent to the field theoretic model of a set of three fields $\mathbf{v}$, $\mathbf{b}$, and $\mathbf{b}'$ with the action functional

$$
\begin{aligned}
S\left[\mathbf{v},\mathbf{b},\mathbf{b}'\right] = & -\frac{1}{2}\int \mathrm{d}x_1\,\mathrm{d}x_2 v_i(x_1) D^{-1}_{ij}(x_1;x_2) v_j(x_2) \\
& + \frac{1}{2}\int \mathrm{d}x_1\,\mathrm{d}x_2 b'_i D^b_{ij}(x_1;x_2) b'_j(x_2) \\
& + \int \mathrm{d}x \mathbf{b}' \cdot \left[-\partial_t \mathbf{b} + \nu_0 \Delta \mathbf{b} - (\mathbf{v}\cdot\partial)\,\mathbf{b} + (\mathbf{b}\cdot\partial)\,\mathbf{v}\right]
\end{aligned} \tag{2}
$$

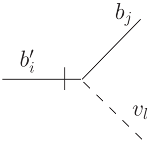
Model (2) corresponds to a standrad Feynman diagrammatic perturbation theory with propagators

$$\Delta_{ij}^{bb'}(k) = \langle b_i b'_j \rangle_0 = \frac{P_{ij}(\mathbf{k})}{-\mathrm{i}\omega_k + \nu_0 k^2} = \left(\Delta_{ij}^{b'b}\right)^*, \quad \langle b_i b'_j \rangle_0 =$$

$$\Delta_{ij}^{vv}(k) = \langle v_i v_j \rangle_0 = \frac{\nu_0 g_0 R_{ij}(\mathbf{k})}{k^{d+\varepsilon}}, \quad \langle v_i v_j \rangle_0 =$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - \dfrac{k_i k_j}{k^2}$ is the ordinary transverse projector, and the interaction vertex in a frequency-momentum representation $V_{ijl} = \mathrm{i}(k_l \delta_{ij} - k_j \delta_{i,l})$

$V_{ijl} =$



RG Analysis

Q	$\mathbf{v}$	$\mathbf{b}$	$\mathbf{b}'$	m, Λ, μ	ν_0, ν	g_0	g, α, ρ
d_Q^k	-1	0	d	1	-2	ε	0
d_Q^ω	1	0	0	0	1	0	0
d_Q	1	0	d	1	0	ε	0

Table 1: Canonical dimensions of the fields and parameters of the model under consideration.

- logarithmic for $\varepsilon = 0$
- the only superficially divergent function is the 1-irreducible Green's function $\langle b_i' b_j \rangle_{1-\mathrm{ir}}$
- parameters renormalization

$$\nu_0 = \nu Z_\nu, \; g_0 = g\mu^\varepsilon Z_g, \; Z_g = Z_\nu^{-1}$$

- the only independent renormalization constant is given by a diagram shown on the right

$$Z_\nu = 1 - \frac{S_d}{(2\pi)^d} \frac{d-1+\alpha}{2d} \frac{g}{\varepsilon}, \qquad (3)$$

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

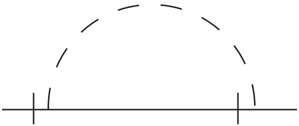
$\Sigma_{b_i' b_j} =$ 

Figure 1: The only self-energy Feynman diagram that contributes to the UV renormalization of the model.

- Equation (3) is exact (no corrections of order $g^n, n \geq 2$)

RG Analysis

- RG functions

$$\beta_g \equiv \mu\partial_\mu g = g(-\varepsilon + \gamma_\nu),$$
$$\gamma_\nu \equiv \mu\partial_\mu \ln Z_\nu$$
$$\gamma_\nu = \frac{S_d}{(2\pi)^d}\frac{d-1+\alpha}{2d} g$$

- inertial range scaling behaviour is driven by the exact one-loop stable fixed point of RG funcions, namely

$$g_* = \frac{(2\pi)^d}{S_d}\frac{2d}{d-1+\alpha}\varepsilon,$$

which is obtained by the requirement of vanishing of β_g. Note that the exact value is $\gamma_\nu^* = \varepsilon$, which is IR stable for $\varepsilon > 0$ and corresponds to the so-called kinetic regime in the genuine MHD turbulence

RG Analysis

- we are interested in the scaling behaviour of single-time two-point correlation function of the magnetic field

$$B_{N-m,m}(r) \equiv \left\langle b_r^{N-m}(t,\mathbf{x}) b_r^m(t,\mathbf{x}')) \right\rangle, \quad r = \left|\mathbf{x}-\mathbf{x}'\right|,$$

where b_r denotes the component of the magnetic field along $\mathbf{r} = \mathbf{x} - \mathbf{x}'$

- general correlation function with IR asymptotic form

$$G(r) \simeq \nu_0^{d_G^\omega} l^{-d_G} (r/l)^{-\Delta_G} R(r/L),$$

where d_G and d_G^ω are the corresponding canonical dimensions of G, $R(r/L)$ is a scaling function, $l = 1/\Lambda$ represents the viscous scale, $L = 1/k_{\text{min}}$ is the integral scale, and Δ_G denotes the critical dimension defined as

$$\Delta_G = d_G^k + \Delta_\omega d_G^\omega + \gamma_G^*.$$

γ_G^* represents fixed point value of $\gamma_G \equiv \mu\partial_\mu \ln Z_G$, Z_G is the renormalization constant of $G = Z_G G^R$, and $\Delta_\omega = 2 - \gamma_v^* = 2 - \varepsilon$

$$\Delta_{\mathbf{v}} = 1 - \varepsilon, \quad \Delta_{\mathbf{b}} = 0, \quad \Delta_{\mathbf{b}'} = d$$

- using the relations for generalized correlation function one obtains

$$B_{N-m,m}(r) \simeq \nu_0^{-N/2}(r/l)^{-\gamma^*_{N-m}-\gamma^*_m} R_{N,m}(r/L),$$

where γ^*_{N-m} and γ^*_m are the anomalous dimensions of the composites operators b_r^{N-m} and b_r^m, respectively, taken at the fixed point g_*

- deep inside the inertial region $(r/L \to 0)$ scaling function $R_{N,m}(r/L)$ takes the form

$$R_{N,m}(r/L) = \sum_i C_{F_i}(r/L)(r/L)^{\Delta_{F_i}},$$

where summation over all possible renormalized composite operators F_i with corresponding critical dimensions Δ_{F_i} is performed

- leading contribution is given by operators constructed solely from $\mathbf{b}(x)$ in the form

$$F_{N,p} = [\mathbf{n}\cdot\mathbf{b}]^p\,(\mathbf{b}\cdot\mathbf{b})^l\,, \quad N = 2l + p$$

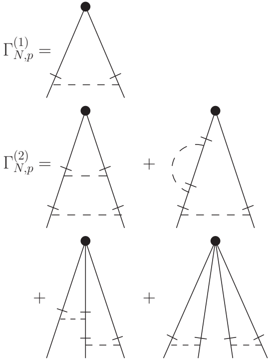


Figure 2: The Feynman diagrams for the function $\Gamma_{N,p}(x;\mathbf{b})$ in the two-loop approximation following the rules mentioned previously.

- final form of the asymptotic inertial range behaviour of the correlation functions is then

$$B_{N-m,m}(r) \propto r^{\zeta_{N,m}} = r^{\zeta^{(1)}_{N,m}\varepsilon + \zeta^{(2)}_{N,m}\varepsilon^2}$$

- for both N and m either even or odd

$$\zeta^{(1)}_{N,m} = -\frac{m(N-m)(d-1)[1+\alpha(d+1)]}{(d+2)(d-1+\alpha)}$$

- for even values of N and odd values of m

$$\zeta^{(1)}_{N,m} = -\frac{(d-1)\{m(N-m)[1+\alpha(d+1)] + d + 1 + \alpha\}}{(d+2)(d-1+\alpha)}$$

- the two-loop corrections $\zeta^{(2)}_{N,m}$ have the following form

$$\begin{aligned}\zeta^{(2)}_{N,m} = -\frac{S_{d-1}}{S_d}\frac{d}{(d+2)(d-1+\alpha)^2}\int_0^1 \mathrm{d}x(1-x^2)^{\frac{d-3}{2}}\Big\{\sqrt{1-x^2} \\ \times\ [(d-2)D_1(W_1Y_1 + 2\rho^2\delta_{3d}Y_3) + D_2W_2Y_1] \\ -\ \frac{2}{d+4}(D_3W_3 + D_4W_4)Y_2\Big\}\end{aligned}$$

Anomalous scaling of $B_{N-m,m}(r)$

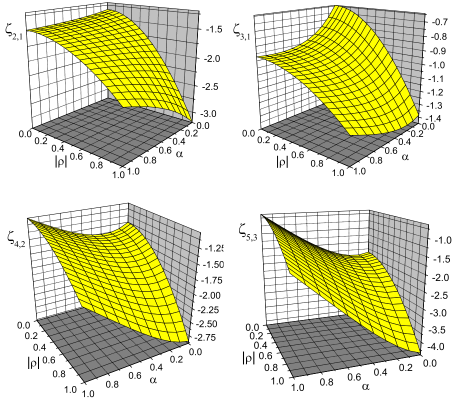


Figure 3: Dependencies of the total two-loop scaling exponents $\zeta_{2,1}$, $\zeta_{3,1}$, $\zeta_{4,2}$, and $\zeta_{5,3}$ on α and ρ for $d = 3$ and $\varepsilon = 1$.

- the scaling properties of the correlation function $B_{N-m,m}$ become more anomalous due to the impact of helicity
- in agreement with recent experimental measurements[a]
- behaviour of $\zeta_{2,1}$ as a function of α for fixed $|\rho|$
- unique behaviour of $\zeta_{3,1}$ as a function of $\alpha \ll 1$ for $|\rho| \approx 1$
- decreasing tendencies of $\zeta_{4,2}$ and $\zeta_{5,3}$ for small enough α and $|\rho|$
- for large enough value of α the scaling exponents become increasing functions of α regardless of the value of $|\rho|$

[a] D. A. Schaffner *et al*, Phys. Rev. Lett. **112**, (2014) 165001

Anomalous scaling of $B_{N-m,m}(r)$

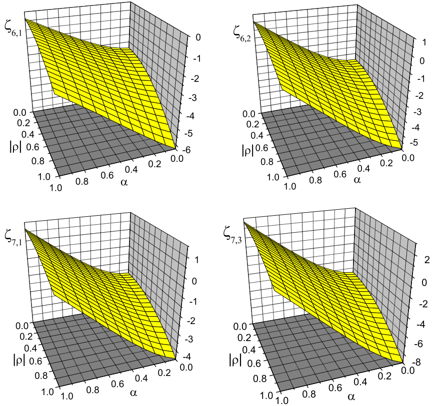


Figure 4: Dependencies of the total two-loop scaling exponents $\zeta_{6,1}$, $\zeta_{6,2}$, $\zeta_{7,1}$, and $\zeta_{7,3}$ on α and ρ for $d = 3$ and $\varepsilon = 1$.

- $\zeta_{N,m}, N = 6, 7$ are universally increasing functions of α regardless of the value of the helicity parameter ρ
- although not shown here, similar behaviour is valid for all scaling exponents $N \geq 8$

Conclusion

- ▷ scaling properties of $B_{N,m}(r)$ within the framework of helical and compressible Kazantsev-Kraichnan model were investigated using field theoretic RG technique and the OPE up to the two-loop approximation
- ▷ IR asymptotic behaviour in the inertial interval is dependent on α but not on ρ
- ▷ presence of helicity can significantly decrease the scaling exponents of the magnetic correlation functions
- ▷ influence of compressibility is also investigated but exhibits more complicated behaviour
    - for small order correlation functions the corresponding scaling exponents decrease as functions of the compressibility parameter at least for $\alpha \ll 1$ and $|\rho| \ll 1$
    - however, for higher order correlation functions the scaling exponents become increasing functions of α regardless of the value of the helicity parameter

Thank you for your attention!

Simultaneous influence of helicity and compressibility on anomalous scaling of the magnetic field in the Kazantsev-Kraichnan model
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Phys. Rev. E **95**, (2017) 053210

$$C_1 = (d+1)(N-p)(d+N+p-2) - 2N(N-1 \quad (4)$$

$$C_2 = -(N-p)(d+N+p-2) + dN(N-1), \quad (5)$$

$$C_3 = (N-2)C_1, \quad (6)$$

$$C_4 = (N-2)[-3(N-p)(d+N+p-2) + (d+2)N(N-1)], \quad (7)$$

and

$$W_1 = 2 + \alpha - \alpha^2 \quad (8)$$

$$W_2 = 2(1-x^2) + \alpha[d(d-3) + 4x^2] - \alpha^2[d(d-1) - 2(1-x^2)], \quad (9)$$

$$W_3 = (1-x^2)(9-5d+4x^2) + \alpha[9(1-2x^2) + x^2(d^2+8x^2) + 5d(1-x^2)] - \alpha^2(10-3d-11x^2+4x^4), \quad (10)$$

$$W_4 = -2(1-x^2)^2 + 4\alpha(1-x^2)(d-x^2) + \alpha^2[d^2(d+1-x^2) - 2(1-x^2)^2 + d(2x^2-3)]. \quad (11)$$

In addition,

$$Y_1 = x\left[\arctan\left(\frac{1+x}{\sqrt{1-x^2}}\right) - \arctan\left(\frac{1-x}{\sqrt{1-x^2}}\right)\right], \quad (12)$$

$$Y_2 = \frac{x}{\sqrt{4-x^2}}\left[\arctan\left(\frac{2+x}{\sqrt{4-x^2}}\right) - \arctan\left(\frac{2-x}{\sqrt{4-x^2}}\right)\right], \quad (13)$$

$$Y_3 = \pi - \arctan\left(\frac{1+x}{\sqrt{1-x^2}}\right) - \arctan\left(\frac{1-x}{\sqrt{1-x^2}}\right). \quad (14)$$

$$D_1 = D_2 = 2m(N-m), \quad D_3 = m(N-m)(3N+2d-4), \tag{15}$$
$$D_4 = 3m(N-4)(N-m) \tag{16}$$

for even values of N and m,

$$D_1 = 2[m(N-m)+d+1], \quad D_2 = 2[m(N-m)-1], \tag{17}$$
$$D_3 = m(N-m)(3N+2d-4)+(N-4)(d+1), \tag{18}$$
$$D_4 = 3(N-4)[m(N-m)-1] \tag{19}$$

for even N and odd m, and

$$D_1 = D_2 = 2m(N-m), \quad D_3 = (N-m)[m(3N+2d-4)-d-1], \tag{20}$$
$$D_4 = 3(N-m)[m(N-4)+1], \tag{21}$$